MATH6520 Differentiable Manifolds

Taught by Reyen Sjamaar
Notes by Yun Liu

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1 Introduction

1.1 Topological manifolds

Let A be an open subset of $\mathbb{R}^n$, $f : A \to \mathbb{R}$ a function, $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ a multi-index. The α-th partial derivative of f is $\partial^\alpha f = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_1^{\alpha_n}} f : A \to \mathbb{R}$ if exists. The order of α is $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Let $i \geq 0$, we say f is C^i if $\partial^\alpha f$ exists and is continuous for all α of order $\leq i$.

$$C^i(A) = \{f : A \to \mathbb{R} | f \text{ is } C^i\}$$

We say f is C^∞or ***smooth*** if $f \in C^i(A)$ for all $i \geq 0$.

$$C^\infty(A) = \cap_{i \geq 0} C^i(A)$$

Now let f be a map $f : A \to \mathbb{R}^m$, $f(x) = \begin{pmatrix} f_1(x) \\ \cdots \\ f_m(x) \end{pmatrix}$. For $0 \leq i < \infty$, we say f is C^i if f_j is C_j for $1 \leq j \leq m$. Let B be open in $\mathbb{R}^m$,$f : A \to B$ is ***diffeomorphism*** if f is smooth, bijective and $f^{-1} : B \to A$ is smooth.

Example 1.1. $A = B = \mathbb{R}$, $f(x) = x^3$, then f is smooth and bijective, but f^{-1} is not smooth.

Remark 1.2. If $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ are open and $f : A \to B$ is a diffeomorphism, then $n = m$ by the chain rule:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \cap & & \cap & & \cap \\ \mathbb{R}^n & & \mathbb{R}^m & & \mathbb{R}^l \end{array}$$

if $f \in C^i$ and $g \in C^i$ then $g \circ f \in C^i$ and $D(g \circ f)(x) = D(g(f(x)))Df(x)$. So Df is invertible in this case because $I_n = D(f^{-1}(f(x)))Df(x)$ and $I_m = D(f(f^{-1}(x))Df^{-1}(x).$, which indicates $n \leq m$ and $m \leq n$ and therefore $n = m$.

Corollary 1.3. *If f is a diffeomorphism, then $Df(x)$ is invertible for all x.*

Proof. Denote $g = f^{-1} : B \to A$, then $f \circ g = Id_B$ and $g \circ f = Id_A$, so $Dg(f(x))Df(x) = I_n$ and $Df(g(x))Dg(x) = I_m$. □

Definition 1.4. Let M be a topological space. A ***chart*** on M is a pair (U, φ) where U is an open subset of M(domain), and $\varphi : U \to \mathbb{R}^n$ is a map with the following properties:

1. $\varphi(U)$ is open in $\mathbb{R}^n$;
2. $\varphi : U \to \varphi(U)$ is a homeomorphism (i.e.f is continuous, bijective and f^{-1} is continuous)

φ is a ***coordinate map***.

Definition 1.5. If $x \in U$, we say (U, φ) is a chart at x. If $x \in U$, and $\varphi(x) = 0$, we say (U, φ) is ***centered*** at x. M is l***ocally Euclidean*** on a topological manifold if M admits a chart at every point.

Give two charts (U, φ) and (V, ϕ) on M, we can form the ***transition map***:

$$\phi \circ (\varphi|_{U \cap V})^{-1} : \varphi(U \cap V) \to \phi(U \cap V)$$

which is a homeomorphism with inverse

$$\varphi \circ (\phi|_{U \cap V})^{-1} : \phi(U \cap V) \to \varphi(U \cap V)$$

Remark 1.6. We can abbreviate the map as $\varphi \circ \phi^{-1}$ and $\phi \circ \varphi^{-1}$.

The charts are ***compatible*** if $\varphi \circ \varphi^{-1}$ and $\phi \circ \varphi^{-1}$ is smooth (equivalently, either is a diffeomorphism). Trivially true if $U \cap V = \emptyset$. If $(U, \varphi : U \to \mathbb{R}^n)$ and $(V, \phi : V \to \mathbb{R}^m)$ are compatible and $U \cap V \neq \emptyset$, then $n = m$.

1.2 Smooth structure

Definition 1.7. An ***atlas*** $\mathscr{A}$ on M is a collection of charts

$$\mathscr{A} = \{(U_\alpha, \varphi_\alpha) | \alpha \in I\}$$

with the properties:

1. $\cup_\alpha U_\alpha = M$
2. every pair of charts $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$ is compatible. I.e. the transition map $\varphi_{\beta\alpha} : \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_{\alpha\beta}) \to \varphi_\beta(U_{\alpha\beta})$ is smooth.

The set of atlas on M is partially ordered by inclusion. (If $\mathscr{A}$, $\mathscr{B}$ are two atlas on M, we say $\mathscr{A} \leq \mathscr{B}$ if $\mathscr{A} \subseteq \mathscr{B}$.)

Definition 1.8. A ***smooth structure*** on M is a maximal atlas, i.e. if $\mathscr{B}$ is an atlas on M and $\mathscr{A} \subseteq \mathscr{B}$, then $\mathscr{A} = \mathscr{B}$. A ***smooth manifold*** is a pair $(M, \mathscr{A})$ where $\mathscr{A}$ is a maximal atlas on M.

Lemma 1.9. *Let $\mathscr{A}$ be an atlas on M, then it is contained in a unique maximal atlas.*

Proof. Define $\bar{\mathscr{A}} = \{(U, \varphi) | (U, \varphi)$is a chart on M and is compatible with every chart in $\mathscr{A}\}$. If $\mathscr{B}$ is an atlas on M and $\mathscr{A} \leq \mathscr{B}$, then $\mathscr{B} \subseteq \bar{\mathscr{A}}$. So it's enough to show that $\bar{\mathscr{A}}$ is an atlas.

Let $c_0 = (U_0, \varphi_0)$ and $c_1 = (U_1, \varphi_1)$ are two charts in $\bar{\mathscr{A}}$, we need to show that they are compatible, i.e. $\varphi_{10} = \varphi_1 \circ \varphi_0^{-1} : \varphi_0(U_{01}) \to \varphi_1(U_{01})$ is smooth. Enough to show that for any $fx \in \varphi(U_{01})$, φ_{10} is smooth in a neighborhood of x. Choose chart $c_2 = (U_2, \varphi_2) \in \mathscr{A}$ at $\varphi_0^{-1}(x)$, then c_0, c_1 are compatible with c_2, so

$$\varphi_{10} = \varphi_1 \circ \varphi_0^{-1} = (\varphi_1 \circ \varphi_2^{-1}) \circ (\varphi_2 \circ \varphi_0^{-1}) = \varphi_{12} \circ \varphi_{02}^{-1} : \varphi_0(U_0 \cap U_1 \cap U_2) \to \varphi_1(U_0 \cap U_1 \cap U_2)$$

is smooth. □

Remark 1.10. Smoothness is a local property.

1.3 Dimension

Let $(M, \mathscr{A})$ be a smooth manifold, $(U, \varphi : U \to \mathbb{R}^n), (V, \phi : V \to \mathbb{R}^m) \in \mathscr{A}$ be charts at $x \in M$. By compatibility, $n = m$.

Define $dim_x M = n$ to be the ***dimension*** of M at x. Note that $dim_y M = n$ for all $y \in U \cup V$. So for each n, the set $M_n = \{x \in M | dim_x M = n\}$ is open, and $\{x \in M | dim_x M \neq n\} = \cup_{m \in \mathbb{N} \setminus \{n\}} M_m$ is open. So M_n is open and closed. M_n is a union of connected components of M. (Q: Infinitely many?)

We say M is ***pure*** of dimension n if each component has the same dimension n.

Fact 1.11. *Not every topological manifold has a smooth structure. [Kervaire,1960, Kervaire manifold, a 10-dimensional PL manifold with no differentiable structure]*

Remark 1.12. Usually we denote a manifold $(M, \mathscr{A})$ by M. A chart on M is a chart in $\mathscr{A}$.

1.4 Examples of manifolds

1. Let E be a finitely-dimensional real vector space. Choose a linear isomorphism $\varphi : E \to \mathbb{R}^n$. Declare $U \subseteq E$ to be open if $\varphi(U) \subseteq \mathbb{R}^n$ is open. This defines a topology on E and (E, φ) is a chart. Let $\mathscr{A}$ be the smooth structure defined by this single chart, then $(E, \mathscr{A})$ is an n-manifold with smooth structure independent of φ. (I.e. if you choose another linear isomorphism you can get the same smooth structure on E.)

   Reason: If $\phi : E \to \mathbb{R}^n$ is another linear isomorphism, then $\phi \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism and hence smooth, so the two charts are compatible with each other and therefore defines the same smooth structure.

2. Any set M equipped with the discrete topology is a 0-dim manifold, and the charts are of the form $(\{x\}, \varphi : \{x\} \to \mathbb{R}^0)$.

3. $(M, \mathscr{A})$ is a smooth manifold, $U \subseteq M$ is open. Then $\mathscr{A}_U = \{(V, \phi) | V \subseteq U\}$ is a smooth structure on U, called the *induced* smooth structure. $(U, \mathscr{A}_U)$ is an *open submanifold* of M.

4. The product $M = M_1 \times M_2$ of manifolds M_1, M_2 is a manifold.

   Reason: Given charts (U_1, φ_1) on M_1 and (U_2, φ_2) on M_2, we can form the product chart (U, φ) with $U = U_1 \times U_2, \varphi = \varphi_1 \times \varphi_2$.The product charts form an atlas on M (not maximal), giving rise to a unique smooth structure om M.

5. (Not manifold) Let $\hat{M} = \mathbb{R} \times \{0, 1\}$. Given the equivalence relation $(a, t) \sim (b, s) \Leftrightarrow a = b \neq 0$, we can get $M = \hat{M}/\sim$. The topology on M is: $U \subseteq M$ is open if $\varphi^{-1}(U)$ is open in $\hat{M}.M = U_0 \cup U_1$ is a union of two open sets where $U_0 = \pi(\mathbb{R} \times \{0\})$ and $U_1 = \pi(\mathbb{R} \times \{1\})$ and the corresponding coordinate map are $\varphi_0 : U_0 \to \mathbb{R}, [x, 0] \mapsto x$, and $\varphi_1 : U_1 \to \mathbb{R}, [x, 1] \mapsto x$. $\varphi_{10} = \varphi_1 \circ \varphi_0^{-1} : \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}, x \mapsto x$. But M is not Hausdorff.

6. The n-sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} | \|x\| = \sqrt{\sum_{i=1}^n x_i^2} = 1\}$ is equipped with Euclidean norm. Give $\mathbb{S}^n$ the subspace topology. Define open subset $U_i^+ = \{x \in \mathbb{S}^n | x_i > 0\}$ and $U_i^- = \{x \in \mathbb{S}^n | x_i < 0\}$ in $\mathbb{S}^n$ and coordinate maps $\varphi_i^\pm : U_i^\pm \to \mathbb{R}^n, x \mapsto (x_1, \cdots, x_{i-1}, \hat{x}_i, x_{i+1}, \cdots, x_{n+1})$ is a homeomorphism onto $\{y \in \mathbb{R}^n | \|y\| < 1\}$ and the transition map
   $$\varphi_j^\pm \circ \varphi_i^\pm(y) = (y_1, \cdots, y_{i-1}, \pm\sqrt{1 - \|y\|^2}, y_i, \cdots \hat{y}_j, \cdots, y_n)$$
   for $i < j$ is smooth for $\|y\| < 1$. $\mathbb{S}^n$ is Hausdorff and second-countable, hence an n-manifold.

7. The n-dimensional real projective space $\mathbb{P}^n(\mathbb{R})$ or $\mathbb{RP}^n$ or $\mathbb{P}^n_{\mathbb{R}}$ is the set of all lines (1-dimensional linear subspace) in $\mathbb{R}^{n+1}$. For $x \in \mathbb{R}^{n+1}$, let $[x] = \mathbb{R}x$ (line spanned by x). Define
   $$\tau : \mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{PR}^n, x \mapsto [x],$$
   τ is surjective and $\tau(x) = \tau(y) \Leftrightarrow y = \lambda x$ for some $\lambda \in \mathbb{R}\backslash\{0\}$. Give $\mathbb{P}^n(\mathbb{R})$ the quotient topology with respect to τ, i.e. $U \subseteq \mathbb{P}^n(\mathbb{R})$ is open $\Leftrightarrow \tau^{-1}(U)$ is open in $\mathbb{R}\backslash\{0\}$. Charts are all the lines $U_i = \{[x] \in \mathbb{P}^n(\mathbb{R}) | x_i \neq 0\}$ which are open subsets of $\mathbb{P}^n(\mathbb{R})$. Define
   $$\varphi_i : U_i \to \mathbb{R}^n, [x] \mapsto (\frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_{n+1}}{x_i})$$
   and
   $$\rho_i : \mathbb{R}^n \to \mathbb{R}^{n+1}, y \mapsto (y_1, \cdots, y_{i-1}, 1, y_i, \cdots, y_n)$$
   then $\rho_i \circ \varphi_i([x]) = (\frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, 1, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_{n+1}}{x_i})$, so $\tau(\rho_i(\varphi_i([x])) = [x]$ and conversely $\varphi_i(\tau(\rho_i(y))) = y$. Hence $\tau \circ \rho_i = \varphi_i^{-1} : \mathbb{R}^n \to U_i$. So
   $$\varphi_j \circ \varphi_i^{-1}(y) = (\frac{y_1}{y_j}, \cdots, \frac{\hat{y_j}}{y_j}, \cdots, \frac{y_{i-1}}{y_j}, \frac{1}{y_j}, \frac{y_i}{y_j}, \cdots, \frac{y_n}{y_j})$$
   is smooth on its domain $y_j \neq 0$, and therefore $\mathbb{P}^n(\mathbb{R})$ is an n-manifold.

8. $\mathbb{P}^n(\mathbb{C})$ is the space of all lines in $\mathbb{C}^{n+1}$, and this is a $2n$-manifold.

9. $\mathbb{P}^n(\mathbb{H})$ is the space of all lines in $\mathbb{H}^{n+1}$, and this is a $4n$-manifold.

Problem. Why is $\mathbb{P}^n(\mathbb{R})$ Hausdorff and second-countable?

Lemma 1.13. *(i). The quotient map $\tau : \mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{P}^n(\mathbb{R})$ is open.*
(ii). $\mathbb{P}^n(\mathbb{R})$ is second-countable.

Proof. (i). Let $V \subseteq \mathbb{R}^{n+1}\backslash\{0\}$ be open, then $\tau^{-1}(\tau(V)) = \cup_{\lambda \neq 0} \lambda V$ is open, so $\tau(V)$ is open.
(ii). Let $\{V_i\}$ be a countable basis of the topology on $\mathbb{R}^{n+1}\backslash\{0\}$, then by (i), $\{\tau(V_i)\}$ is a countable basis for $\mathbb{P}^n(\mathbb{R})$. □

Lemma 1.14. *Let X be a topological space and R an equivalent relation on X. Let $Y = X/R$ be the quotient space. If the graph of R is closed in $X \times X$, and if the quotient map $X \to Y$ is open, then Y is Hausdorff.*

Proof. Let X be a topological space. The following are equivalent: (1) X is Hausdorff, (2) the diagonal $\Delta(X) \subset X \times X$ is closed.

Suppose X is a Hausdorff space. We need to show that the complement of the diagonal, $\Delta^c := X \times X \backslash \Delta$ is open. So let $(x, y) \in \Delta$. Then $x \neq y$, and so there are disjoint open sets U and V , containing x and y, respectively. By definition of the product topology, $U \times V$ is an open subset of $X \times X$, and clearly $U \times V \subset \Delta^c$ (for otherwise $U \cap V \neq \varnothing$). This shows that Δ^c is open. Conversely, suppose Δ is closed, that is to say, Δ^c is open. Let x and y be two distinct elements of X. Then $(x, y) \in \Delta^c$, and so there is a basis open set $U \times V \subset \Delta^c$ containing (x, y). Now note that U and V are open, disjoint subsets of X, containing x and y, respectively. This shows that X is Hausdorff.

So if we construct the product quotient map $\pi \times \pi : X \times X \to Y \times Y$, is an open map, So $\pi \times \pi(\Gamma(R)^c) = \Delta(Y)^c$ is open, i.e. $\Delta(Y)$ is closed in $Y \times Y$, so Y is Hausdorff. $\square$

2 Smooth manifolds

2.1 The smooth category

Let $(M, \mathscr{A})$ and $(N, \mathscr{B})$ be smooth manifolds and $f : M \to N$ a map.

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \varphi \downarrow & & \downarrow \phi \\ \varphi(U \cap f^{-1}(V)) & \xrightarrow{\phi \circ f \circ \varphi^{-1}} & \phi(V) \end{array}$$

$\phi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \to \phi(V)$ is called the ***empression*** for f in the charts (U, φ) and (V, ϕ).

Definition 2.1. f is ***smooth*** if f is continuous and $\phi \circ f \circ \varphi^{-1}$ is smooth for any $(U, \varphi) \in \mathscr{A}$ and $(V, \phi) \in \mathscr{B}$.

We call the charts (U, φ) and (V, ϕ) are adapted to f if $U \subseteq f^{-1}(V)$. To check that f is smooth, need only to check whether $\phi \circ f \circ \varphi^{-1}$ is smooth for all pairs of adapted charts $(U, \varphi) \in \mathscr{A}$ and $(V, \phi) \in \mathscr{B}$.

Example 2.2. $M = \mathbb{R}^{n+1} \backslash \{0\}$, $N = \mathbb{P}^n(\mathbb{R})$, $f = \tau : M \to N, x \mapsto [x]$ is the quotient map. Then f is continuous. Let $V_i = \{x \in \mathbb{R}^{n+1} \backslash \{0\} | x_i \neq 0\}$, then $\mathscr{A} = \{(V_i, Id_{V_i}) | 1 \leq i \leq n+1\}$ is an atlas on $\mathbb{R}^{n+1} \backslash \{0\}$. Moreover, $\tau(V_i) = U_i = \{[x] \in N | x_i \neq 0\}$. Recall that $\varphi_i([x]) = \frac{1}{x_i}(x_1, \cdots, x_i, \cdots \hat{} x_{n+1}) \in \mathbb{R}^n$, $\{(U_i, \varphi_i) | 1 \leq i \leq n+1\}$ is an atlas on N. The empression for f in these charts

$$\varphi_i \circ f \circ Id(x) = \varphi_i([x]) : V_i \to \mathbb{R}^n$$

is smooth. so f is smooth.

Example 2.3. $M = \mathbb{S}^n$, $N = \mathbb{R}^{n+1}$, f is the inclusion map, then f is smooth.

The composition $g \circ f$ of smooth maps $M \xrightarrow{f} N \xrightarrow{g} P$ is smooth by chain rule.
Smooth manifolds form a category C^∞:

- Objects: smooth manifolds
- Arrows (morphisms): smooth maps

Notation:
$C^\infty(M, N)$ =set of all smooth maps from M to N.
$C^\infty(M) = C^\infty(M, \mathbb{R})$, or sometimes $C^\infty(M, \mathbb{C})$, where $\mathbb{C} = \mathbb{R}^2$ is a smooth manifold.

2.2 Tangent vectors

Given a manifold M, consider the set of triples (c, x, h) where $c = (U, \varphi : U \to \mathbb{R}^n)$ is a chart, $x \in M$ and $h \in \mathbb{R}^n$. Call two triples *equivalent* $(c_1, x_1, h_1) \sim (c_2, x_2, h_2)$ if $x_1 = x_2$ and $h_2 = D(\varphi_2 \circ \varphi_1^{-1})(\varphi_1(x_1)h_1$.

Definition 2.4. An equivalent class $[c, x, h]$ is a ***tangent vector*** to M at x. The ***tangent bundle*** TM is the collection of all tangent vectors to M. The ***tangent bundle mojection*** is

$$\begin{array}{rccc} \pi = \pi_M : & TM & \to & M \\ & [c, x, h] & \mapsto & x \end{array}$$

The tangent space to M at x is $T_xM = \pi^{-1}(x)$. Then $TM = \coprod_{x \in M} T_xM$ is the disjoint union.

Lemma 2.5. *Let $x \in M$, and $c = (U, \varphi)$ be a chart at x.*

1. *The map $\mathbb{R}^n \to T_xM$ defined by $h \mapsto [c, x, h]$ is a bijection.*
2. *Let $d_x\varphi : T_xM \to \mathbb{R}^n$ be the inverse of the bijection. Let $c_1 = (U_1, \varphi_1)$ be another chart at x, then $d_x\varphi_1 \circ d_x\varphi^{-1} = D(\varphi_1 \circ \varphi^{-1})(\varphi(x))$ is an invertible linear map.*

$$\begin{array}{ccccc} & & T_xM & & \\ & \swarrow & & \searrow & \\ \mathbb{R}^n & & \to & & \mathbb{R}^n \end{array}$$

3. *If $[c,x,h_1]$=$[c,x,h_2]$ then $h_2 = D(\varphi\circ\varphi^{-1})h_1 = h_1$, so the map is injective. If $v \in T_xM$, $v = [c_0, x, h_0]$ for some chart $c_0 = (U_0, \varphi_0)$ at x, then $v = [c,x,h]$ with $h = D(\varphi\circ\varphi_0^{-1})(\varphi_0(x))h_0$, so the map is surjective.*

4. $d_x\varphi_1 \circ d_x\varphi^{-1}(h) = d_x\varphi_1([c,x,h]) = d_x\varphi_1([c_1, x, D(\varphi_1\circ\varphi^{-1})(\varphi(x))h]) = D(\varphi_1\circ\varphi^{-1})(\varphi(x))h.$

Endow T_xM with a vector space structure by declaiming $d_x\varphi$ to be a linear isomorphism. (Why can?)

Let V be an open submanifold of M, then each chart $c = (U,\varphi)$ on V is a chart on M. So for $x \in U$, $h \in \mathbb{R}^n$, we have tangent vectors

$$\begin{array}{ccc} [c,x,h]_V \in T_xV & \xrightarrow{\cong} & [c.x.h]_M \in T_xM \\ d_x\varphi \searrow & & \swarrow d_x\varphi \\ & h \in \mathbb{R}^n & \end{array}$$

so $T_xV \cong T_xM$. The map $[c,x,h]_V \mapsto [c,x,h]_M$ is a linear isomorphism.

We identify T_xV with T_xM, then $TV = \pi_M^{-1}(V) \subseteq TM$.

Special case: $M = U$ an open subset of $\mathbb{R}^n$, let $c = (U, Id_U)$ be the identity chart, then $d_x(Id_U) : T_xU \xrightarrow{\sim} \mathbb{R}^n$ is an isomorphism, so the map $\begin{array}{ccc} TU & \to & U\times\mathbb{R}^n \\ [c,x,h] & \mapsto & (x,h) \end{array}$ is a bijection. We identify $TU = U\times\mathbb{R}^n$, then $\pi : TU \to U$ is given by $\pi(x,h) = x$.

2.3 Tangent maps

Let $f : M \to N$ be a smooth map, $\dim(M) = m$ and $\dim(N) = n$. Choose f-adapted charts $c = (U,\varphi)$ and $c' = (V,\phi)$.

For $x \in U$, $h \in \mathbb{R}^n$, $v = [c,x,h]$, we put

$$Tf(v) = Tf([c,x,h]) = [c', f(x), D(\phi\circ f\circ\varphi^{-1})(\varphi(x))h].$$

Lemma 2.6. *$Tf : TM \to TN$ maps T_xM linearly to $T_{f(x)}N$.*

Proof.
$$\begin{array}{ccc} T_xM & \xrightarrow{T_xf} & T_{f(x)}N \\ d_x\varphi \downarrow \wr & & d_x\phi \downarrow \wr \\ \mathbb{R}^m & \xrightarrow{D(\phi\circ f\circ\varphi^{-1})\varphi(x)} & \mathbb{R}^n \end{array}$$

T_xf is composite of linear maps. ∎

Lemma 2.7. *The tangent map is well-defined.*

Proof. Let $x \in M$, consider two pairs of adapted charts

$$\begin{array}{ll} c_1 = (u_1, \varphi_1), & c_1' = (V_1, \phi_1) \\ c_2 = (u_2, \varphi_2), & c_2' = (V_2, \phi_2) \end{array}$$

Let $\tilde{f}_i : \phi_i \circ f \circ \varphi_i^{-1}$ be the empression for f, then $\tilde{f}_2 = \phi_{21}\circ f\circ\varphi_{21}^{-1}$.

Let $h_1, h_2 \in \mathbb{R}^n$, suppose $[c_1, x, h_1] = [c_2, x, h_2]$, then $h_1 = D(\varphi_{12})\varphi_2(x)h_2$, so

$$\begin{array}{rcl} [c_2', f(x), D\tilde{f}_2\varphi_2(x)h_2] & = & [c_2', f(x), D(\phi_{21}\circ\tilde{f}_1\circ\varphi_{21}^{-1})\varphi_2(x)h_2] \\ & = & [c_2', f(x), D\phi_{21}(\phi_1(f(x))\circ D\tilde{f}_1(\varphi_1(x))h_1] \\ & = & [c_1', f(x), D\tilde{f}_1(\varphi_1(x))h_1] \end{array}$$

∎

Let $U \subseteq \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^n$ be smooth, then $Tf : U\times\mathbb{R}^m \to \mathbb{R}^n\times\mathbb{R}^m$ is given by

$$Tf(x,h) = Tf([c,x,h]) = [c', f(x), Df(x)h] = (f(x), Df(x)h).$$

Tf records f and Df.

Let $c = (U, \varphi : U \to \mathbb{R}^n)$ be a chart on a manifold M, then $T\varphi : TU \to \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ is given by

$$T\varphi([c, x, h]) = (\varphi(x,), h),$$

writing $v = [c, x, h]$, $T\varphi(v) = (\varphi(x), d_x\varphi(v))$.

For each $x \in U$, $T_x\varphi : T_xU \xrightarrow{\sim} \{\varphi(x)\} \times \mathbb{R}^n$ is bijective. So $T\varphi : TU \to \varphi(U) \times \mathbb{R}^n$ is a bijection, where $\varphi(U) \times \mathbb{R}^n$ is open in $\mathbb{R}^{2n}$. So the pair $Tc = (TU, T\varphi)$ is a chart on TM (in the sense of homework 1-1, called a tangent chart.

Theorem 2.8. *(i) The tangent charts .Tc form an atlas on TM, hence TM is a 2n-manifold.*
(ii) For any smooth $f : M \to N$, the tangent map $Tf : TM \to TN$ is smooth.

Proof. (i) Let $c_1 = (U_1, \varphi_1)$ and $c_2 = (U_2, \varphi_2)$ be charts on M. Transition map $Tc_1 \to Tc_2$?

$$\begin{array}{ccc} & T_xM & \\ d_x\varphi_1 \swarrow & & \searrow d_x\varphi_2 \\ \mathbb{R}^m & \xrightarrow[D(\varphi_2\circ\varphi_1^{-1})(\varphi_1(x))]{\sim} & \mathbb{R}^m \end{array}$$

For $x \in U_1 \cap U_2$, $v \in T_xM$, $T_x\varphi_i(v) = (\varphi_i(x), d_x\varphi_i(v))$, so

$$T_x\varphi_2 \circ (T_x\varphi_1(x))^{-1}(y, h) = (\varphi_2 \circ \varphi_1^{-1}(y), D(\varphi_2 \circ \varphi_1^{-1})(y)h)$$

is smooth (C^∞). So

$$T\varphi_2 \circ (T\varphi_1)^{-1} : \varphi_1(U_1 \cap U_2) \times \mathbb{R}^m \to \varphi_2(U_1 \cap U_2) \times \mathbb{R}^m$$

is smooth, and $\varphi_1(U_1 \cap U_2) \times \mathbb{R}^m$ and $\varphi_2(U_1 \cap U_2) \times \mathbb{R}^m$ are open in $\mathbb{R}^{2m}$.

(ii) Let $c = (U, \varphi : U \to \mathbb{R}^m)$, $c' = (V, \phi : V \to \mathbb{R}^n)$ be adapted charts on M and N, then Tc is a chart on TM and Tc' is a chart on TN.

Empression for Tf is

$$\begin{array}{rccc} T\phi \circ Tf \circ (T\varphi)^{-1} : & \varphi(U) \times \mathbb{R}^m & \to & \phi(V) \times \mathbb{R}^n \\ & (y, h) & \mapsto & (\phi \circ f \circ \varphi^{-1}(y), D(\phi \circ f \circ \varphi^{-1})(y)h) \end{array}$$

And

$$\begin{array}{rcl} T\phi \circ Tf \circ (T\varphi)^{-1}(y, h) & = & T\phi \circ Tf([c, \varphi^{-1}(y), h]) \\ & = & T\phi([c', f \circ \varphi^{-1}(y), D(\phi \circ f \circ \varphi^{-1})(y)h] \\ & = & (\phi \circ f \circ \varphi^{-1}(y), D(\phi \circ f \circ \varphi^{-1})(y)h) \end{array}$$

is smooth. ☐

Theorem 2.9. *(chain rule) T is a functor from C^∞ to itself, i.e.*

1. *For every smooth manifold M, TM is a smooth manifold, and for every smooth map $f : M \to N$, $Tf : TM \to TN$ is smooth.*
2. *$T(id_M) = id_{TM}$*
3. *$T(g \circ f) = Tg \circ Tf$*

Proof. Only need to prove 3.

Choose adapted charts $c = (U, \varphi)$ at x, $c' = (V, \phi)$ at $y = f(x)$, $c'' = (W, \psi)$ at $z = g(y)$. May assume $f(U) \subseteq V$, $g(V) \subseteq W$. Let $\tilde{f} = \phi \circ f \circ \varphi^{-1}$, $\tilde{g} = \psi \circ g \circ \phi^{-1}$, $h = g \circ f$, $\tilde{h} = \psi \circ h \circ \varphi^{-1}$. Then $\tilde{h} = \tilde{g} \circ \tilde{f}$.

$$\begin{array}{ccccc} U & \overset{f}{\to} & V & \overset{g}{\to} & W \\ x & \mapsto & y = f(x) & \mapsto & z = g(y) \\ \varphi \downarrow & & \phi \downarrow & & \psi \downarrow \\ \mathbb{R}^m & \overset{\tilde{f}}{\Leftarrow} & \mathbb{R}^n & \overset{\tilde{g}}{\to} & \mathbb{R}^l \\ \varphi(x) & \mapsto & \phi(y) & \mapsto & \psi(z) \end{array}$$

So for $v = [c, x, k] \in T_xM$, we have

$$Th(v) = [c'', h(x), D\tilde{h}(\varphi(x))k] = [c'', z, D\tilde{h}(\varphi(x))k].$$

while

$$Tg \circ Tf(v) = Tg[c', y, D\tilde{f}(\varphi(x))k] = [c'', z, D\tilde{g}(\phi(y))D\tilde{f}(\varphi(x))k].$$

By the chain rule, we have $Th(v) = Tg \circ Tf(v)$. $\square$

Example 2.10. $M = N = P = \mathbb{R}^n$
$(g \circ f)'(x) = g'(f(x))f'(x)$
$Tf(x, t) = (f(x), f'(x)t),\ Tg(y, u) = (g(y), g'(y)u)$

Example 2.11. $N = \mathbb{R}^k$, $f : M \to \mathbb{R}^k$ is smooth., then

$$Tf : TM \to T\mathbb{R}^k = \mathbb{R}^k \times \mathbb{R}^k$$

sends T_xM to $\{f(x)\} \times \mathbb{R}^k$.

$$\begin{array}{cccccc} TM & \xrightarrow{Tf} & \mathbb{R}^k & \times & \mathbb{R}^k \\ & f \searrow & \mathrm{pi}_1 \downarrow & & \downarrow \mathrm{pi}_2 \\ & & \mathbb{R}^k & & \mathbb{R}^k \end{array}$$

In other words, $\mathrm{pi}_1(Tf(v)) = f(x)$ if $v \in T_xM$.

Define $df(v) = \mathrm{pi}_2(Tf(v))$ for $v \in T_xM$, then $T_xf(v) = (f(x), d_xf(v))$. So for any $x \in M$, $d_xf : T_xM \to \mathbb{R}^k$ is linear. d_xf is the directional derivative of f at x along v.

If $c = (U, \varphi)$ is a chart at x, and if $v = [c.x.h]$, $\tilde{f} = f \circ \varphi^{-1}$, then $T_xf(v) = (f(x), D\tilde{f}(\varphi(x))h)$.

So $d_xf(v) = D\tilde{f}(\varphi(x))h = \lim\limits_{t \to 0} \frac{\tilde{f}(\varphi(x)+th) - \tilde{f}(\varphi(x))}{t}$.

A linear map $T_xM \to \mathbb{R}$ is a ***cotangent vector*** to M at x. The cotangent space at x is $(T_xM)^*$, usually written T_x^*M. So if $f : M \to \mathbb{R}$ is smooth, we have $d_xf \in T_x^*M$ for $x \in M$.

Example 2.12. Let $I \subseteq \mathbb{R}$ be an open interval and $\gamma : I \to M$ a smooth map (a smooth path in M), then $T\gamma : TI = I \times \mathbb{R} \to TM$.

For each $t \in I$, $T_t\gamma : \{t\} \times \mathbb{R} \to T_{\gamma(t)}M$ is linear, so determined by its value $T_t\gamma(t, 1) = \gamma'(t)$. (velocity vector at time t)

Lemma 2.13. *For every $x \in M$, $v \in T_xM$, there is a path $\gamma : I \to M$ with $0 \in I$, $\gamma(0) = x$ and $\gamma'(0) = v$.*

2.4 Derivations

Definition 2.14. Let $x \in M$, U, V be open neighborhoods of x. $f : U \to \mathbb{R}, g : V \to \mathbb{R}$ are smooth functions. f and g are equivalent if there exists an open neighborhood $W \subseteq U \cap V$ such that $f = g$ on W. The equivalence class of f is called the ***germ*** of f at x, denoted $[f]$ or $[f]_x$. The set of all germs at x is denoted $C^\infty_{M,x}$. $C^\infty_{M,x} = \mathrm{colim}_{U \ni x} C^\infty(U)$, the colimit taken over all open $U \ni x$.

If $[f]_x$ and $[g]_x$ are germs at x, then $[f+g]_x$, $[fg]_x$, $[cf]_x (c \in \mathbb{R})$ are well-defined germs, i.e. $C^\infty_{M,x}$ is a commutative algebra over $\mathbb{R}$ with unit.

Definition 2.15. The ***evaluation map*** $ev = ev_x : C^\infty_{M,x} \to \mathbb{R}$ is defined by $ev_x([f]_x) = f(x)$. ev is a morphism of $\mathbb{R}$-algebra with unit.

Definition 2.16. A ***derivation*** of M at x is an $\mathbb{R}$-linear map $l : C^\infty_{M,x} \to \mathbb{R}$ satisfying the Leibniz rule $l([fg]) = l([f])g(x) + f(x)l([g])$.

Let D_xM be the set of all derivatives of M at x. $D_xM \subseteq \mathrm{Hom}_{\mathbb{R}}(C^\infty_{M,x}, \mathbb{R})$ is a linear subspace.

Lemma 2.17. *Let $l \in D_xM$, then $l([1]) = 0$.*

Proof. $l(1) = l(1 \cdot 1) = l(1) + l(1)$, hence $l(1) = 0$. $\square$

Example 2.18. Let $M = \mathbb{R}^n$, $x = 0$, define $l_i([f]_0) = \frac{\partial f}{\partial x_i}(0)$, $l_i = \frac{\partial}{\partial x_i}|_{x=0}$ for $i = 1, \cdots, n$. Then $l_i \in D_0\mathbb{R}^n$. Hence $l = \sum_{i=1}^n c_i l_i \in D_0\mathbb{R}^n$ for $c_i \in \mathbb{R}$. Note

$$l([f]) = \sum_{i=1}^n c_i \frac{\partial f}{\partial x_i}(0) = Df(0)c$$

where $c = \begin{pmatrix} c_1 \\ \cdots \\ c_n \end{pmatrix} \in \mathbb{R}^n$. l is the directional derivative along c at 0.

Lemma 2.19. *$l_1, \cdots, l_n$ defined above form a basis of $D_0\mathbb{R}^n$, so for any $l \in D_0\mathbb{R}^n$, $l([f]) = Df(0)v$ for some unique $v \in \mathbb{R}^n$.*

Proof. Let $l \in D_0\mathbb{R}^n$, $f \in C^\infty_{\mathbb{R}^n,0}$, write

$$f(x) = f(0) + \sum_{i=1}^n x_i f_i(x)$$

where $f_i \in C^\infty(U)$ and $f_i(0) = \frac{\partial f}{\partial x_i}(0)$ using Taylor's formula. So

$$l([f]_0) = l([f(0)]) + \sum_{i=1}^n l([x_i f_i(x)]_0) = \sum_{i=1}^n l([x_i]) f_i(0) = \sum_{i=1}^n l([x_i]) \frac{\partial f}{\partial x_i}(0) = Df(0)v$$

with $v_i = l[x_i]$.

Independence: if $l = \sum_{i=1}^n v_i l_i = 0$, then $Df(0)v = 0$ for any $[f] \in C^\infty_{\mathbb{R}^n,0}$. Take $f = x_i$, then $v_i = 0$. □

For M an n-manifold, $x \in M$, $v \in T_xM$, define

$$\begin{array}{rccc} l_v : & C^\infty_{M,x} & \to & \mathbb{R} \\ & [f] & \mapsto & d_xf(v) \end{array}$$

Choose chart $c = (U, \varphi)$ at x, then $v = [c, x, h]$. Let $\tilde{f} = f \circ \varphi^{-1}$, then $d_xf(v) = D\tilde{f}(\varphi(x))h$. So for each $v \in T_xM$, l_v is a derivation.

Theorem 2.20. *The map $\mathscr{L}_x : T_xM \to D_xM$ defined by $\mathscr{L}_x(v) = l_v$ is an isomorphism.*

Proof. For a smooth map $F : M \to N$ and $x \in M$, $y = F(x) \in N$, define $F^* : C^\infty_{N,y} \to C^\infty_{M,x}$ by $F^*([g]_y) = [g \circ F]_x$ and $F_* : D_xM \to D_yN$ by $F_*(l)([g]_y) = l([F^*[g]_y])$, then $F_*(l)$ is a derivation of N at y. Can also denote $F_* = D_xF : D_xM \to D_yN$.

Chain rule: if

$$M \xrightarrow{F} N \xrightarrow{G} P$$

are smooth, and $x \in M$, $y = F(x) \in N$, $z = G(y) \in P$, then

$$C^\infty_{P,z} \xrightarrow{G^*} C^\infty_{N,y} \xrightarrow{F^*} C^\infty_{M,x}$$

satisfies $(G \circ F)^* = F^* \circ G^*$ because

$$(G \circ F)^*([h]_z) = [h \circ G \circ F]_x = F^*([h \circ G]_y) = F^*(G^*([h]_z) = F^* \circ G^*([h]_z).$$

for any $[h]_z \in C^\infty_{P,z}$. And

$$D_xM \xrightarrow{F_*} D_yN \xrightarrow{G_*} D_zP$$

satisfies $(G \circ F)_* = G_* \circ F_*$ because

$$(G \circ F)_*(l) = l \circ (G \circ F)^* = l \circ F^* \circ G^* = G_*(l \circ F^*) = G_* \circ F_*(l)$$

for any $l \in D_xM$.

Choose chart $c = (U, \varphi)$ centered at x, apply the following lemma to $F = \varphi$ gives a commutative diagram

$$\begin{array}{ccc} T_xM & \xrightarrow[\cong]{d_x\varphi} & \mathbb{R}^n \\ \mathscr{L}_x \downarrow & & \cong\downarrow \mathscr{L}_0 \\ D_xM & \xrightarrow[\cong]{\varphi_* = D_x\varphi} & D_0\mathbb{R}^n \end{array}$$

$\varphi^* : C^\infty_{\mathbb{R}^n,0} \to C^\infty_{M,x}$ is an isomorphism, so $\varphi_* : D_xM \to D_0\mathbb{R}^n$ is an isomorphism, hence $\mathscr{L}_x$ is an isomorphism. □

Lemma 2.21. *Let $F : M \to N$ be smooth, $x \in M$, $y = f(x)$, then the diagram*

$$\begin{array}{ccc} T_xM & \xrightarrow{TF} & T_yN \\ \mathscr{L}_x \downarrow & & \downarrow \mathscr{L}_y \\ D_xM & \xrightarrow{F_*} & D_yN \end{array}$$

commutes.

Proof. Let $v \in T_xM$, $[g]_y \in C^\infty_{N,y}$, then

$$D_xF(\mathscr{L}_x(v))([g]_y) = \mathscr{L}_x(v)(F^*[g]_y) = \mathscr{L}_x(v)([g \circ F]_x) = d_x(g \circ F)(v) = d_yg \circ T_xF(v).$$

while

$$\mathscr{L}_y(T_xF(v))([g]_y) = d_yg(T_xF(v)).$$

So the diagram commutes by chain rule. □

Remark 2.22. So now we have a natural isomorphism of vector spaces

$$\begin{array}{cccc} \mathscr{L}_x : & T_xM & \to & D_xM \\ & v & \mapsto & l_v \end{array}$$

Naturality means for any smooth map $F : M \to N$, $x \in M$ and $y = F(x) \in N$, the diagram

$$\begin{array}{ccc} T_xM & \xrightarrow{\mathscr{L}_x} & D_xM \\ T_xF \downarrow & & \downarrow D_xF \\ T_yN & \xrightarrow{\mathscr{L}_y} & D_yN \end{array}$$

commutes.

From now on, we identify $T_xM = D_xM$ and $T_xF = D_xF$.

In our next homework, we will see that $D_xM \cong (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ where $\mathfrak{m}_x = \ker(\mathrm{ev}_x : C^\infty_{M,x} \to \mathbb{R}) = \{[f]_x \in C^\infty_{M.x} | f(x) = 0\}$ is an ideal in the algebra of germs $C^\infty_{M,x}$.

Hence the cotangent space $T_x^*M = \mathrm{Hom}_{\mathbb{R}}(T_xM, \mathbb{R}) \cong D_x^*M \cong \mathfrak{m}_x/\mathfrak{m}_x^2$ is called the Zariski cotangent space.

2.5 Submanifolds

For $k \leq n$, we identify $\mathbb{R}^k$ with the subspace $\{x \in \mathbb{R}^n | x_{k+1} = \cdots = x_n = 0\}$ of $\mathbb{R}^n$.

Definition 2.23. Let A be a subset of M, $x \in A$. A chart $c = (U, \varphi)$ at x is a ***submanifold chart*** of dimension k for A at x if $A \cap U = \varphi^{-1}(\mathbb{R}^k)$. Let $\varphi_A = \varphi|_{A \cap U}$, then $c_A = (U \cap A, \varphi_A)$ is the *restriction* of the chart to A. We call A a ***submanifold*** (or ***embedded submanifold***) of dimension k if there exists a collection of submanifold charts of dimension k for A whose domains cover A.

The restrictions of these charts to A define an atlas, and hence a smooth structure on A, called the ***induced*** smooth structure. The induced topology is the subspace topology.

Definition 2.24. If A is a k-dimensional submanifold of an n-dimensional manifold M, then the codimension of A is $codim_M(A) = n - k$.

Example 2.25. Open subsets of M are submanifolds of codimension zero.

Definition 2.26. Let X be a topological space, and $Y \subseteq X$. Then Y is ***locally closed*** if for every $y \subseteq Y$, there is a open neighborhood $U \ni y$ in X, such that $Y \cap U$ is closed in U.

Equivalently, Y is of the form $Y = C \cap V$ with C closed in X and V open in X.

Lemma 2.27. *Let A be a k-dimensional submanifold of an n-manifold M. Then let $i : A \to M$ be the inclusion. Then*

(i) A is a locally closed subset of M;

(ii) the inclusion map $i : A \to M$ is smooth and $Ti : TA \to TM$ is injective.

(iii) Let $c = (U, \varphi)$ be a submanifold chart for A. Then $Tc = (TU, T\varphi)$ is a submanifold chart for $Ti(TA)$. For $x \in A$, we have $T_x i(T_x A) = (d_x \varphi)^{-1}(\mathbb{R}^k)$.

(iv) $Ti(TA)$ is a submanifold of TM of dimension $2k$.

Proof. *(i)* Taking a submanifold chart (U, φ) at $x \in A$, we have $U \cap A = \varphi^{-1}(\mathbb{R}^k)$. And $\varphi : U \to \varphi(U)$ is a homeomorphism, so $\varphi^{-1}(\mathbb{R}^k)$ is closed in U.

(ii) i is continuous. Let $c = (U, \varphi)$ be a submanifold chart for A. Then the pair of charts $c_A = (U \cap A, \varphi_A)$, $c = (U, \varphi)$ is adapted for i, and

$$\begin{array}{ccc} U \cap A & \xrightarrow{i} & U \\ \varphi_A \downarrow & & \downarrow \varphi \\ \mathbb{R}^k & \xrightarrow[\text{inclusion}]{\tilde{i}} & \mathbb{R}^n \end{array}$$

$\tilde{i} = \varphi \circ i \circ \varphi_A^{-1}$ is linear and hence smooth. By the chain rule(i.e. apply the functor T to the above graph), the diagram

$$\begin{array}{ccc} T(U \cap A) & \xrightarrow{Ti} & TU \\ T\varphi_A \downarrow\cong & & \cong\downarrow T\varphi \\ \varphi(U \cap A) \times \mathbb{R}^k & \xrightarrow{T\tilde{i}} & \varphi(U) \times \mathbb{R}^n \end{array}$$

commutes. Here $T\tilde{i}$ is the restriction to $\varphi(U \cap A) \times \mathbb{R}^k$ of the inclusion $\mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^n \times \mathbb{R}^n$. For $x \in U \cap A$, we have the commutative diagram

$$\begin{array}{ccc} T_x A & \xrightarrow{T_x i} & T_x M \\ d_x \varphi_A \downarrow\cong & & \cong\downarrow d_x \varphi \\ \mathbb{R}^k & \xrightarrow{\textit{inclusion}} & \mathbb{R}^n \end{array}$$

so $T_x i : T_x A \hookrightarrow T_x M$ is injective.

(iii)&(iv) $Ti(T(U \cap A)) = T\varphi^{-1}(\mathbb{R}^k \times \mathbb{R}^k)$. This shows that $Tc = (TU, T\varphi)$ is a submanifold chart for $Ti(TA)$, and therefore $Ti(TA)$ is a submanifold of TM of dimension $2k$. □

Henceforth, we identify TA with $Ti(TA) \subseteq TM$, and identify $T_x A$ with $T_x i(T_x A) \subseteq T_x M$. (ERROR)

The next theorem gives us an alternative way of looking at charts.

Theorem 2.28. *Let V be an open neighborhood of $x \in M$, $\varphi_1, \cdots, \varphi_n \in C^\infty(V)$. Suppose $d_x\varphi_1, \cdots, d_X\varphi_n \in T_x^* M$ form a basis of $T_x^*(M)$. Let*

$$\varphi = (\varphi_1, \cdots, \varphi_n) : V \to \mathbb{R}^n,$$

then there exists an open neighborhood $U \subseteq V$ of x such that $(U, \varphi|_U)$ is a chart at x.

Proof. We need to show that there exists an open neighborhood $U \subseteq V$ of x such that $\varphi|_U ; U \to \mathbb{R}^n$ is a diffeomorphism.

Choose chart (W, ψ) at x, It's enough to show that

$$\zeta : \varphi \circ \psi^{-1} : \psi(U) \to \varphi(U)$$

is a diffeomorphism for a small enough neighborhood $U \subseteq V \cap W$ of x.

Notice that because the $d_x\varphi_i$ are independent, then $d_x\varphi : T_x M \to \mathbb{R}^n$ is an isomorphism. Now let $y = \psi(x)$. By the chain rule,

$$d_y \zeta = d_x \varphi \circ (d_x \psi)^{-1},$$

so $d_y\zeta$ is an isomorphism. So by the inverse function theorem, there is an open neighborhood $U' \subseteq \psi(W)$ of y such that $\zeta(U')$ is open and $\zeta : U' \to \zeta(U')$ is a diffeomorphism. Now take $U = \psi^{-1}(U')$ and we have the desired chart. $\square$

Fact 2.29. *$\forall \alpha \in T_x^*M$, there exists a germ $[f]_x \in C^\infty_{M,x}$ such that $d_x f = \alpha$, i.e. $d_x : C^\infty_{M,x} \to T_x^*M$ is surjective.*

This will be proved in next homework.

Corollary 2.30. *A subset M of an n-manifold M is a submanifold of codimension $l = n - k$ if and only if for all $x \in A$, there is an open $U \ni x$ and $\zeta_1, \cdots, \zeta_l \in C^\infty(U)$ satisfying*
(i) $d_x\zeta_1, d_x\zeta_2, \cdots, d_x\zeta_n \in T_x^*M$ are linearly independent
(ii) for all $y \in U$, $y \in A \Leftrightarrow \zeta_1(y) = \zeta_2(y) = \cdots = \zeta_l(y) = 0$.

Proof. " $\Rightarrow$ " Choose submanifold chart (U, φ) at x, then $d_x\varphi : T_xM \xrightarrow{\cong} \mathbb{R}^n$, so $d_x\varphi_1, \cdots, d_x\varphi_n \in T_x^*M$ are linearly independent and also $U \cap A = \varphi^{-1}(\mathbb{R}^k)$. Let $\zeta_1 = \varphi_{k+1}, \cdots, \zeta_l = \varphi_n$ and we can verify this works.

" $\Leftarrow$ " Put $\varphi_{k+1} = \zeta_1, \cdots, \varphi_n = \zeta_l$. We can complete $\alpha_{k+1} = d_x\varphi_{k+1}, \cdots, \alpha_n = d_x\varphi_n \in T_x^*M$ to a basis $\alpha_1, \cdots, \alpha_n \in T_x^*M$. By the fact above we can find functions $\varphi_1, \cdots, \varphi_k \in C^\infty(U)$ for a small enough open neighborhood U of x with $d_x\varphi_1 = \alpha_1, \cdots, d_x\varphi_k = \alpha_k$. By the previous theorem, $\varphi = (\varphi_1, \cdots, \varphi_n) : U \to \mathbb{R}^n$ is a chart (often shrinking U) and $U \cap A = \varphi^{-1}(\mathbb{R}^k)$. $\square$

Remark 2.31. Keep the same setup as in the corollary. Let $\zeta = (\zeta_1, \cdots, \zeta_l) \in C(U, \mathbb{R}^l)$, then condition (i) is equivalent to $d_x\zeta : T_xM \to \mathbb{R}^l$ surjective, and condition (ii) is equivalent to $U \cap A = \zeta^{-1}(0)$.

T_xA consists of all $v \in T_xM$ with $d_x\zeta_1(v) = \cdots = d_x\zeta_1(v) = 0$. So $T_xA = \{v \in T_xM | d_x\zeta(v) = 0\} = \ker(d_x\zeta)$. In the special case of $M = \mathbb{R}^n$, $d_x\zeta$ manifests itself as the Jacobian matrix $D\zeta_x$, which is an $l \times n$ matrix. Then $d_x\zeta_j = (D\zeta_j)_x$ is the j-th row of $D\zeta_x$. So

$$\begin{aligned} T_xA &= \ker(D\zeta_x) \\ &= \ker((D\zeta_1)_x) \cap \cdots \cap \ker((D\zeta_l)_x) \\ &= <(\nabla_x\zeta_1)>^\perp \cap \cdots \cap <(\nabla_x\zeta_l)>^\perp \end{aligned}$$

2.6 Rank

Definition 2.32. Let M, N be manifolds of dimension m and n, respectively, and let $bef : M \to N$ be smooth. The ***rank*** of f at $x \in M$ is

$$\mathrm{rank}_x(f) = \mathrm{rank}(T_xf : T_xM \to T_{f(x)}N) = \dim(T_xf(T_{f(x)}N)).$$

Fact 2.33. *A linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ of rank $r \le \min\{m, n\}$ has an $m \times n$ matrix representation*

$$A = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to suitable bases of $\mathbb{R}^n$ and $\mathbb{R}^m$.

In coordinate charts with respect to these bases, $T(x_1, \cdots, x_n) = (x_1, \cdots, x_r, 0, \cdots, 0)$.

Theorem 2.34. *Let $f : M \to N$ be a smooth map of smooth manifolds, $m = \dim(M)$, $n = \dim(N)$. Choose $a \in M$, let $r = rank_a(f)$, then there exist charts (U, φ) centered at a and (V, ϕ) centered at $f(a)$ and a smooth map $g : \varphi(U) \to \mathbb{R}^{n-r}$ such that $f(U) \subseteq V$ and*

$$\begin{cases} \phi \circ f \circ \varphi^{-1}(x_1, \cdots, x_m) = (x_1, \cdots, x_r, g(x_1, \cdots, x_m)) \\ g(0) = 0 \\ Dg(0) = 0 \end{cases}.$$

Proof. Let $E = T_aM$ and $F = T_{f(a)}N$, $T = T_af : E \to F$, and $F_1 = T(E)$. Let $r = dimF_1$. Choose a basis $v_1, \cdots, v_r$ of F_1 and complete it to a basis $v_1, ..., v_r, v_{r+1}, ..., v_n$ of F. Choose vectors $u_1, \cdots, u_r \in E$ with $T(u_i) = v_i, 1 \le i \le r$, then u_i must be independent since their images are independent, so $E_1 = \mathrm{span}\{u_1, ..., u_r\} \subseteq E$ has dimension r, and $T|_{E_1} : E_1 \to F_1$ is an isomorphism.

If $u \in E_1 \cap \ker T$, then $Tu = 0$, so $u = 0$ because $u \in E_1$. So $E_1 \cap \ker T = \{0\}$.

Choose basis $\{u_{r+1}, \cdots, u_m\}$ of $\ker T$. Then $u_1, \cdots, u_r, u_{r+1}, \cdots, u_m$ form a basis of E. (rank + nullity $= m = \dim E$)

The matrix of T with respect to these bases is

$$A = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}_{n\times m}.$$

Let $\begin{cases} u_1^*, \cdots, u_m^* \in E^* \\ v_1^*, \cdots, v_n^* \in F^* \end{cases}$ be the dual basis such that $\begin{cases} u_i^*(u_j) = \delta_{ij} & 1 \le i,j \le m \\ v_i^*(v_j) = \delta_{ij} & 1 \le i,j \le n \end{cases}$, then the dual map satisfies $T^*(v_i^*) = v_j^* \circ T = u_i^*$

By the previous theorem, there is a chart (V, ϕ) centered at $f(a)$ such that $d_{f(a)}\phi_j = v_j^*$ for $j = 1, \cdots, n$. Let $U_0 = f^{-1}(V)$, this is an open neighborhood of a. So for $j = 1, \cdots, r$, define

$$\varphi_j = \phi_j \circ f,$$

then $d_a\varphi_j = d_{f(a)}\phi_j \circ T_a f = v_j^* \circ T = T^*(v_j^*) = u_j^*$ for $j = 1, \cdots, r$.

Then choose functions $\varphi_{r+1}, \cdots, \varphi_m \in C^\infty(U_0)$ such that

$$\begin{cases} \varphi_j(a) = 0 \\ d_a\varphi_j = u_j^* \end{cases}, r < j \le m.$$

Again by previous theorem, there exists an open neighborhood $U \subseteq U_0$ of x such that $\varphi = (\varphi_1, \cdots, \varphi_m) : U \to \mathbb{R}^m$ is a coordinate map.

Let $g_i = \phi_i \circ f \circ \varphi^{-1} : U \to \mathbb{R}$ for $r+1 \le i \le n$, then $g = (g_{r+1}, \cdots, g_n) : U \to \mathbb{R}^{n-r}$ satisfies that $g(0) = 0$.

Let $x \in \varphi(U)$, then

$$\begin{cases} \phi_j \circ f \circ \varphi^{-1}(x) = \phi_j \circ \phi^{-1}(x) = x_j & 1 \le j \le r \\ \phi_j \circ f \circ \varphi^{-1}(x) = g_j(x) & r+1 \le j \le n \end{cases},$$

so $\phi \circ f \circ \varphi(x) = (x_1, \cdots, x_r, g(x))$.

Finally, $d_a\varphi_j = u_j^*$ for $i \le j \le m$ implies that $d_a\varphi_j(u_i) = \delta_{ji}$. So $d_a\varphi(u_i) = e_i$ is the standard basis vector of $\mathbb{R}^m$. Hence for $j > r$,

$$Dg_j(0)e_i = d_{f(a)}\phi_j \circ T \circ (d_a\varphi)^{-1}e_i = d_{f(a)}\phi_j \circ T(u_i) = \begin{cases} d_{f(a)}\phi_j(v_i) = v_j^*(v_i) = \delta_{ji} = 0 & 1 \le i \le r \\ d_{f(a)}\phi(0) = 0 & r+1 \le i \le n \end{cases},$$

so $Dg_j(0) = 0$. □

Corollary 2.35. *(**Semicontinuity** of the rank) Every $a_0 \in M$ has a neighborhood U such that $\mathrm{rank}_a(f) \ge \mathrm{rank}_{a_0}(f)$ for all $a \in U$.*

Proof. Choose the chart as in previous theorem, we have

$$\phi \circ f \circ \varphi^{-1}(x) = \begin{pmatrix} x_1 \\ \cdots \\ x_r \\ g(x)^{\mathrm{T}} \end{pmatrix}$$

$$D(\phi \circ f \circ \varphi^{-1})(x) = \begin{pmatrix} I_r & 0 \\ D_1g(x) & D_2g(x) \end{pmatrix}$$

where $D_1g(x)$ is the Jacobian matrix of g with respect to $x_1, \cdots, x_r$, and $D_2g(x)$ is the Jacobian matrix of g with respect to $x_{r+1}, \cdots, x_m$. So

$$\mathrm{rank}_a(f) = \mathrm{rank}_{\varphi(a)}(\phi \circ f \circ \varphi^{-1}) = \mathrm{rank}(D(\phi \circ f \circ \varphi^{-1})(\varphi(a))) \ge r.$$

□

Definition 2.36. $f : M \to N$ is a ***subimmersion*** at $a \in M$ (or has constant rank at a) if $\text{rank}_a(f) = \text{rank}_b(f)$ for b in a neighborhood of a.

Theorem 2.37. *(Subimmersion or constant rank theorem). Suppose f is a subimmersion at a. There exists adapted chart (U, φ) centered at a and (V, ϕ) centered at $f(a)$ such that $\phi \circ f \circ \varphi^{-1}(x_1 . \cdots x_m) = (x_1, \cdots x_r, 0, \cdots, 0)$.*

Proof. Choose adapted charts $(U, \varphi), (V, \phi)$ as in previous theorem. Then f has constant rank near a if and only if $\tilde{f} = \phi \circ f \circ \varphi^{-1}$ has constant rank near $0 \in \mathbb{R}^m$. So $D_2 g(x) = 0$ for x in a neighborhood of $0 \in \mathbb{R}^m$. Hence, $g(x)$ is independent of$x_{r+1}, \cdots, x_m$ for $x \in W = W_1 \times W_2$, where W_1 is a neighborhood of $0 \in \mathbb{R}^r$ and W_2 is a neighborhood of $0 \in \mathbb{R}^{n-r}$. So there is a smooth $h : W_1 \to \mathbb{R}^{n-r}$ with $g(x_1, \cdots, x_m) = h(x_1, \cdots, x_r)$. Hence,

$$\phi \circ f \circ \varphi^{-1}(x) = (x_1, \cdots, x_r, h(x_1, \cdots, x_r)).$$

Define shear transformation

$$\begin{array}{rccc} \sigma : & W_1 \times \mathbb{R}^{n-r} & \to & W_1 \times \mathbb{R}^{n-r} \\ & (u, v) & \mapsto & (u, v - h(u)) \end{array}$$

then it is a

$$\begin{array}{rccc} \sigma^{-1} : & W_1 \times \mathbb{R}^{n-r} & \to & W_1 \times \mathbb{R}^{n-r} \\ & (y, z) & \mapsto & (y, z + h(y)) \end{array}$$

is the inverse. Now we have $\sigma \circ \phi \circ f \circ \varphi^{-1}(x_1, \cdots, x_m) = (x_1, \cdots, x_r, 0, \cdots, 0)$. Replace ϕ with $\sigma \circ \phi$. Need also to restrict φ to $U \bigcap \varphi^{-1}(W)$ and ϕ to $V \bigcap \phi^{-1}(W_1 \times \mathbb{R}^{n-r})$. (ERROR) □

2.7 Immersions and Submersions

Definition 2.38. Let $f : M \to N$ be smooth, f is called an ***immersion*** (***submersion***) at $a \in M$ if $T_a f : T_a M \to T_{f(a)} N$ is injective (surjective).

Theorem 2.39. *(Immersion Theorem). If f is an immersion at a, then $rank_b(f) = m$ for b in a neighborhood of a, and there exist adapted charts (U, φ) centered at a and (V, ϕ) centered at $f(a)$ with*

$$\phi \circ f \circ \varphi^{-1}(x_1, \cdots, x_m) = (x_1, \cdots, x_m, 0, \cdots, 0).$$

Proof. If f is an immersion at a, then $\text{rank}_a(f) = m = \dim(M)$, so $m \leq n$ and for b closed to a we have $m \leq \text{rank}_b(f) \leq n$, so $rank_b(f) = m$ is a constant. Take $r = m$ in the subimmersion theorem. □

Theorem 2.40. *(Submersion Theorem). If f is an immersion at a, then $rank_b(f) = n$ for b in a neighborhood of a, and there exist adapted charts (U, φ) centered at a and (V, ϕ) centered at $f(a)$ with*

$$\phi \circ f \circ \varphi^{-1}(x_1, \cdots, x_m) = (x_1, \cdots, x_n).$$

Proof. If f is an submersion at a, then $\text{rank}_a(f) = n = \dim(N)$, so $n \leq m$ and for b closed to a we have $n \leq \text{rank}_b(f) \leq m$, so $rank_b(f) = n$ is a constant. Take $r = n$ in the subimmersion theorem. □

Definition 2.41. If $f : M \to N$, then $A = f^{-1}(c) = \{a \in M | f(a) = c\}$ is the ***fibre*** of f over c.

Theorem 2.42. *(Fibre Theorem) Assume that $f : M \to N$ is a subimmersion of constant rank r at all points of $A = f^{-1}(c), c \in N$, then the fibre A is a closed submanifold of M of dimension $\dim(A) = m - r$ and the tangent bundle equal to*

$$TA = \ker(Tf) = \coprod_{a \in A} \ker(T_a f).$$

Proof. Let $a \in A$, choose adapted charts (U, φ) centered at a and (V, ϕ) centered at $f(a)$ as in the subimmersion theorem. Let $F \subseteq \mathbb{R}^m$ be the linear subspace $F = \{x \in \mathbb{R}^m | x_1 = \cdots = x_r = 0\} \cong \mathbb{R}^{m-r}$, then

$$x \in \varphi(U \cap A) \Leftrightarrow \tilde{f}(x) = 0 \Leftrightarrow x \in F$$

where $\tilde{f}(x) = (x_1, \cdots, x_r, 0, \cdots, 0)$. So $U \cap A = \varphi^{-1}(F)$. This shows that A is a submanifold of M at a: up to renumbering the coordinates (U, φ) is a submanifold chart for A. We have that $\dim(A) = \dim(F) = m - r$. We also have the following commutative diagram

$$\begin{array}{ccc} T_aM & \xrightarrow{T_a f} & T_{f(a)}N \\ d_a\varphi \downarrow\cong & & \cong\downarrow d_{f(a)}\phi \\ \mathbb{R}^m & \xrightarrow{\tilde{f}=D\tilde{f}(0)} & \mathbb{R}^n \end{array}.$$

Notice that $d_a\varphi$ and $d_f(a)\phi$ are isomorphism, hence we can write

$$T_aA = (d_a\varphi)^{-1}(F) = (d_a\varphi)^{-1}(\ker \tilde{f}) = \ker(d_a\varphi \circ \tilde{f}) = ker(d_{f(a)}\phi \circ T_af) = \ker(T_af)$$

for all $a \in A$. □

Definition 2.43. The number $e(f, A) = n - r$ is called the **excess** of f at $A = f^{-1}(c)$.

Remark 2.44. Interpretation: Near $a \in A$, $\varphi(A)$ is the solution set of n equations in m variables

$$\begin{cases} \tilde{f}_1(x_1, \cdots x_m) = 0 \\ \cdots \\ \tilde{f}_n(x_1, \cdots x_m) = 0 \end{cases}$$

If the equations were functionally independent, we would have $\dim(A) = m - n$. Instead, only r equations are independent, then $\dim(A) = m - r = m - n + e(f, A)$.

In the case where the excess is zero, we have the following important definition.

Definition 2.45. $c \in N$ is called a ***regular value*** of f if f is a submersion at a for all $a \in A = f^{-1}(c)$.

Theorem 2.46. *(Regular Value Theorem). If $c \in N$ is a regular value of f , then $A = f^{-1}(c)$ is a closed submanifold of dimension $m - n$, and $TA = \ker(Tf)$.*

Example 2.47. Let $M = M(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$ be the $n \times n$ matrices with entries in $\mathbb{R}$. Define $f : M \to M$ by $f(X) = XX^T$, then $f^{-1}(I) = \{X \in M | XX^T = I\}$ is the orthogonal group of $n \times n$ orthogonal matrices, denoted $\mathcal{O}(n)$ or $\mathcal{O}(n, \mathbb{R})$.

Claim 2.48. $\mathcal{O}(n)$ is a submanifold of M of dimension $n(n-1)/2$ and $T_I\mathcal{O}(n) = \{H \in M | H^T = -H\}$, the Lie algebra of skew-symmetric matrices.

Proof. Method 1: check that f is a subimmersion of appropriate rank $< n^2$ later! (Lie groups)

Method 2: Let $N = \{Y \in M | Y = Y^T\}$ be the subspace of symmetric matrices, then f is a map $f : M \to N$. Check that I is a regular value of f.

Let $A \in M$, then $Df(A) : M \to N$ is linear. For $H \in M$,

$$f(A + H) = AA^T + AH^T + HA^T + HH^T = f(A) + L(H) + R(H)$$

where $L(H) = AH^T + HA^T$ is linear and $R(H) = HH^T$. So considering the matrix norms,

$$\frac{\|R(H)\|}{\|H\|} = \frac{\|HH^T\|}{\|H\|} \leq \frac{\|H\|^2}{\|H\|} = \|H\| \to 0 \text{ as } H \to 0.$$

So we get $Df(A)H = AH^T + HA^T = L(H)$. To check that f is a submersion at $A \in \mathcal{O}(n)$, we need to know that $L : M \to N$ is surjective. Let $B \in N$. We want to solve the equation

$$AH^T + HA^T = B$$

for H. To do this, rewrite the above equation as

$$AH^T + HA^T = \frac{1}{2}B + \frac{1}{2}B^T$$

and then we can find that half of the above equation is easy to solve. If we set $HA^T = \frac{1}{2}B$, then

$$H = HA^TA = \frac{1}{2}BA$$

satisfies the initial equation $AH^T + HA^T = B$. So I is a regular point of f. And

$$\dim(\mathcal{O}(n)) = \dim(M) - \dim(N) = n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1) = \binom{n}{2}$$

$$T_I\mathcal{O}(n) = \ker(Df(I)) = \ker(H \to H + H^T) = \{H \in M | H^T = -H\}$$

$\square$

Remark 2.49. $\mathcal{O}(n)$ is our first example of a Lie group. For any $A \in \mathcal{O}(n)$, $1 = \det(AA^T) = \det(A)^2 \Rightarrow det(A) = \pm 1$. So $\mathcal{O}(n)$ has two connected components, namely the preimage under det of $+1$ and the preimage under det of -1. The *special orthogonal group* is $SO(n) = \{A \in O(n) | det(A) = 1\}$, the connected component of O(n) of determinant 1 matrices, and the other connected component of $\mathcal{O}(n)$ is the coset $A_0 S\mathcal{O}(n) = \{A \in \mathcal{O}(n) | det(A) = -1\}$, where A_0 is the matrix

$$A_0 = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \cdots & \\ & & & 1 \end{pmatrix}.$$

2.8 Embeddings

Definition 2.50. $f : M \to N$ is an embedding if it is an immersion (that is, an immersion at every point of M) and a homeomorphism onto its image.

Equivalently, f : M $\to$ N is an embedding if
(i) f is injective,
(ii) T_af is injective for all $a \in M$,
(iii) $f^{-1} : f(M) \to M$ is continuous.

Theorem 2.51. *(Embedding Theorem).*
(i) The inclusion of a submanifold $i : A \to M$ is an embedding.
(ii) If $f : P \to M$ is an embedding, then $A = f(P)$ is a submanifold of M and $f : P \to A$ is a diffeomorphism and $TA = Tf(TP)$.

Proof. (i) Recall that Ti is injective, so i is an immersion. By definition, the inclusion is injective. Also the topology on A induced by the smooth structure is the subspace topology inherited from M, i.e. $i : A \to M$ is a homeomorphism onto its image.

(ii) Let $k = \dim(P)$ and $n = \dim(M)$. Let $p \in P$, $a = f(p) \in A$. Immersion Theorem gives adapted charts (U, φ) centered at p and (V, ϕ) centered at a such that

$$\phi \circ f \circ \varphi^{-1}(x_1, \cdots x_k) = (x_1, \cdots, x_k, 0, \cdots, 0)$$

is the inclusion $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$.

We would like $f(U) = V \cap A$ and $f(U) = \phi^{-1}(\mathbb{R}^k)$, but this may fail.

First replace (V, ϕ) with (V_0, ϕ_0) where $V_0 = V \cap \phi^{-1}(\varphi(U) \times \mathbb{R}^{n-k})$ and $\phi_0 = \phi|_{V_0}$.

To get $f(U) = V_0 \cap A$, use the fact that f is a homeomorphism onto A. So there is open $V' \subseteq M$ with $f(U) = V' \cap A$. Replace (V_0, ϕ_0) with (V_1, ϕ_1) where $V_1 = V_0 \bigcap V'$ and $\phi_1 = \phi_0|_{V_1}$. Then (V_1, ϕ_1) is a submanifold chart for A at a. The map $f^{-1} : A \to P$ is represented near a by

$$\begin{array}{cccc} \varphi \circ f \circ \phi^{-1} : & \phi(V_1 \cap A) & \to & \varphi(U) \subseteq \mathbb{R}^k \\ & x & \mapsto & x \end{array},$$

so f^{-1} is smooth, f is a diffeomorphism. And $Tf : TP \to TM$ sends TP onto TA. $\square$

Example 2.52. (Non-example). Let P $= (-\frac{\pi}{2}, \frac{3\pi}{2})$, $M = \mathbb{R}^2$, and

$$f(t) = \begin{pmatrix} \cos(t) \\ \cos(t)\sin(t) \end{pmatrix}.$$

We have chosen the domain such that this is injective. The graph of this function for $t \in P$ is the *lemniscate*. f is an injective immersion, and

$$f'(t) = \begin{pmatrix} -\sin(t) \\ -\sin(t)^2 + \cos(t)^2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then $f^{-1} : f(P) \to P$ is not continuous: f is not an embedding, $f(P)$ is not a submanifold.

Example 2.53. 6 (Non-example). Let $P = \mathbb{R}$ and $M = \mathbb{S}^1 \times \mathbb{S}^1$ be the 2-torus. Let

$$f(t) = \begin{pmatrix} e^{it} \\ e^{i\alpha t} \end{pmatrix}.$$

Hence we think of $\mathbb{S}^1$ as the unit circle $\{z \in \mathbb{C} || z| = 1\}$. Then

$$f'(t) = \begin{pmatrix} ie^{it} \\ i\alpha e^{i\alpha t} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so f is an immersion.

1. If $\alpha = \frac{p}{q} \in \mathbb{Q}$, then f is not injective: $f(t) = f(t + 2\pi q)$. But f descends to a map

   $$\bar{f} : \mathbb{R}/2\pi q\mathbb{Z} \to \mathbb{S}^1 \times \mathbb{S}^1.$$

   called a ***torus knot***. $\bar{f}$ is an injective immersion, and in fact $\bar{f}$ is an embedding because the inverse is continuous. To check that $\bar{f}^{-1} : f(\mathbb{R}) \to \mathbb{R}/2\pi q\mathbb{Z}$ is continuous, one must check that if $C \subseteq \mathbb{R}/2\pi q\mathbb{Z}$ is closed, then $f(C)$ is closed in $\mathbb{S}^1 \times \mathbb{S}^1$. True because $\mathbb{R}/2\pi q\mathbb{Z} \cong \mathbb{S}^1$ is compact.

2. (Kronecker). If $\alpha \notin Q$, then f is injective and $A = f(P)$ is dense in M, so it is certainly not a submanifold.

Definition 2.54. An ***immersed submanifold*** of a manifold M is a pair (P, f) where P is a manifold and $f : P \to M$ is an injective immersion. Then $f : P \to A = f(P)$ is a continuous bijection, but $f^{-1} : A \to P$ is not necessarily continuous for the subspace topology on A.

Remark 2.55. Identifying P with A, we see an immersed submanifold is a subset A of M equipped with a smooth structure such that the inclusion $i : A \to M$ is an immersion. But the topology on A induced by this smooth structure may be finer (bigger, stronger) than the subspace topology.

3 Vector Fields

Let M be an n-manifold, and let $\pi = \pi_M : TM \to M$ be the tangent bundle projection. Then for any $a \in M$, $\pi^{-1}(a) = T_aM$ for all $a \in M$.

Definition 3.1. A ***vector field*** on M is a smooth section of π, that is, a map $\xi : M \to TM$ satisfying $\pi \circ \xi = Id_M$. So $\xi(a) \in T_aM$ for all $a \in M$. We often write $\xi(a) = \xi_a$.

Definition 3.2. A point $a \in M$ is a ***zero*** or ***equilibrium*** of ξ if $\xi_a = 0 \in T_aM$.

Let $c = (U, \varphi)$ be a chart on M. Then $\xi(U) \subseteq TU = \pi^{-1}(U)$, so c and Tc are adapted for ξ. An expression for ξ in coordinates is then

$$T\varphi \circ \xi \circ \varphi^{-1} : \varphi(U) \to T\varphi(U) = \varphi(U) \times \mathbb{R}^n.$$

So this defines a vector field on $\varphi(U) \subseteq \mathbb{R}^n$. For each $x \in \varphi(U)$, $T\varphi \circ \xi \circ \varphi^{-1}(x) = (x, h)$ for some $h \in \mathbb{R}^n$. We write $h = \tilde{\xi}(x)$, with $\tilde{\xi} : \varphi(U) \to \mathbb{R}^n$. (ERROR)

A vector field ξ is smooth if and only if $\tilde{\xi}$ is smooth for all charts c on M.

Remark 3.3. This meshes with our definition of smooth morphisms of manifolds: $f : M \to N$ is smooth if and only if it's expression in any pair of charts is smooth as a map $\mathbb{R}^m \to \mathbb{R}^n$. We see that ξ is smooth if and only if it's expression $T\pi \circ \xi \circ \varphi^{-1}$ is smooth, if and only if $\tilde{\xi}$ is smooth.

For vector fields ξ, η and a function $f : M \to \mathbb{R}$, we define vector fields $\xi + \eta$ and $f\xi$ by $(\xi + \eta)(a) = \xi(a) + \eta(a)$ and $(f\xi)(a) = f(a)\xi(a)$. With respect to a chart c we have $\widetilde{(\xi + \eta)} = \tilde{\xi} + \tilde{\eta}$ and $\widetilde{(f\xi)} = \tilde{f}\tilde{\xi}$.

Denote $\mathcal{T}(M) = \{\xi : M \to TM | \pi \circ \xi = Id_M, \xi \text{ is smooth}\}$. $\mathcal{T}(M)$ is a module over the algebra $C^\infty(M)$.

3.1 k-frame

Definition 3.4. A ***k-frame*** on M is an ordered k-tuple $(\xi_1, \cdots, \xi_k)$ of smooth vector fields on M such that for all $a \in M$, $(\xi_1)_a, \cdots, (\xi_k)_a \in T_aM$ are linearly independent. In particular, A 1-frame on M is a nowhere vanishing vector field ξ. That is, $\xi(a) \neq 0$ for all a.

An n-manifold M is called ***parallelizable*** if it has an n-frame.

Example 3.5. $M = U$ is an open subset of $\mathbb{R}^n$, a smooth vector field on U is a smooth map

$$\xi : U \to TU = U \times \mathbb{R}^n$$

of the form $\xi(x) = (x, \tilde{\xi}(x))$ where $\tilde{\xi} : U \to \mathbb{R}^n$ is smooth.

U is parallelizable: the constant vector field $e_1, \cdots, e_n : U \to U \times \mathbb{R}^n$ form a n-frame.

Example 3.6. Lie groups are parallelizable.

Example 3.7. $M = \mathbb{S}^2$, $k_M = 0$.

Fact 3.8. *(F.Adams, 1962) Let $M = \mathbb{S}^{n-1}$, $m = max\{l | 2^l \text{ divides } n\}$. Write $m = 4b + a$ with $b \in \mathbb{Z}$, $a \in \{0, 1, 2, 3\}$. (Euclidean Algorithm)*

*Let $\varrho(n) = 2^a + 8b$, called **Radon-Hurwitz number**.*

n	1	2	3	4	5	6	7	8	9	10	$\cdots$
$\varrho(n)$	1	2	1	4	1	2	1	8	1	2	$\cdots$

Theorem 3.9. *(Adam's theorem) $k_{\mathbb{S}^{n-1}} = \varrho(n) - 1$.*

If n is odd, then $m = 0$, $a = b = 0$, so $k_m = 0$. So every smooth vector field on an even-dimensional sphere has zeros.

$\mathbb{S}^{n-1}$ is parallelizable if and only if $k_{\mathbb{S}^n} = n - 1$, if and and only if $\varrho(n) = n$

Exercise 3.10. (Next homework)$\mathbb{S}^{n-1}$ is parallelizable if and only if $\mathbb{R}^n$ has the structure of a division algebra.

Theorem 3.11. *(Kerlaire, 1956) $\rho(n) = n$ if and only if n = 1, 2, 4, 8. Therefore, $\mathbb{S}^{n-1}$ is parallelizable if and only if n = 1, 2, 4, 8.*

3.2 Flows

3.2.1 Trajectories

Definition 3.12. Let $\xi \in \mathcal{T}(M)$. An ***integral curve*** or ***trajectory*** of ξ is a smooth map $\gamma : J \to M$ where $J \subseteq \mathbb{R}$ is a an open interval, which satisfies $\gamma'(t) = \xi_{\gamma(t)}$ for all $t \in J$. We say γ starts at $a \in M$ if $0 \in J$ and $\gamma(0) = a$.

For $s \in \mathbb{R}$, let $J_{+s} = \{t + s | t \in J\}$.

Lemma 3.13. *(Time Translation Lemma) Let $\gamma : J \to M$ be a trajectory of ξ, define $\delta : J_{-s} \to M$ by $\delta(t) = \gamma(t+s)$ then δ is a trajectory. If $s \in J$, then δ starts at $a = \gamma(s)$.*

Proof. Recall tangent map $T_t\gamma : T_t\mathbb{R} \to T_{\gamma(t)}M$, we put $\gamma'(t) = T_t\gamma(t, 1)$.
δ is smooth and $\delta'(t) = \gamma'(t + s) = \xi_{\gamma(t+s)} = \xi_{\delta(t)}$, so δ is a trajectory. If $s \in J$, then $0 \in J_{-s}$, and $\delta(0) = \gamma(s) = a$. ☐

Trajectories are partially ordered (by inclusion).

Definition 3.14. A trajectory $\gamma : J \to M$ is ***maximal*** if for any trajectory $\gamma_1 : J_1 \to M$ with $J \subseteq J_1$ and $\gamma = \gamma_1|_J$, we have $J = J_1$.
The ***initial value problem*** for ξ is to find for each $a \in M$ a maximal trajectory $\gamma : J \to M$, with $\gamma'(t) = \xi_{\gamma(t)}$, and starting at $a = \gamma(0)$.

Theorem 3.15. *(Existence and Uniqueness) For each $a \in M$, the IVP has a unique solution.*

Proof. 1. Existence of a trajectory:
Choose a chart (U, φ) at $a \in M$. For a path $\gamma : J \to M$, write $\tilde{\gamma} = \varphi \circ \gamma : J \to \varphi(U) \subseteq \mathbb{R}^n$. Write

$$\begin{array}{rccc} T\varphi \circ \xi \circ \varphi^{-1} : & \varphi(U) & \to & \varphi(U) \times \mathbb{R}^n \\ & x & \mapsto & (x, \tilde{\xi}(x)) \end{array}$$

where $\tilde{\xi} : \varphi(U) \to \mathbb{R}^n$ is the empression for ξ in the chart. Then γ is a trajectory of ξ starts at a if and only if

$$\begin{cases} \tilde{\gamma}'(t) = \tilde{\xi}_{\gamma(t)} \\ \tilde{\gamma}(0) = \varphi(a) \end{cases}$$

which is a vector-valued ODE with smooth RHS $\tilde{\xi}$. By the existence theorem of ODE, a solution $\tilde{\gamma} : J \to \varphi(U)$ exists. So a trajectory $\gamma = \varphi^{-1} \circ \tilde{\gamma} : J \to U$ starting at a exists.
2. Uniqueness
Suppose $\gamma_1 : J_1 \to M_1$ $\gamma_2 : J_2 \to M_2$ are two trajectories starting at a. Let $I = \{t \in J_1 \cap J_2 | \gamma_1(t) = \gamma_2(t)\}$ then $0 \in I$ because $\gamma_1(0) = \gamma_2(0) = a$.
Let $\Gamma(t) = (\gamma_1(t), \gamma_2(t))$, then $\Gamma : J_1 \cap J_2 \to M \times M$ is smooth and $I = \Gamma^{-1}(\Delta_M)$, where $\Delta_M = \{(x, x) | x \in M\}$. Since M is Hausdorff, I is closed in $J_1 \cap J_2$.
By the uniqueness of solutions of ODE, I is open.
So $I = J_1 \cap J_2$ is a connected, non-empty component of $J_1 \cap J_2$, and $\gamma_1 = \gamma_2$ on $J_1 \cap J_2$.
3. Maximality
Let $\{\gamma_\alpha : J_\alpha \to M | \alpha \in \Lambda\}$ be the collection of trajectories starting at a. For any $\alpha, \beta \in \Lambda$ we have $\gamma_\alpha = \gamma_\beta$ on $J_\alpha \cap J_\beta$.
Define $J = \cup_\alpha J_\alpha$ an open interval containing a, and $\gamma(t) = \gamma_\alpha(t)$ for some $\alpha \in \Lambda$, $t \in J$ such that $t \in J_\alpha$, then γ is well-defined and smooth, also maximal. ☐

Corollary 3.16. *Let $\gamma : J \to M$ be the maximal trajectory of ξ starting at a . Let $s \in J$ and $b = \gamma(s)$, $\delta(t) = \gamma(t+s)$, then $\delta : J_{-s} \to M$ is the maximal trajectory starting at b.*

3.2.2 Flow

For $a \in M$, let $0 \in \mathcal{D}^a = \mathcal{D}^a(\xi) \subseteq \mathbb{R}$ be the domain of the maximal trajectory starting at a.

Definition 3.17. The flow domain of ξ is $\mathcal{D} = \mathcal{D}(\xi) = \{(t,a) | t \in \mathcal{D}^a\} \subseteq \mathbb{R} \times M$. The ***flow*** of ξ is the map $\Theta : \mathcal{D} \to M$ defined by $\Theta(t,a) = \gamma(t)$, where γ is the maximal trajectory starting at a.

Note $\{0\} \times M \subseteq \mathcal{D}$.
Also put $\mathcal{D}_t = \{a \in M | (t,a) \in \mathcal{D}\}$ and $\Theta_t(a) = \Theta^a(t) = \Theta(t,a)$ for $t \in \mathbb{R}, a \in M$. Then we have

$$\begin{cases} \Theta^a : \mathcal{D}^a \to M \\ \Theta_t : \mathcal{D}_t \to M \end{cases}$$

and $t \in \mathcal{D}^a \Leftrightarrow (t,a) \in \mathcal{D} \Leftrightarrow a \in \mathcal{D}_t$.

Example 3.18. $M = \mathbb{R}$, $\xi(x) = x^2$. Find the flow domain and the flow of ξ, and solve the IVP for ξ.

Solution. We have the IVP

$$\begin{cases} x'(t) = x^2 \\ x(0) = x_0 \end{cases}$$

To solve this ODE, we have

$$\frac{x'(t)}{x^2(t)} = 1$$

for $x \neq 0$,so by integration of t on both sides

$$-\frac{1}{x(t)} = t + c$$

where c is a constant. Take the initial value $x(0) = -\frac{1}{c} = x_0$, so

$$x(t) = \frac{x_0}{1 - tx_0}$$

holds for both $x(0) \neq 0$ and $x(0) = 0$.
Flow $\Theta(t,x) = \frac{x}{1-tx}$.
Domain:
For $x_0 = 0$, $x(t) = 0$ for all $t \in \mathbb{R}$, so $\mathcal{D}^0 = \mathbb{R}$.
For $x_0 > 0$, $x(t)$ exists for $t < \frac{1}{x_0}$, so $\mathcal{D}^{x_0} = (-\infty, \frac{1}{x_0})$.
For $x_0 < 0$, $x(t)$ exists for $t > \frac{1}{x_0}$, so $\mathcal{D}^{x_0} = (\frac{1}{x_0}, \infty)$.

Theorem 3.19. *(Flow)*

1. *The flow domain $\mathcal{D} \subseteq \mathbb{R} \times M$ of ξ is open and the flow $\Theta : \mathcal{D} \to M$ is smooth. In particular, for each $t \in \mathbb{R}$, the set $\mathcal{D}_t \subseteq M$ is open in M and $\Theta_t : \mathcal{D}_t \to M$ is smooth.*

2. *$\mathcal{D}^{\Theta(t,a)} = \mathcal{D}^a - t$ for all $(t,a) \in \mathcal{D}$.*

3. *$\mathcal{D}_{s+t} \supseteq \mathcal{D}_s \cap \Theta_s^{-1}(\mathcal{D}_t)$ and on $\mathcal{D}_s \cap \Theta_s^{-1}(\mathcal{D}_t)$ the flow law $\Theta_{s+t} = \Theta_t \circ \Theta_s$ holds.*

Proof. (Flow)

1. Use charts to reduce to the case $M = U$ an open subset in $\mathbb{R}^n$, then the result is true by theory of ODE: for every $a \in U$, there is an open neighborhood V_a of a in U and an $\varepsilon > 0$ such that for all $x \in V_a$, the trajectory $\Theta(t,x)$ exists for all $t \in (-\varepsilon, \varepsilon)$. This shows $\mathcal{D}$ is open. Also $\Theta(t,x)$ depends smoothly on t and x.

2. For $a \in M$, let $\gamma(t) = \Theta(t,a)$. For $s \in \mathcal{D}^a$, let $\delta(t) = \gamma(t+s)$. The domain of δ is $\mathcal{D}^a - s$ by the time translation lemma, and δ starts at $\gamma(s) = \Theta(s,a)$, i.e. $\delta(t) = \Theta(t, \Theta(s,a))$.The definition interval of $\mathcal{D}$ is $\mathcal{D}^{\Theta(s,a)}$

3. Fix $a \in M$, let $\gamma(t) = \Theta(t,a)$. For $s \in \mathcal{D}^a$, let $\delta(t) = \gamma(t+s)$. We get $\gamma(s+t) = \Theta(t, \Theta(s,a))$, i.e. $\Theta(t,a) = \Theta(t, \Theta(s,a))$. So
$$\Theta_{s+t}(a) = \Theta_t \circ \Theta_s(a).$$
For this to hold, we have $s \in \mathcal{D}^a$, $t \in \mathcal{D}^{\Theta(s,a)} = \mathcal{D}^a - s$, which is equivalent to
$$\begin{cases} a \in \mathcal{D}_s \\ \Theta(s,a) \in \mathcal{D}_t \end{cases} \Leftrightarrow \begin{cases} a \in \mathcal{D}_s \\ a \in \Theta_s^{-1}(\mathcal{D}_t) \end{cases}$$
On the other hand, if t, s satisfies these conditions, then $s + t \in \mathcal{D}^a$, i.e. $a \in \mathcal{D}_{s+t}$.

$\square$

Corollary 3.20. *$\Theta_t(\mathcal{D}_t) = \mathcal{D}_{-t}$ and $\Theta_t : \mathcal{D}_t \to \mathcal{D}_{-t}$ is a diffeomorphism with inverse Θ_{-t}.*

Proof. By the third property in previous theorem, for any $(s,t,a) \in \mathbb{R} \times \mathbb{R} \times M$,
$$s + t \in \mathcal{D}^a \Leftrightarrow s \in \mathcal{D}^a - t = \mathcal{D}^{\Theta t,a)} \Leftrightarrow \Theta(t,a) \Leftrightarrow \Theta_t(a) \in \mathcal{D}_s$$
Set $s = -t$, we get $0 \in \mathcal{D}^a \Leftrightarrow \Theta_t(a) \in \mathcal{D}_{-t}$, where LHS is always true, so $\Theta_t(\mathcal{D}_t) \subseteq \mathcal{D}_{-t}$. And $\Theta_{-t}(\Theta_t(a)) = a$. Replace t with $-t$, we get $\Theta_{-t}(\mathcal{D}_{-t}) \subseteq \mathcal{D}_t$, $\Theta_t(\Theta_{-t}(a)) = a$. $\square$

Definition 3.21. ξ is complete if $\mathcal{D} = \mathcal{D}(\xi) = \mathbb{R} \times M$, i.e. $\mathcal{D}^a = \mathbb{R}$ for all $a \in M$ and $\mathcal{D}_t = M$ for all $t \in \mathbb{R}$.

If ξ is complete, each Θ_t is a diffeomorphism $M \to M$, i.e. $\Theta_t \in \mathrm{Diff}(M)$ the group of diffeomorphisms $M \to M$.In this case, $t \mapsto \Theta_t$ is a homeomorphism $\mathbb{R} \to \mathrm{Diff}(M)$ by the flow law. It also defines an action of $\mathbb{R}$ on M.

Lemma 3.22. *(Uniform Time Lemma) If there is $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \times M \subseteq \mathcal{D}$, then ξ is complete.*

Proof. Let $a \in M$, $s = \sup(\mathcal{D}^a)$. Suppose $s < +\infty$, let $t_0 \in (s - \varepsilon, s) \subseteq \mathcal{D}^a$, $b = (t_0, a)$. Since $(-\varepsilon, \varepsilon) \subseteq \mathcal{D}^b = \mathcal{D}^{\Theta(t_0,a)} = \mathcal{D}^a - t_0$, we have $(t_0 - \varepsilon, t_0 + \varepsilon) \subseteq \mathcal{D}^a$. But $t_0 + \varepsilon > s$. It's a contradiction, hence $s = +\infty$.
Similarly, $\inf(\mathcal{D}^a) = -\infty$. $\square$

Definition 3.23. The support of ξ is $\mathrm{supp}(\xi) = \overline{\{a \in M | \xi(a) \neq 0_a\}}$.

Theorem 3.24. *If supp(ξ) is compact, then ξ is complete.*

Proof. Let $K = \mathrm{supp}(\xi)$. If $a \notin K$, then $\xi(a) = 0$. So $\Theta(t,a) = a$ for all t and $\mathcal{D}^a = \mathbb{R}$.
For every $a \in K$, there exists $\varepsilon_a > 0$ and an open neighborhood U_a of a such that $(-\varepsilon_a, \varepsilon_a) \times U_a \subseteq \mathcal{D}$, i.e. for any $b \in U_a$, $(-\varepsilon_a, \varepsilon_a) \subseteq \mathcal{D}^b$.
Cover K with finitely many $U_{a_1}, \cdots, U_{a_p}$, put $\varepsilon = \min\{\varepsilon_{a_1}, \cdots \varepsilon_{a_p}\}$, then $(-\varepsilon, \varepsilon) \times M \subseteq \mathcal{D}$,by the uniform time lemma, ξ is complete. $\square$

Corollary 3.25. *On a compact manifold, every vector field is complete.*

Example 3.26. An example of complete vector fields: linear vector fields on vector spaces.
Let $\xi : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map, i.e. a matrix A.
Flow of ξ: $\Theta : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$.
$$\begin{cases} x'(t) = \xi(x(t)) = Ax(t) \\ x(0) = x_0 \end{cases}$$
The solution to this equation is $x(t) = e^{At}x_0$. This is defined for all t

Example 3.27. A ***time-dependent linear vector field*** on $\mathbb{R}^n$ is a vector field ξ on $\mathbb{R} \times \mathbb{R}^n$ of the form $\xi(t,x) = (\frac{\partial}{\partial t}, A(t)x)$, where $A : \mathbb{R} \to M(n, \mathbb{R})$ is a smooth map. Such a vector space is also complete.

4 Derivations

Let A be a commutative ring with 1, B be a commutative algebra with 1 over A, C be a B-module.

Definition 4.1. An A-derivation of B into C is a map $l : B \to C$ satisfying:

1. A-linearity: $l(a_1b_1 + a_2b_2) = a_1l(b_1) + a_2l(b_2)$
2. Leibniz rule: $l(b_1b_2) = l(b_1)b_2 + b_1l(b_2)$

for all $a_i \in A$, $b_i \in B$.

Remark 4.2. We should really write $b_2l(b_1)$ instead of $l(b_1)b_2$, because C is a left B-module, but that is wrong if B is noncommutative. In that case, need to assume C is a B-bimodule. For commutative B, can make C a bimodule by putting $b_1cb_2 = b_1b_2c$.

The set $\mathrm{Der}_A(B, C)$ is the set of all A-derivations of B into C. It is a B-module (as well as an A-module): If $b_1, b_2 \in B$ and $l_1, l_2 \in \mathrm{Der}_A(B, C)$, then $b_1l_1 + b_2l_2 \in \mathrm{Der}_A(B, C)$ for commutative B.

Lemma 4.3. *If $l \in Der_A(B, C)$, then $l(a1_B) = 0$ for all $a \in A$.*

Proof. $l(a1_B) = al(1_B)$. Only need to show that $l(1_B) = 0$.
Note that $l(1_B) = l(1_B \cdot 1_B) = l(1_B)1_B + 1_Bl(1_B)$, so $l(1_B) = 0$. □

If $C = B$, we write $\mathrm{Der}_A(B) = \mathrm{Der}_A(B, B)$.

Example 4.4. Let M be an n-manifold, $A = \mathbb{R}$.

1. Let $a \in M$ and $B = C^\infty_{M,a}$ be the algebra of germs at a, and $C = \mathbb{R}$. Here $A = \mathbb{R}$. Recall the evaluation map

$$\begin{array}{rccc} \mathrm{ev}: & C^\infty_{M,a} & \to & \mathbb{R} \\ & [f] & \mapsto & f(a) \end{array}$$

   is an algebra homomorphism. This makes C a B-module: $[f]c = f(a)c$.

   We call $\mathrm{Der}_{\mathbb{R}}(C^\infty_{M,a}, \mathbb{R})$ the derivations of M at a. The map

$$\begin{array}{rccc} \mathscr{L}_a: & T_aM & \to & \mathrm{Der}_{\mathbb{R}}(C^\infty_{M,a}, \mathbb{R}) \\ & v & \mapsto & \mathscr{L}_a(v) \end{array}$$

   where $\mathscr{L}_a(v)([f]) = d_af(v)$, is an isomorphism.

2. Let $B = C^\infty(M) = C$, We call $\mathrm{Der}_{\mathbb{R}}(C^\infty(M))$ the ***derivation*** of M.

   Each $\xi \in \mathcal{T}(M)$ defines a derivation $\mathscr{L}_\xi$ of M, $\mathscr{L}_\xi(f) = df(\xi)$.

   That is, $\mathscr{L}_\xi(f)$ is the function defined by $\mathscr{L}_\xi(f)(a) = d_af(\xi_a)$. This is the ***Lie derivative*** or ***directional derivative*** of f along ξ.

   $\mathscr{L}_\xi$ satisfies the Leibniz rule. Need to check $\mathscr{L}_\xi(f)$ is smooth: in a chart (U, φ), write $\tilde{f} = f \circ \varphi^{-1}$, then $(T\varphi \circ \xi \circ \varphi^{-1})(x) = (x, \tilde{\xi}(x))$, i.e. $T\varphi \circ \xi \circ \varphi^{-1} = \mathrm{id}_{\varphi(U)} \times \tilde{\xi}$ where $\tilde{\xi} : \varphi(U) \to \mathbb{R}^n$ is smooth. Then $\mathscr{L}_\xi(f)(a) = d_af(\xi_a) = D\tilde{f}(\varphi(a))\tilde{\xi}(\varphi(a))$ for $a \in U$. Since $\tilde{f}$ and $\tilde{\xi}$ are smooth, so is $\mathscr{L}_\xi(f)$.

Theorem 4.5. *The map*

$$\begin{array}{rccc} \mathscr{L}: & \mathcal{T}(M) & \to & Der_{\mathbb{R}}(C^\infty(M)) \\ & \xi & \mapsto & \mathscr{L}_\xi \end{array}$$

is an isomorphism.

Ingredients of proof:

Lemma 4.6. *For all $0 < p < q$, there is a C^∞ function $\lambda : \mathbb{R}^n \to [0, 1]$ with*

$$\lambda(x) = \begin{cases} 1 & \|x\| < p \\ 0 & \|x\| < q \end{cases}.$$

Proof. Define

$$\alpha(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

α is smooth: check $\alpha^k(0) = 0$ for all $k \geq 0$. (We say α is flat at 0. The k-th Taylor polynomial of α at $x = 0$ is the zero polynomial.)

Define $\beta : \mathbb{R} \to [0, \infty)$ by $\beta(x) = \alpha(x-p)\alpha(q-x)$.

Define $\gamma : \mathbb{R} \to [0,1]$ by $\gamma(x) = \frac{\int_x^q \beta(y)dy}{\int_p^q \beta(y)dy}$. Then γ is smooth.

Let $\lambda(x) = \gamma(\|x\|)$. $\square$

Theorem 4.7. *(Extension Theorem) For every point $a \in M$, the restriction map*

$$\begin{array}{ccc} C^\infty(M) & \to & C^\infty_{M,a} \\ f & \mapsto & [f]_a \end{array}$$

is surjective.

Proof. Let $[g]_a \in C^\infty_{M,a}$, then $g \in C^\infty(U)$ for some open neighborhood U of a.

Without lose of generality, may assume U is a domain of a chart (U, φ) centered at a.

Choose $0 < p < q$ such that $B_q(0) \subseteq \varphi(U)$. Choose λ as in previous lemma, and let $\varrho = \lambda \circ \varphi \in C^\infty(U)$, then $\text{supp}(\varrho) = \varphi^{-1}(B_q(0))$ is compact, hence closed in M because M is Hausdorff. Hence $V = M \ supp(\varrho)$ is open and U, V form an open cover of M. Now define

$$f(x) = \begin{cases} 0 & x \in V \\ \varrho g & x \in U \end{cases}$$

so f is well-defined. f is smooth on both U and V, hence smooth on M. Also $f = g$ on $\varphi^{-1}(B_p(0))$, so $[f]_a = [g]_a$. $\square$

Taking $g = 1$ in the Extension theorem, we get the following lemma.

Lemma 4.8. *(Bump Function Lemma) For all $a \in M$ and every neighborhood U of a, there is a smooth $p \in C^\infty(M)$ with $supp(p) \subseteq U$ and $p = 1$ on a neighborhood of a.*

Lemma 4.9. *(Locality of Derivations) Let $l \in Der_{\mathbb{R}}(C^\infty(M))$ and $f \in C^\infty(M)$, let $U \subseteq M$ be open. If $f = 0$ on U, then $l(f) = 0$ on U.*

Proof. Let $a \in U$, $\varrho \in C^\infty(M)$ a bump function at a supported at U, then $\varrho \cdot f \equiv 0$, so $f = \varrho f + (1-\varrho) f = (1-\varrho) f$. Then $l(f) == l((1-\varrho)f) = l(1-\varrho) \cdot f + (1-\varrho) \cdot l(f) = 0$ on a neighborhood of a. Since $a \in U$ is arbitrary, $l(f) = 0$ on U. $\square$

So now we can prove the theorem: $\mathscr{L} : \mathcal{T}(M) \to \text{Der}(M)$ is an isomorphism. Here $\mathcal{T}(M)$ is the smooth vectors fields on a manifold M, and $\text{Der}(M) = \text{Der}_{\mathbb{R}}(C^\infty(M)$ and $\mathscr{L}_\xi(f) = df(\xi)$.

Proof. Injectivity: suppose $\mathscr{L}_\xi = 0$, then $\forall f \in C^\infty(M)$, $df(\xi) = 0$.

Let $a \in M$, by extension lemma,

$$\begin{array}{ccc} C^\infty(M) & \to & C^\infty_{M,a} \\ f & \mapsto & [f]_a \end{array}$$

is surjective, so for any $[f]_a \in C^\infty_{M,a}$,

$$\mathscr{L}_a(\xi_a)([f]_a) = d_a f(\xi_a) = 0$$

Now using the point wise version of the isomorphism: $\mathscr{L}_a : T_aM \to \text{Der}_a(M) = \text{Der}(C^\infty_{M,a}, \mathbb{R})$, we see that $\xi_a = 0 \in T_aM$, so $\xi = 0$.

Surjectivity: Use locality of derivations and charts to reduce to the case $M = U \subseteq \mathbb{R}^n$ an open subset. In this case, use Taylor's theorem. (Next homework) $\square$

5 Intermezzo: Point-set topology of manifolds

5.1 Paracompactness

Definition 5.1. A topological space X is ***paracompact*** if it is Hausdorff and every open cover $\mathscr{U}$ of X admits a locally finite refinement $\mathscr{U}'$.

Locally finite means: $\forall x \in X$, there is a neighborhood which intersects only finitely many members of $\mathscr{U}'$.

Refinement means: every member of $\mathscr{U}'$ is a subset of a member of $\mathscr{U}$.

In the next statement, "manifold" means merely a topological space with a smooth structure. In particular, we do not demand that it is Hausdorff or second countable.

Theorem 5.2. *. Let M be a Hausdorff manifold. Then the following are equivalent:*

1. *Every connected component of M is second countable;*

2. *M is metrizable;*

3. *M is paracompact.*

This theorem tells us that paracompactness is a bit weaker than the property second countable. The following example is a manifold which is paracompact but not second countable.

Example 5.3. Let M be the disjoint union of uncountably many copies of $\mathbb{R}$ (with its standard smooth structure). Then M is paracompact but not second countable.

Sketch: If we consider the plane $\mathbb{R}^2$ but with a different topology: choose a line, and all the lines parallel to it, and say that intervals of these lines are the basic connected components. This makes M a one-dimensional manifolds.

Proof. Proof of the theorem:

$1 \Rightarrow 2$: Let C be a connected component of M, C is Hausdorff and second countable. M is also locally compact, hence regular. (Why?) For each $x \in M$, and each closed subset $F \subseteq M$ with $x \notin A$, there are open U, V with $x \in U$, $A \subseteq V$ and $U \cap V = \emptyset$.

Urysohn Metrization Theorem: Every regular space with a countable basis is metrizable.

So C is metrizable. Then M is metrizable. (Let d_C be the metric on C, let $\delta_C(x,y) = \frac{d_C(x,y)}{1+d_C(x,y)}$, then δ_C is a metric on C. Now for $x, y \in M$, set

$$d(x,y) = \begin{cases} \delta_C(x,y) & x, y \in C \text{ for some connected component } C \\ 1 & \text{o.w.} \end{cases}$$

Then d is a metric on M compatible with its topology.)

$2 \Rightarrow 3$: Every metrizable space is paracompact. (Why?)

$3 \Rightarrow 1$: May assume M is connected. Every point $x \in M$ has a second countable neighborhood. (what does it mean?) To get M is second countable, we show M is σ-compact, i.e. a countable union of compact subsets.

M has open cover $\mathscr{U}$ whose member U has compact closure. By paracompactness, may assume $\mathscr{U}$ is locally finite. Let $\emptyset \neq U_0 \subseteq U$, then $\bar{U}_0$ being compact, intersects only finitely many members $U_1, \cdots, U_{n_1}$ of $\mathscr{U}$. Likewise, $\bar{U}_0 \cup \cdots \cup \bar{U}_{n_1}$ intersects only finitely many $U_{n_1+1}, \cdots, U_{n_2}$ of $\mathscr{U}$. Continuing in this way we obtain a sequence $U_0, \cdots, U_{n_1}, \cdots$ of $\mathscr{U}$ with $\cup_{j=0}^{\infty} \bar{U}_j = \cup_{j=0}^{\infty} U_j$.

The collection $\{\bar{U}_j\}$ is locally finite, therefore $\cup_{j=0}^{\infty} \bar{U}_j$ is locally finite. (Why?) The union of a locally finite family of closed sets is closed, so this implies that $\cup_{j=0}^{\infty} \bar{U}_j$ is closed. (Why?)

But $\cup_{j=0}^{\infty} \bar{U}_j = \cup_{j=0}^{\infty} U_j$ is open, so $\cup_{j=0}^{\infty} \bar{U}_j = M$. So M is σ-compact. □

Example 5.4. Let ω_1 be the ***first uncountable ordinal***. The elements of ω_2 are all countable ordinals. The ***long ray*** is

$$R = \omega_1 \times [0, 1)$$

equipped with lexicographic order:

$$\begin{cases} (a,s) < (b,t) & a < b \\ (a,s) < (a,t) & s < t \end{cases}$$

and the order topology (subbasis consisting of all segments $R_{<x}$ and $R_{>x}$).

R is a connected 1-manifold with boundary point $(0,0)$. The long line is obtained by gluing two copies of R together along their boundary. The long line is not second countable, so it is not paracompact.

5.2 Partition of Unity

M is a manifold in the weak sense, i.e. a topological space equipped with a smooth structure.

Definition 5.5. Let $\mathscr{U} = \{U_i | i \in I\}$ be an open cover of M. A partition of unity subordinate to $\mathscr{U}$ is a family of smooth functions $\{\lambda_i : M \to [0,1] | i \in I\}$ satisfying

1. $\text{supp}(\lambda_i) \subseteq U_i$,
2. $\{\text{supp}(\lambda_i) | i \in I\}$ is a locally finite collection of subset of M,
3. $\sum\limits_{i \in I} \lambda_i = 1$.

Note the second condition implies for any $a \in M$, there is a neighborhood V of a such that only finitely many λ_i are nonzero on V, so the sum $\sum\limits_{i\in I}\lambda_i$ make sense, and for any $J \subseteq I$, the sum $\sum\limits_{i \in J}\lambda_i$ is well-defined and smooth.

Theorem 5.6. *Let M be paracompact and let $\mathscr{U}$ be an open cover. Then there exists a partition of unity subordinate to $\mathscr{U}$. In particular, this theorem holds when M is Hausdorff and second countable.*

Lemma 5.7. . *Let M be paracompact. Then*

1. *M is regular, and*
2. *every open cover $\mathscr{U} = \{U_i | i \in I\}$ of M has a shrinking: a locally finite refinement $\mathscr{V} = \{V_i | i \in I\}$, indexed by the same set, with $\bar{V}_i \subseteq U_i$.*

Proof. 1. Let $A \subseteq M$ be closed, $x \in M \backslash A$. For each $y \in A$, choose open sets U_y, V_y such that $x \in U_y$, $y \in V_y$, $U_y \cap V_y = \emptyset$. Then $\{U_y, V_y | y \in A\} \cup \{M\backslash(A \cup \{x\})\}$ is an open cover of M.

Let $\{W_i | i \in I\}$ be a locally finite refinement, $J = \{i \in I | W_i \cap V_y \neq \emptyset \text{ for some } y \in A\}$.

Choose open $U \ni x$ such that U intersects only finitely many W_j, say $W_{j_1}, \cdots, W_{j_n}$. Let $U' = U \backslash (\bar{W}_{j_1} \cup \cdots \cup \bar{W}_{j_n})$, $W = \cup_{j \in J} W_j$. Then $U' \cap W = \emptyset$, and $U' \ni x$ is open, W is an open neighborhood of A.

2. Let $A_i = M \backslash U_i$, $x \in U_i$. Use regularity to find a neighborhood $W_{i,x} \ni x$ such that $W_{i,x} \cap U_{A_i} = \emptyset$ for some open neighborhood U_{A_i} of A_i, so $\bar{W}_{i,x} \subseteq U_i$. Then $\mathscr{W} = \{W_{i,x} | i \in I, x \in M\}$ is an open cover of M and a refinement of $\mathscr{U}$. Choose a locally finite refinement $\mathscr{W}'$ of $\mathscr{W}$. $\mathscr{W}' = \{W'_j | j \in J\}$.

Choose functions $f : J \to I$ such that $\bar{W}'_j \subseteq W_{f(j),x}$ for some $x \in U_{f(j)}$, then $\bar{W}'_j \subseteq U_{f(j)}$.

For $i \in I$, set $V_i = \cup_{j \in f^{-1}(i)} W'_j \subseteq U_i$, so $\mathscr{V} = \{V_i | i \in I\}$ is a shrinking of $\mathscr{U}$. □

Proof. Now we can prove the theorem.

Choose a locally finite atlas $\{(V_j, \varphi_j) | j \in J\}$ of M, such that

1. $\mathscr{V} = \{V_j | j \in J\}$ refines $\mathscr{U}$ and
2. $\bar{V}_j$ is compact with $\varphi_j(V_j)$ is bounded in $\mathbb{R}^n$.

It suffices to define a partition of unity subordinate to $\mathscr{V}$. Choose a shrinking $\mathscr{W} = \{W_j | j \in J\}$ of $\mathscr{V}$. Then $\bar{W}_j \subseteq V_j$, so $\bar{W}_j$ is compact. Cover $\varphi_j(\bar{W}_j)$ with finitely many closed balls $B_{j,1}, \cdots, B_{j,k(j)} \subseteq \varphi_j(V_j)$.

Choose smooth functions $v_{j,k} : \mathbb{R}^n \to [0,1]$ such that $v_{j,k} > 0 \Leftrightarrow x \in \text{int}(B_{j,k})$. Let

$$v_j = \sum_{k=1}^{k(j)} v_{j,k} : \mathbb{R}^n \to [0, \infty)$$

then

$$v_j(x) \begin{cases} > 0 & x \in \varphi_j(W_j) \\ = 0 & x \in \mathbb{R}^n \backslash \cup_{k=1}^{k(j)} B_{j,k} \end{cases}$$

let

$$\mu_j = \begin{cases} v_j \circ \varphi_j(a) & a \in V_j \\ 0 & M \backslash \varphi^{-1}(\cup_{j=1}^{k(j)} B_{j,k}) \end{cases}$$

then μ_j is well-defined and smooth. $\mu_j > 0$ on W_j , $\mathrm{supp}(\mu_j) \subseteq V_j$. The collection $\{\mathrm{supp}(v_j) | j \in J\}$ is locally finite. We put

$$\lambda_j = \frac{\mu_j}{\sum\limits_{i \in J} \mu_i}$$

$\square$

5.2.1 Applications

Lemma 5.8. *(Smooth Urysohn's Lemma). Let A, B be disjoint closed subsets of M. Then there exists a smooth $f : M \to [0, 1]$ with$f = 0$ on A and $f = 1$ on B. Hence, there exist open sets $U \supseteq A$ and $V \supseteq B$ and $U \cap V = \emptyset$.*

Proof. Let $U' = M \backslash A$, $V' = V \backslash B$, then $\{U', V'\}$ is an open cover of M. Let $\{f, g\}$ be a partition of unity subordinate to $\{U', V'\}$. Then

$$\begin{array}{lcl} \mathrm{supp}(f) \subseteq U' & \Rightarrow & f = 0 \text{ on } A \\ \mathrm{supp}(g) \subseteq V' & \Rightarrow & g = 0 \text{ on } B \end{array}$$

Since $f + g = 1$, we have $f = 1 - g = 1$ on B.

Now take $U = f^{-1}([0, \varepsilon))$, $V = f^{-1}((1 - \varepsilon, 1])$ for small enough ε. $\square$

Definition 5.9. Let $A \subseteq M$ be closed. A ***germ*** at A is an equivalence class $[f]_A$ of smooth functions where $f \in C^\infty(U)$ for some $U \supseteq A$ open, and $f, g \in C^\infty(U)$ are equivalent, $f \sim_A g$, if $f = g$ for some $W \supseteq A$ open, $W \subseteq U \cap V$.

Let $C^\infty_{M,A}$be the algebra of germs $[f]_A$ at A.

If M is paracompact, any $[f] \in C^\infty_{M,A}$ is the germ of a globally defined $f \in C^\infty(M)$.

Lemma 5.10. *(Generalized Extension Lemma) Let $A \subseteq M$ be closed and $U \subseteq M$ an open neighborhood of A. Let $f : A \to \mathbb{R}$ be smooth, then there exists a smooth $\tilde{f} : M \to \mathbb{R}$ satisfying $\tilde{f}|_A = f$, supp$(\tilde{f}) \subseteq U$.*

Recall: "$f : A \to \mathbb{R}$ is smooth" means $\forall a \in A$, there exists an open neighborhood V_a of a and $f_a \in C^\infty(V_a)$ such that $f_a|_{A \cap V_a} = f$.

Proof. For each $a \in A$, choose V_a, f_a as above. Without lose of generality, we may assume $V_a \subseteq U$.

Let $\mathscr{U} = \{V_a | a \in A\} \cup \{M \backslash A\}$ an open cover of M. Let $\{\lambda_a | a \in A\} \cup \{\lambda_0\}$ be a partition of 1 subordinate to $\mathscr{U}$. Then $\lambda_a, \lambda_0 : M \to [0, 1]$ is smooth and the collection $\{\mathrm{supp}(\lambda_a), \mathrm{supp}(\lambda_0) | a \in A\}$ is locally finite, and $\mathrm{supp}(\lambda_a) \subseteq V_a$, $\mathrm{supp}(\lambda_0) \subseteq M \backslash A$. Also $\sum\limits_{a \in A} \lambda_a + \lambda_0 = 1$.

Define $\tilde{f}_a : M \to \mathbb{R}$ by

$$\tilde{f}_a(x) = \begin{cases} \lambda_a(x) f_a(x) & x \in V_a \\ 0 & x \in M \backslash \mathrm{supp}(\lambda_a) \end{cases}$$

, and $\tilde{f}_0 = 0$. Then$\tilde{f}_a$ and $\tilde{f}_0$ are smooth, put

$$\tilde{f} = \sum_{a \in A} \tilde{f}_a$$

then for $x \in A$, $\lambda_0(x) = 0$, so

$$\tilde{f}(x) = \sum_{a \in A} \tilde{f}_a(x) + \tilde{f}_0(x) = (\sum_{a \in A} \lambda_a + \lambda_0) f(x) = f(x)$$

on A. Also $\mathrm{supp}(\tilde{f}) \subseteq \cup_{a \in A} \mathrm{supp}(\tilde{f}_a) \subseteq \cup_{a \in A} \mathrm{supp}(\lambda_a) \subseteq \cup_{a \in A} V_a \subseteq U$. $\square$

Corollary 5.11. *Let $A \subseteq M$ be closed, let $C^\infty_{M,A}$ be the algebra of germs of smooth functions at A, the the restriction map*

$$\begin{array}{ccc} C^\infty(M) & \to & C^\infty_{M,A} \\ f & \mapsto & [f]_A \end{array}$$

is surjective.

Proof. Let $[g]_A \in C^\infty_{M,A}$ be represented by a smooth function $g : U \to \mathbb{R}$ where $U \supseteq A$ is open. Then $M \backslash U$ and A are nonintersecting closed subsets in M, by Urysohn Lemma, there is an open $V \supseteq A$ and $\bar{V} \subseteq U$. (Why need V?)

By Generalized Extension Lemma, the constant function 1 on $\bar{V}$extends to a smooth function

$$\varrho : M \to \mathbb{R}$$

with $\operatorname{supp}(\varrho) \subseteq U$. Put

$$f = \begin{cases} \varrho g & x \in U \\ 0 & x \in M \backslash \operatorname{supp}(\varrho) \end{cases}$$

then f is well-defined and smooth, and $f = g$ on V, so $[f]_A = [g]_A$. $\square$

6 Riemannian Metric

Definition 6.1. A ***Riemannian metric*** on M is a collection $\mathcal{G} = \{g_a | a \in M\}$ where for each $a \in M$, $g_a : T_aM \times T_aM \to \mathbb{R}$ is an inner product (positive definite, symmetric bilinear form) with the property that for each chart (U, φ), the map $\tilde{g} : \varphi(U) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

$$\tilde{g}_x(h,k) = g_{\varphi^{-1}(x)}((d_{\varphi^{-1}(x)}\varphi)^{-1}(h), (d_{\varphi^{-1}(x)}\varphi)^{-1}(k))$$

is smooth.

For $x \in \varphi(U)$, let $A_x = (\tilde{g}_x(e_i, e_j))_{n\times n}$ be the matrix of the inner product $\tilde{g}_x$ on $\mathbb{R}^n$ with respect to the standard basis. Then $\tilde{g}_x(h,k) = h^T A_x k$.

So $\tilde{g}$ is smooth $\Leftrightarrow$ the map $A : \varphi(U) \to M_n(\mathbb{R}), x \mapsto A_x$ is smooth.

Proposition 6.2. *M has a Riemannian metric.*

Proof. Choose atlas $\{U_i.\varphi_i | i \in I\}$ on M. For each $i \in I$, $a \in U_i$, $v, w \in T_aM$, define

$$(g_i)_a(v,w) =< d_a\varphi_i(v), d_a\varphi_i(w) >= d_a\varphi_i(v)^T d_a\varphi_i(w)$$

the standard inner product in $\mathbb{R}^n$. Matrix of $\tilde{g}_i$ is I_n, so on U_i, g_i is a smooth Riemannian metric.

Choose partition of 1 subordinate to $\mathscr{U} = \{U_i | i \in I\}$, put

$$(h_i)_a = \begin{cases} \lambda_i(a)(g_i)_a & a \in U_i \\ 0 & a \in M \backslash \operatorname{supp}(\lambda_i) \end{cases}$$

then h_i is a smooth, symmetric, positive semi-definite bilinear form on TM.

Let

$$g = \sum_{i \in I} h_i,$$

then g is smooth, symmetric, positive semi-definite bilinear form.

Let $a \in M$, $v \in T_aM$, then

$$g_a(v,v) = \sum_{i \in I, a \in \operatorname{supp}(\lambda_i)} \lambda_i(a)(g_i)_a(v,v) \geq 0.$$

Since $(g_i)_a(v,v) > 0$ for at least one i, we get $g_a(v,v) > 0$. So g is positive definite. (Why?) $\square$

Fact 6.3. *Every Riemannian manifold, i.e. a Hausdorff manifold M equipped with a Riemannian metric g, is paracompact.*

Reason: Without lose of generality, assume M is connected.

Let $\gamma : [0,1] \to M$ be a smooth path, the length of γ is

$$length(\gamma) = \int_0^1 \|\gamma'(t)\| dt.$$

Define for $x, y \in M$,

$$d(x,y) = \inf\{length(\gamma) | \gamma(0) = x, \gamma(1) = y\},$$

then d is a metric on M compatible with the topology. (Why?)

So M is metrizable, hence paracompact.

7 Lie Groups

7.1 Lie Groups

Definition 7.1. A ***Lie group*** is a set G equipped with a smooth structure and a group structure which are compatible in the sense that multiplication

$$\begin{array}{rccc} \mu = \mu_G : & G \times G & \to & G \\ & (g,h) & \mapsto & gh \end{array}$$

and inversion

$$\begin{array}{rccc} \tau = \tau_G : & G & \to & G \\ & g & \mapsto & g^{-1} \end{array}$$

are smooth. The identity (or unit) of G is denoted by 1 or 1_G.

Example 7.2. Let $\mathbb{F}$ be a finite-dimensional associative division algebra with unit over $\mathbb{R}$, then $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ by Frobenius theorem. Let $M(m \times n, \mathbb{F})$ = set of $m \times n$ matrices with entries in $\mathbb{F}$. In particular, $M(n, \mathbb{F}) = M(n \times n, \mathbb{F})$.

Under matrix addition and multiplication, $M(n, \mathbb{F})$ is a ring. The general linear group in n-dimension over $\mathbb{F}$ is its group of units. $GL(n, \mathbb{F}) = \{X \in M(n, \mathbb{F}) | X \text{ is invertible}\}$.

$M(n, \mathbb{F})$ is a vector space over $\mathbb{R}$ of dimension $n^2 \dim(\mathbb{F})$.

X is invertible $\Leftrightarrow \det(X) \neq 0$.

So $GL(n, \mathbb{F})$ is an open submanifold of $M(n, \mathbb{F})$.

The matrix multiplication

$$M(n, \mathbb{F}) \times M(n, \mathbb{F}) \to M(n, \mathbb{F})$$

is bilinear and therefore smooth.

The inversion

$$\tau : GL(n, \mathbb{F}) \to GL(n, \mathbb{F})$$

is smooth because $X^{-1} = \frac{1}{\det(X)} adj(X)$ by Cramer's Rule.

For $n = 1$, $GL(1, \mathbb{F}) = \mathbb{F}^* = \mathbb{F} \backslash \{0\}$.

Example 7.3. $\mathbb{F}^n$ equipped with addition is a Lie group of dimension $n \dim_{\mathbb{R}}(\mathbb{F})$.

Let G be a fixed Lie group.

For $a \in G$, define $L_a, R_a, Ad_a : G \to G$ by $L_a(g) = ag, R_a(g) = ga, Ad_a(g) = aga^{-1}$.

$Ad_a(g) = L_a \circ R_{a^{-1}} = R_{a^{-1}} \circ L_a$.

Let $i_a(g) = (a, g)$. Then

$$\begin{array}{ccc} G & \xrightarrow{i_a} & G \times G \\ & L_a \searrow & \downarrow \mu \\ & & G \end{array}$$

commutes, i.e. $L_a = \mu \circ i_a$, so L_a is smooth. Also $L_a^{-1} = L_{a^{-1}}$,so L_a is a diffeomorphism.

Same for R_a and Ad_a.

For subsets U, V of G, define

$$aU = L_a(U), Ua = R_a(U), aUa^{-1} = Ad_a(U), U^{-1} = \tau(U), UV = \mu(U, V).$$

If U, V are open, so are the above sets. Also define *the identity component of* G, G_0= connected component of 1.

Lemma 7.4. *Let G be a Lie group.*

1. *Suppose G is connected, let U be an open neighborhood of 1, then U generates G. Hence G is second-countable.*
2. *G_0 is an open and closed subgroup of G and the connected components are the (left) cosets aG_0.*
3. *G is a pure manifold and a paracompact topological space.*

Proof. G is a Lie group.

1. Let H be the subgroup generated by U, i.e.

$$\begin{array}{rcl} H & = & \cap\{K|K \text{ is a subgroup of } G \text{ containing } U\} \\ & = & \bigcup\limits_{k\in\mathbb{N}} (U \cdot U^{-1})^{(k)} \end{array}$$

where $U \cdot U^{-1} = \mu(U \times U^{-1})$, $V^{(k)} = \underbrace{V \cdot \cdots \cdot V}_{k \text{ times}}$, so H is open, and $gH = L_g(H)$ is open. So $H = G \backslash \bigcup\limits_{g \notin H} gH$ is closed. Therefore $H = G$. Taking U to be second-countable, we see $G = \bigcup\limits_{k} (U \cdot U^{-1})^{(k)}$ is second-countable.

2. G_0 is open and closed connected component of G. $\mu(G_0 \times G_0) \subseteq G_0$, $\tau(G_0) \subseteq G_0$, so G_0 is open and closed subgroup of G, and every coset gG_0 is open and closed connected submanifold, hence a connected component of G.

3. $\dim(gG_0) = \dim(L_g(G_0)) = \dim(G_0)$, so G is pure. Let $\Delta_G = \{(g, g) \in G \times G | g \in G\}$ be the diagonal, then $\Delta_G = f^{-1}(1)$ where

$$\begin{array}{rccc} f: & G \times G & \to & G \\ & (g, h) & \mapsto & gh^{-1} \end{array}.$$

G is a manifold, so $\{1\}$ is closed. f is continuous, so Δ_G is closed, hence G is Hausdorff. Combined 1&2, G is paracompact.

□

Let G, H be Lie groups, a morphism $f : G \to H$ is a map that is smooth and a group homomorphism.

Example 7.5. Let G be a Lie group,

1. For all $g \in G$, $Ad_G : G \to G$ is an automorphism of G, called the inner automorphism defined by g.
2. Morphisms $G \to GL(n, \mathbb{F})$ are called representation of G over $\mathbb{F}$.
3. $\det_{\mathbb{R}} : GL(n, \mathbb{R}) \to \mathbb{R}^\times$, $\det_{\mathbb{C}} : GL(n, \mathbb{C}) \to \mathbb{C}^\times$, $\det_{\mathbb{H}} : GL(n, \mathbb{H}) \to \mathbb{R}^\times$ are morphisms.

7.2 Lie Subgroup

Definition 7.6. A Lie subgroup (or embedded Lie subgroup) of G is a subset that is both a subgroup and an (embedding) submanifold.

Lemma 7.7. *Let H be a Lie subgroup of G, then the smooth structure and group structure on H are compatible, hence H is a Lie group.*

Proof. The multiplication and inversion on H

$$\mu_H : H \times H \to H, \tau_H : H \to H$$

are obtained by restricting domain and codomian of μ_G and τ_G , hence by lemma below μ_H and τ_H are smooth. □

Lemma 7.8. *Let M, N be manifolds, $f : M \to N$ smooth. Let $A \subseteq M$, $B \subseteq N$ be submanifolds. Suppose $f(A) \subseteq B$, let $g : A \to B$ be the restriction of f, then g is smooth.*

Proof. Restrict domain: $f|_A : A \to N$ can be regarded as the composition of two smooth functions $f|_A = f \circ i_A$,

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[dr, "f|_A"'] & M \arrow[d, "f"] \\ & N \end{tikzcd}$$

Restrict codomain $g : A \to B$ an be regarded as the composition $r_B \circ f|_A$

$$\begin{tikzcd} A \arrow[r, "f|_A"] \arrow[dr, "g"'] & N \arrow[d, "r_B"] \\ & B \end{tikzcd}$$

If we have a smooth global retraction $r_B : N \to B$ of $i_B : B \to N$, then $g = r_B \circ f|_A$ is smooth. We may not have global retractions, but we have them locally. □

Example 7.9. Lie subgroup

1. G_0 is a Lie subgroup of any Lie group G.
2. $\mathbb{S}^1 = \{z \in \mathbb{C} | |z| = 1\}$ is a Lie subgroup of $\mathbb{C}^\times = GL(n, \mathbb{C})$.
3. $\mathbb{S}^3$ is a Lie subgroup of $\mathbb{H}^\times = GL(1, \mathbb{H})$. (Homework)

Example 7.10. Non-example
Let $G = \mathbb{S}^2 \times \mathbb{S}^2$, $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, the map

$$\begin{array}{rccc} f : & \mathbb{R} & \to & G \\ & t & \mapsto & (e^{it}, e^{i\alpha t}) \end{array}$$

is a morphism of Lie groups. It's also an injective immersion with dense image. $H = f(\mathbb{R})$ is not a Lie subgroup. We say H is an ***immersed Lie subgroup***.

Lemma 7.11. *Let H be a Lie subgroup of G, then H is a closed subset of G.*

Proof. H is a submanifold of G, so locally closed.
Choose open neighborhood U of $1 \in H$ such that $U \cap H$ is closed in U. After replacing U with UU^{-1}, may assume $U = U^{-1}$.
Let $x \in \bar{H}$, the open neighborhood $xU = L_x(U)$ intersects H. Let $y \in xU \cap H$, then $x \in yU^{-1} = yU$ and $y(U \cap H) = L_y(U \cap H) = yU \cap H$ is closed in yU.
But $x \in yU \cap \bar{H}$, and $yU \cap \bar{H}$ is the closure of $yU \cap H$ in yU, so $x \in yU \cap H$, hence $x \in H$. □

Fact 7.12. *(converse of the above lemma) If H is a subgroup of G, and is a closed subset of G, then H is a Lie subgroup.*

7.3 Vector Fields on Lie groups

The multiplication

$$\begin{array}{rccc} \mu : & G \times G & \to & G \\ & (g, h) & \mapsto & gh9 \end{array}$$

is smooth, so the partial tangent map with respect to the second factor

$$\begin{array}{ccc} G \times TG & \to & TG \\ (g, \xi) & \mapsto & TL_g(\xi) \end{array}$$

is also smooth. Put $\mathfrak{g} = T_1 G$, the tangent space at the identity. The restriction to

$$\begin{array}{ccc} G \times \mathfrak{g} & \to & TG \\ (g, \xi) & \mapsto & T_1 L_g(\xi) \in T_g G \end{array}$$

is also smooth. Hence for each $\xi \in \mathfrak{g}$, the map

$$\begin{array}{rccc} \xi_L : & G & \to & TG \\ & g & \mapsto & T_1 L_g(\xi) \end{array}$$

is smooth and $\xi_L \in \mathcal{T}(G)$. Hence, if $\xi_1, \cdots, \xi_n$ is a basis of $\mathfrak{g}$, then $(\xi_1)_L(g), \cdots, (\xi_n)_L(g)$ is a basis of $T_g G$ for any $g \in G$. So we have shown the following.

Lemma 7.13. *Every Lie group is parallelizable.*

Similarly, we have $\xi_R \in \mathcal{T}(G)$ given by $\xi_R(g) = T_1 R_g(\xi)$.

Lemma 7.14. *ξ_L is complete for every $\xi \in \mathfrak{g}$.*

Proof. Let $\gamma : (-\varepsilon, \varepsilon) \to G$ be a trajectory of ξ_L starting at 1_G.

$$\begin{cases} \gamma'(t) = \xi_L(\gamma(t)) = T_1 L_{\gamma(t)}(\xi) \\ \gamma(0) = 1_G \end{cases}$$

Let $g \in G$ and $\delta = L_g \circ \gamma : (-\varepsilon, \varepsilon) \to G$, then $\delta(t) = g\gamma(t)$, so $\delta(0) = g 1_G = g$ and

$$\begin{array}{rcc} \delta'(t) & = & T_{\gamma(t)} L_g(\gamma'(t)) \\ & = & T_{\gamma(t)} L_g(\xi_L(\gamma(t)) \\ & = & T_{\gamma(t)} L_g \circ T_1 L_{\gamma(t)}(\xi) \\ & = & T_1(L_g \circ L_{\gamma(t)})(\xi) \\ & = & T_1 L_{g\gamma(t)}(\xi) \\ & = & \xi_L(g\gamma(t)) \\ & = & \xi_L(\delta(t)). \end{array}$$

So δ is a trajectory of ξ_L starting at g. So ξ_L is complete by the Uniform Time Lemma. □

Definition 7.15. For $\xi \in \mathfrak{g} = T_1 G$, let $\gamma : \mathbb{R} \to G$ be the maximal trajectory of $\xi_L$starting at 1_G. Define $\exp(\xi) = \gamma(1) \in G$. We have a map $\exp : \mathfrak{g} \to G$.

Example 7.16. Let $G = GL(n, \mathbb{F})$ where $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, then $\mathfrak{g} = T_1 G = M(n, \mathbb{F})$. Let $A \in \mathfrak{g}$. To find the trajectory A_L, solve the ODE

$$\begin{cases} \gamma'(t) = L_{\gamma(t)} = \gamma(t) A \\ \gamma(0) = I \end{cases}$$

with solution $\gamma(t) = e^{tA}$. So $\exp(A) = e^A$.

Notice that A_R has the same trajectory starting at 1.

7.4 Actions of Lie groups on manifolds

Definition 7.17. Let M be a manifold and G a Lie group. A (left) action of G on M is a smooth map

$$\begin{array}{rccc} \theta : & G \times M & \to & M \\ & (g, a) & \mapsto & g \cdot a \end{array}$$

with the properties

1. $(gh)a = g(ha)$, and
2. $1 \cdot a = a$

for $g.h \in G$ and $a \in M$. A G-manifold is a manifold M equipped with a G-action.

Remark 7.18. (Notation) Write

$$\theta(g, a) = \theta_g(a) = \theta^a(g) = ga = g \cdot a$$

then $\theta_{gh} = \theta_g \circ \theta_h$, $\theta_1 = \mathrm{id}_M$, or $(gh)a = g(ha)$, $1 \cdot a = a$

For each $g \in G$, θ_g is invertible with inverse $(\theta_g)^{-1} = \theta_{g^{-1}}$. So each θ_g is a diffeomorphism and $g \mapsto \theta_g$ is a group homomorphism $G \to \mathrm{Diff}(M)$.

The set

$$Ga = G \cdot a = \{ga | g \in G\} = \theta^a(G)$$

is the ***G-orbit*** of $a \in M$.

The action of G on M is ***transitive*** if $M = Ga$ for some (and hence every) $a \in M$.

The set

$$G_a = \{g \in G | ga = a\} = (\theta^a)^{-1}(a)$$

is a closed subgroup of G, called the ***isotropy*** or ***stabilizer subgroup*** of a.

The action is ***free at*** a if $G_a = \{1\}$, and ***free*** if it is free at all $a \in M$.

Example 7.19. Let $M = \mathbb{S}^2$, $G = \mathbb{S}^1$, then G spins M about the vertical axis with uniform angular velocity 1 and the action is free at all $a \in M \backslash \{N, S\}$.

$$G_a = \begin{cases} 1 & a \notin \{N, S\} \\ G & a \in \{N, S\} \end{cases}$$

Example 7.20. More examples.

1. A complete vector field $\xi \in \mathcal{T}(M)$ has a flow $\theta : \mathbb{R} \times M \to M$ which, by the flow law, defines an $\mathbb{R}$-action on M.

2. $L : G \times G \to G$ given by $L(g, h) = gh$ is the ***left-translation action*** of G on itself.

3. $R : G \times G \to G$ given by $R(g, h) = hg$ is the ***right-translation action*** of G on itself. It is not a (left) action, but a right-action: $(gh)a = h(ga)$. However $(g, h) \mapsto h^{-1}g$ is a left-action of G on itself.

4. $\mathrm{Ad} : G \times G \to G$ given by $\mathrm{Ad}(g, h) = ghg^{-1}$ is the ***adjoint*** or ***conjugation action*** of G on itself.

5. Let H be another Lie group and $f : G \to H$ a Lie group homomorphism. Then the map $G \times H \to H$ given by $(g, h) \mapsto f(g)h$ is a G-action on H.

6. A representation of G over $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ is a Lie group homomorphism $G \to GL(n, \mathbb{F})$. This defines an action of G on $M = \mathbb{F}^n$ with the property that each $\theta_g : M \to M$ is $\mathbb{F}$-linear.

Remark 7.21. For $\mathbb{F} = \mathbb{H}$, we regard $\mathbb{H}^n$ as a right vector space. For $q \in \mathbb{H}$, $x \in \mathbb{H}^n$, we put

$$xq = \begin{pmatrix} x_1 q \\ \cdots \\ x_n q \end{pmatrix}$$

Matrix $A \in M(n, \mathbb{F})$ act on $\mathbb{F}^n$ be left multiplication:

$$Ax = \begin{pmatrix} a_{11}x_1 + \cdot + a_{1n}x_n \\ \cdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n \end{pmatrix}$$

Matrix multiplication is not left-linear:

$$A(qx) \neq q(Ax)$$

because multiplication is not commutative. However, it is right-linear:

$$A(xq) = (Ax)q$$

because $\mathbb{H}$ is associative.

Lemma 7.22. *Let $a \in M$. Then*

1. *$\theta^a : G \to M$ has constant rank.*

2. *The stabilizer G_a is a Lie subgroup of G with $T_1 G_a = \ker(T_1\theta^a)$.*

3. *If the action is free at a, then θ^a is an immersion and Ga is an immersed submanifold.*

Proof. Through some notational trickery,

$$(gh)a = g(ha) \Leftrightarrow \theta^a(gh) = g\theta^a(h) \Leftrightarrow \theta^a(L_g(h)) = \theta_g(\theta^a(h)).$$

So the following diagram commutes, expressing that θ a is *G-equivariant*.

$$\begin{array}{ccc} G & \xrightarrow{\theta^a} & M \\ L_g \downarrow & & \downarrow \theta_g \\ G & \xrightarrow{\theta^a} & M \end{array}$$

1. Let's compare the rank of θ a at two points. Take derivatives of both sides of this equation, and use the chain rule. We get the following commutative diagram,

$$\begin{array}{ccc} T_1G & \xrightarrow{T_1\theta^a} & T_aM \\ T_1L_g \downarrow & & \downarrow T_a\theta_g \\ T_gG & \xrightarrow{T_g\theta^a} & T_{ga}M \end{array}$$

   where L_g is a diffeomorphism of G, and θ_g is a diffeomorphism of M, so T_1L_g and $T_a\theta_g$ are linear isomorphisms. Hence, $T_1\theta^a$ and $T_g\theta^a$ have the same rank for each $g \in G$. So θ^a has constant rank at each $g \in G$.

2. Combine part (1) and Constant Rank Theorem to see that $G_a = (\theta^a)^{-1}(a)$ is a submanifold of G with tangent space $T_1G_a = \ker(T_1\theta^a)$.

3. If the action is free at a, then θ^a is injective. Since it's also constant rank, $T_g\theta^a$ is injective for all $g \in G$. So θ^a is an injective immersion, therefore $\theta^a(G) = Ga$ is an immersed submanifold.

□

Remark 7.23. In the lemma, the conclusion still holds if the action is not free at a. In fact, the orbit $Ga = \theta^a(G)$ is always an immersed submanifold.

Corollary 7.24. *Let H be a Lie subgroup and $F : G \to H$ a Lie group homomorphism, then*

1. *f has constant rank.*

2. *The subgroup $N = \ker(f) = f^{-1}(1_H)$ is a (normal) Lie subgroup of G and $T_1N = \ker(T_1f)$.*

3. *If f is injective, then $g(G)$ is an immersed Lie subgroup of H.*

Proof. Apply the lemma above to the action of G on H given by $g \cdot h = f(g)h$. □

7.5 Classical Lie Groups

Example 7.25. Lie groups:

1. Let $G = GL(n,\mathbb{F})$ where $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and

$$H = Z(\mathbb{F})^\times = \begin{cases} \mathbb{R}^\times & \mathbb{F} = \mathbb{R}, \\ \mathbb{C}^\times & \mathbb{F} = \mathbb{C}, \\ \mathbb{R}^\times & \mathbb{F} = \mathbb{H}. \end{cases}$$

   Let $f = \det : G \to H$. The special linear group

$$SL(n,\mathbb{F}) = \ker(\det) = \{X \in M(n,\mathbb{F}) | \det(X) = 1\}$$

   is a Lie subgroup of $GL(n,\mathbb{F})$.

2. Let $G = GL(n,\mathbb{F})$, $M = M(n,\mathbb{F})$,
$$\begin{array}{rccc} \theta: & G\times M & \to & M \\ & (g,X) & \mapsto & gXg^T \end{array}$$
where $g^\top$ is the transpose of $g \in G$. Then θ is an action satisfies
$$\begin{cases} \theta(gh,X) = ghX(gh)^T = ghXh^Tg^T = \theta(g,\theta(h,X)) \\ \theta(1,X) = X \end{cases}$$
if $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Stabilizer of $X = I$ is $O(n,\mathbb{F}) = \{g \in G | gg^T = I\}$ the orthogonal group, which is a Lie group.

3. The tangent space of $O(n,\mathbb{F})$ at identity is
$$T_IO(n,\mathbb{F}) = \{X \in M(n,\mathbb{F}) | X^T = -X\}.$$
The reason is as follows: the action θ at identity is
$$\begin{array}{rccc} \theta^I: & G & \to & M \\ & g & \mapsto & gg^T \end{array}$$
and we want to find the stabilizer of the derivative of θ^I at I.
$$\begin{array}{rccc} T_1\theta^I: & T_IG & \to & T_IM \\ & \wr\!\downarrow & & \wr\!\downarrow \\ & M & & M \end{array}$$
We have that if H is a small matrix,
$$\begin{array}{rcc} \theta^I(I+H) & = & (I+H)(I+H)^T \\ & = & I+H+H^T+HH^T \\ & = & \theta^I(I)+H+H^T+HH^T \end{array}$$
so $D\theta^I(I)H = H + H^T$.

4. To get a version of the orthogonal group over any $\mathbb{F}$, consider $\theta(g,X) = gXg^*$ where $g^* = \bar{g}^T$ is the conjugate transpose. Then
$$G_I = U(n,\mathbb{F}) = \{g \in G | gg^* = I\}$$
is the unitary group over $\mathbb{F}$.
$$\begin{array}{rcc} U(n,\mathbb{R}) & = & O(n,\mathbb{R}) \\ U(n,\mathbb{C}) & = & O(n,\mathbb{C}) \\ U(n,\mathbb{H}) & = & \mathrm{Sp}(n) \end{array}$$
The last of these, $U(n,\mathbb{H})$ is often written $\mathrm{Sp}(n)$, called the ***compact symplectic group***.

7.6 Smooth maps on Vector Fields (Continuation)

Let M, N be manifolds and $f : M \to N$ smooth. Then f induces $Tf : TM \to TN$ on the tangent spaces. Does f induce a map $\mathcal{T}(M) \to \mathcal{T}(N)$?

Example 7.26. $M = \mathbb{R}$,$N = \mathbb{R}^2$,
$$f(t) = \begin{pmatrix} \cos t \\ \cos t \sin t \end{pmatrix}, \xi = \frac{\partial}{\partial t}$$
there is no vector field η on N with the property that $\eta(f(t)) = T_af(\xi(t))$.

Definition 7.27. $\xi \in \mathcal{T}(M)$ and $\eta \in \mathcal{T}(N)$ are f-related, $\xi \sim_f \eta$, if $\eta_{f(a)} = T_af(\xi_a)$ for all $a \in M$, that is,
$$\eta \circ f = Tf \circ \xi$$
and the following diagram commutes.
$$\begin{array}{ccc} TM & \xrightarrow{Tf} & TN \\ \xi \uparrow\downarrow \pi_M & & \pi_N \downarrow\uparrow \eta \\ M & \xrightarrow{f} & N \end{array}$$

Lemma 7.28. *The following are equivalent for $\xi \in \mathcal{T}(M), \eta \in \mathcal{T}(N)$:*

1. *$\xi \sim_f \eta$.*
2. *$\xi(g \circ f) = \eta(g) \circ f$ for all $g \in C^\infty(N)$.*
3. *for every trajectory γ of ξ, $f \circ \gamma$ is a trajectory of η.*

Proof. In 2 we identify $\xi \in \mathcal{T}(M)$ with $\mathcal{L}_\xi \in \mathrm{Der}(M)$ given by $\xi(h) = \mathcal{L}_\xi(h) = dh(\xi)$ for all $h \in C^\infty(M)$. Write $g \circ f = f^*(g)$, then assertion 2 reads $\xi \circ f^* = f^* \circ \eta$.

1. (1) $\Leftrightarrow$ (2) If $a \in M$,

$$\begin{array}{rclcl} \xi(g \circ f)(a) & = & d_a(g \circ f)(\xi_a) & = & d_{f(a)}g(T_a f(\xi_a)) \\ \eta(g) \circ f(a) & = & d_a g(\eta_{f(a)}) & & \end{array}$$

   so the above two equations are equal for all $g \in C^\infty(N)$ if and only if

$$\alpha(T_a f(\xi_a)) = \alpha(\eta_{f(a)})$$

   for all $\alpha \in T^*_{f(a)}N$, if and only if

$$T_a f(\xi_a) = \eta_{f(a)}$$

   if and only if

$$\xi \sim_f \eta.$$

2. (1) $\Rightarrow$ (3) Let $\delta = f(\gamma(t))$. Then

$$\begin{array}{rcl} \delta'(t) & = & (f \circ \gamma)'(t) \\ & = & T_{\gamma(t)} f(\gamma'(t)) \\ & = & T_{\gamma(t)} f(\xi_{\gamma(t)}) \\ & = & \eta_{f(\gamma(t))} \\ & = & \eta_{\delta(t)} \end{array}$$

   so δ is a trajectory of η.

3. (3) $\Rightarrow$ (1) Let $a = \gamma(0)$, $\delta = f \circ \gamma$ is a trajectory of η starting at $f(a)$, so

$$\eta_{f(a)} = \delta'(0) = (f \circ \gamma)'(0) = T_a f(\gamma'(0)) = T_a f(\xi_a)$$

   i.e. $\xi \sim_f \eta$.

$\square$

Corollary 7.29. *Suppose $\xi \sim_f \eta$. Let $\theta_\xi : \mathcal{D}_\xi \to M$, $\theta_\eta : \mathcal{D}_\eta \to N$ be the flow of ξ, respectively, η. For $t \in \mathbb{R}$, let*

$$\begin{array}{rcl} M_t & = & \mathcal{D}_\xi \cap \{t\} \times M \\ N_t & = & \mathcal{D}_\eta \cap \{t\} \times N \end{array}$$

then $f(M_t) \subseteq N_t$ and the diagram

$$\begin{array}{ccc} M_t & \xrightarrow{f} & N_t \\ \theta_{\xi,t} \downarrow & & \downarrow \theta_{\eta,t} \\ M_{-t} & \xrightarrow{f} & N_{-t} \end{array}$$

commutes.

Equivalently, f maps trajectories of ξ to trajectories of η, or the following commutes.

$$\begin{array}{ccc} C^\infty(M) & \xleftarrow{f^*} & C^\infty(N) \\ \mathcal{L}_\xi \downarrow & & \downarrow \mathcal{L}_\eta \\ C^\infty(M) & \xleftarrow{f^*} & C^\infty(N) \end{array}$$

Problem. Given ξ, when can we guarantee the existence of a related η?

Lemma 7.30. *If f is a diffeomorphism, for every $\xi \in \mathcal{T}(M)$, there is a unique $\eta = f_*(\xi) = Tf \circ \xi \circ f^{-1}$which is f-related to ξ.*

Conversely,

Lemma 7.31. *If f is a local diffeomorphism, i.e. $T_x f$ is invertible for all x, then for every $\eta \in \mathcal{T}(N)$ there id a unique f-related $\xi = f^*(\eta) \in \mathcal{T}(M)$, namely $\xi_x = (T_x f)^{-1}(\eta_{f(x)})$. This is smooth because f locally has smooth inverse.*

Now suppose f is an embedding.

Given any $\eta \in \mathcal{T}(N)$, when is there an f-related $\xi \in \mathcal{T}(M)$?

Identify M with the submanifold $A = f(M)$, Let Then we want $\xi_a = \eta_a$ for $a \in A$. So $\eta_a \in T_a A$.

We say η is ***tangent*** to A if $\eta_a \in T_a A$ for all $a \in A$.

Lemma 7.32. *For each $\eta \in \mathcal{T}(N)$ which is tangent to A, there is a unique $\xi = \eta|_A$ which is f-related to η.*

7.7 Lie algebras and the Lie bracket

Let k be a commutative ring (e.g. $k = \mathbb{R}$) and let A be a k-module (e.g. $A = C^\infty(M)$). Let $\mathrm{End}_k(A)$ be the k-algebra of k-linear endomorphisms of A, that is, k-linear maps $f : A \to A$.

Multiplication is composition: $f \cdot g = f \circ g$.

Commutation: $[f, g] = fg - gf$.

Definition 7.33. A Lie algebra over k is a k-module L with a k-bilinear product

$$[\cdot,\cdot] : L \times L \to L$$

satisfying anti-symmetry

$$[x, x] = 0$$

and Jacobi identity

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$$

Remark 7.34. Note that the definition of a Lie algebra does not demand that L has a unit, nor does it demand associativity. Instead of associativity, we have the Jacobi identity, which is an infinitesimal form of associativity.

Lemma 7.35. *$\mathrm{End}_k(A)$ is a Lie algebra over k.*

Example 7.36. Let $L = M(n, k)$, the $n \times n$ matrices over k with $[x, y] = xy - yx$.

Example 7.37. $\mathfrak{g} = T_1 G$ is a Lie algebra, where G is a Lie group.

Now suppose A is a k-algebra, then we have the ***k-derivations*** of A, the set of k-linear $l : A \to A$ satisfying the Leibniz rule:

$$l(a_1 a_2) = l(a_1)a_2 + a_1 l(a_2).$$

So

$$\mathrm{Der}_k(A) \subseteq \mathrm{End}_k(A).$$

Lemma 7.38. *$\mathrm{Der}_k A$ is a Lie subalgebra of $\mathrm{End}_k(A)$, that is, a k-submodule closed under the Lie bracket $[\cdot,\cdot]$.*

Proof. Let $l_1, l_2 \in \mathrm{Der}_k(A)$, $a_1, a_2 \in A$, then

$$\begin{aligned}
[l_1, l_2](a_1 a_2) &= l_1(l_2(a_1)a_2 + a_1 l_2(a_2)) \\
&= l_1 l_2(a_1)a_2 + l_2(a_1)l_1(a_2) + l_1(a_1)l_2(a_2) + a_1 l_1 l_2(a_2) \\
[l_2, l_1](a_1 a_2) &= l_2(l_1(a_1)a_2 + a_1 l_1(a_2)) \\
&= l_2 l_1(a_1)a_2 + l_1(a_1)l_2(a_2) + l_2(a_1)l_1(a_2) + a_1 l_2 l_1(a_2)
\end{aligned}$$

In the difference of these two, the second-order terms cancel and we are left with

$$[l_1, l_2](a_1 a_2) = [l_1, l_2](a_1)a_2 + a_1[l_1, l_2](a_2).$$

□

A consequence of this lemma is that for every manifold M,

$$\mathrm{Der}(M) = \mathrm{Der}_{\mathbb{R}}(C^\infty(M))$$

is a Lie algebra under the commutator bracket. Hence,

$$\mathcal{T}(M) \cong \mathrm{Der}(M)$$

is a Lie algebra: for $\xi, \eta \in \mathcal{T}(M)$, define

$$[\xi, \eta] = \mathcal{L}^{-1}([\mathcal{L}_\xi, \mathcal{L}_\eta]).$$

This makes $\mathcal{L} : \mathcal{T}(M) \to \mathrm{Der}(M)$ a Lie algebra isomorphism.

Restricting ξ, η to a chart (U, φ), write

$$\partial_i = \varphi^*(\frac{\partial}{\partial x_i})$$

for the frame on U induced by the standard basis on $\mathbb{R}^n$, then

$$\xi = \sum_{i=1}^n \xi_i \partial_i$$

$$\eta = \sum_{i=1}^n \eta_i \partial_i$$

for some coefficients $\xi_i, \eta_i \in \mathbb{R}$, For $f \in C^\infty(U)$, we have

$$\begin{aligned}
[\xi, \eta](f) &= \xi(\eta(f)) - \eta(\xi(f)) \\
&= \sum_{i,j=1}^n \xi_i(\partial_i \eta_j \partial_j f + \eta_j \partial_i \partial_j f) - \sum_{i,j=1}^n \eta_j(\partial_j \xi_i \partial_i f + \xi_i \partial_j \partial_i f) \\
&= \sum_{i,j=1}^n (\xi_i \partial_i \eta_j - \eta_i \partial_i \xi_j) \partial_j f
\end{aligned}$$

so as a differential operator,

$$[\xi, \eta] = \sum_{i,j=1}^n (\xi_i \partial_i \eta_j - \eta_i \partial_i \xi_j) \partial_j.$$

Proposition 7.39. *(Naturality) Let $f : M \to N$ be smooth, $\xi_1, \xi_2 \in \mathcal{T}(M), \eta_1, \eta_2 \in \mathcal{T}(N)$. Suppose $\xi_i \sim_f \eta_i (i = 1, 2)$, then $[\xi_1, \xi_2] \sim_f [\eta_1, \eta_2]$.*

Proof. We know that $\xi_i \circ f^* = f^* \circ \eta_i$ for $i = 1, 2$. Hence

$$(\xi_i \circ \xi_j) \circ f^* = \xi_i \circ (\xi_j \circ f^*) = (\xi_i \circ f^*) \circ \eta_j = f^* \circ (\eta_i \circ \eta_j)$$

so

$$[\xi_1, \xi_2] \circ f^* = f^* \circ [\eta_1, \eta_2].$$

□

Example 7.40. (Special case)

1. If f is an embedding, and $\eta_1, \eta_2 \in \mathcal{T}(N)$ are tangent to the submanifold M, then so is $[\eta_1, \eta_2]$.
2. If f is a diffeomorphism, then $f_*[\xi_1, \xi_2] = [f_*\xi_1, f_*\xi_2]$.
3. If f is a local diffeomorphism, then $f^*([\xi_1, \xi_2]) = [f^*(\eta_1), f^*(\eta_2)]$.

Two remarks on pulling back and pushing forward vector fields.

Remark 7.41. Let $f : M \to N$ be a diffeomorphism. We define

$$\begin{array}{cccc} F^* : & \mathcal{T}(N) & \to & \mathcal{T}(M) \\ & \eta & \mapsto & (TF)^{-1} \circ \eta \circ F \\ F_* : & \mathcal{T}(M) & \to & \mathcal{T}(N) \\ & \xi & \mapsto & TF \circ \xi \circ F^{-1} \end{array}$$

then $F^* = (F_*)^{-1}$.

Remark 7.42. Let $f : M \to N$ be a diffeomorphism. We can also define $F^* : \mathrm{Der}(N) \to \mathrm{Der}(M)$ by

$$F^*(l)(f) = F^* l((F^*)^{-1}(f)) = l(f \circ F^{-1}) \circ F$$

$$\begin{array}{ccc} C^\infty(N) & \xrightarrow{F^*} & C^\infty(M) \\ l \downarrow & & \downarrow F^*(l) \\ C^\infty(N) & \xrightarrow{F^*} & C^\infty(M) \end{array}$$

Since $\mathrm{Deer}(M) \cong \mathcal{T}(M)$, we can identify this two, and see the following diagram commutes.

$$\begin{array}{ccc} \mathcal{T}(N) & \xrightarrow{F^*} & \mathcal{T}(M) \\ \mathcal{L}_N \downarrow\cong & & \cong\downarrow \mathcal{L}_M \\ \mathrm{Der}(N) & \xrightarrow{F^*} & \mathrm{Der}(M) \end{array}$$

(homework, chain rule).

7.8 Brackets and Flows

There are three different notations for the derivative of $f \in C^\infty(M)$ along a vector field $\xi \in \mathcal{T}(M)$:

$$df(\xi) = \mathcal{L}_\xi(f) = \xi(f).$$

There is a fourth notation: Let $\theta = \theta_\xi : \mathcal{D}_\xi \to M$ be the flow of ξ. Let $a \in M$, then

$$\xi_a = \frac{d}{dt}\theta^a(0)$$

so

$$\begin{array}{rcl} d_a f(\xi_a) & = & d_a f(\frac{d}{dt}\theta^a(0)) \\ & = & d(f \circ \theta^a)(0) \\ & = & \frac{d}{dt}(f \circ \theta_t)(a)|_{t=0} \\ & = & \frac{d}{dt}\theta_t^*(f)(a)|_{t=0}. \end{array}$$

We write $df(\xi) = \frac{d}{dt}\theta_t^*(f)|_{t=0}$.

This shows us that we may substitute for f any other type of object that can be pulled back under diffeomorphism. So we can take derivatives of vector fields along vector fields, for example.

Definition 7.43. For $\xi, \eta \in \mathcal{T}(N)$, define

$$\mathcal{L}_\xi(\eta) = \frac{d}{dt}\theta_t^*(\eta)|_{t=0}$$

where $\theta = \theta_\xi$ is the flow of ξ.

Interpretation: for each point $p \in M$ and $t \in \mathcal{D}^a$, we have

$$\theta_t^*(\eta)(a) = (T_a\theta_t)^{-1}\eta_{\theta_t(a)} \in T_a M.$$

This is a smooth curve $\mathcal{D}^a \to T_a M$. Hence its derivative at $t = 0$ exists and is again in $T_a M$.

Theorem 7.44. *(Commutate)* $\mathcal{L}_{\xi(\eta)} = [\xi, \eta]$.

Proof. Let $\theta = \theta_\xi$, $\mathcal{D} = \mathcal{D}_\xi$. Let $a \in M$, choose an open neighborhood $U \ni a$ and $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \times U \subseteq \mathcal{D}$. Let $f \in C^\infty(M)$, we will show $\mathcal{L}_\xi(\eta)(f) = [\xi, \eta](f)$.

Define

$$\begin{array}{rccc} F: & (-\varepsilon,\varepsilon) \times U & \to & \mathbb{R} \\ & (t,x) & \mapsto & f(\theta_t(x)) - f(x) \end{array}$$

then $F(0,x) = 0$, so there exists smooth $G : (-\varepsilon, \varepsilon) \times U \to \mathbb{R}$ such that

$$\begin{cases} F(t,x) = tG(t,x) \\ \frac{\partial F}{\partial t}(0,x) = G(0,x) \end{cases} \tag{1}$$

Namely we can take

$$G(t,x) = \int_0^1 \partial_1 F(ts,x)ds.$$

Then (1) is equivalent to

$$\begin{cases} \theta_t^*(f) & = f + tG_t \\ \xi(f) & = G_0 \end{cases} \tag{2}$$

where $G_t(x) = G(t,x)$. Pulling back η (or $\mathcal{L}_\eta$) along θ_t gives

$$\begin{array}{rcc} \theta_t^*(\eta)(f) & = & \theta_t^*(\eta(\theta_{-t}^*(f)) \\ & = & \theta_t^*(\eta(f \circ \theta_{-t}) \\ & = & \theta_t^*(\eta(f - tG_{-t}) \\ & = & \theta_t^*(\eta(f)) - t\theta_t^*(\eta(G_{-t})) \end{array})$$

So

$$\begin{array}{rcc} \mathcal{L}_\xi(\eta)(f) & = & \frac{d}{dt}\theta_t^*(\eta)(f)|_{t=0} \\ & = & \xi(\eta(f)) - \theta_0^*(\eta(G_{-0}) - [t\frac{d}{dt}\theta_t^*(\eta(G_{-t}))]|_{t=0} \\ & = & \xi(\eta(f)) - \eta(G_{-0}) \circ \theta_0 \\ & = & \xi(\eta(f)) - \eta(G_0) \\ & = & \xi(\eta(f)) - \eta(\xi(f)) \\ & = & [\xi,\eta](f) \end{array}$$

□

Corollary 7.45. *For $\xi, \eta \in \mathcal{T}(M)$, $f \in C^\infty(M)$,*

1. $\mathcal{L}_\xi(\eta) = -\mathcal{L}_\eta(\xi)$
2. $\mathcal{L}_\xi(f\eta) = \mathcal{L}_\xi(f)\eta + f\mathcal{L}_\xi(\eta)$
3. $F^*(\mathcal{L}_\xi(\eta)) = \mathcal{L}_{F_*(\xi)}(F_*(\eta))$

for any diffeomorphism $F : M \to N$.

Proof. Use some facts.

1. $[\xi, \eta] = -[\eta, \xi]$.
2. If A is a k-algebra over a commutative ring k, and $l_1, l_2 \in \mathrm{Der}_k(A)$, then

$$[l_1, al_2] = l_1(a) \cdot l_2 + a[l_1, l_2]$$

   holds, because for any $b \in A$,

$$\begin{array}{rcc} [l_1, al_2](b) & = & l_1(al_2)(b) - al_2(l_1(b)) \\ & = & l_1(a)l_2(b) + al_1l_2(b) - al_2l_1(b) \end{array}$$

3. Use naturality of Lie braket $[\cdot, \cdot]$.

□

Lemma 7.46. $\frac{d}{dt}\theta^*_{\xi,t}(\eta)|_{t=s} = \theta^*_{\xi,s}\mathcal{L}_\xi(\eta)$.

Proof. Put $\theta_\xi = \theta$, then

$$\begin{array}{rcl} \frac{d}{dt}\theta^*_t(\eta)|_{t=s} & = & \lim\limits_{t\to s}\frac{1}{t-s}(\theta^*_t(\eta)-\theta^*_s(\eta)) \\ & = & \lim\limits_{t\to 0}\dfrac{1}{t}(\theta^*_{t+s}(\eta)-\theta^*_s(\eta)) \\ \text{(by flow law)} & = & \theta^*_s \lim_{t\to 0}\frac{1}{t}(\theta^*_t(\eta)-\eta) \\ & = & \theta^*_s(\mathcal{L}_\xi(\eta)) \end{array}$$

□

Lemma 7.47. *Let $F : M \to M$ be a diffeomorphism, then $\xi \sim_F \xi$ if and only if $F_*(\xi) = \xi$, if and only if $\theta_{\xi,t}\circ F = F\circ\theta_{\xi,t}$ for all t.*

Theorem 7.48. *(Commuting Flow Theorem)*
Let $\xi, \eta \in \mathcal{T}(M)$, then for all s, t, $\theta_{\xi,s}\circ\theta_{\eta,t} = \theta_{\eta,t}\circ\theta_{\xi,s}$ if and only if $[\xi,\eta] = 0$.

Proof. "⇒" Apply the second lemma to $F = \theta_{\xi,s}$ and the vector field $\theta^*_{\xi,s}(\eta) = \eta$, then $[\xi,\eta] = \frac{d}{ds}\theta^*_{\xi,s}(\eta)|_{s=0} = 0$.
"⇐" Let $a \in M$. Put $\gamma(s) = \theta^*_{\xi,s}(\eta)(a) \in T_aM$. Then

$$\gamma : (-\varepsilon,\varepsilon) \to T_aM$$

is smooth and for any s,

$$\gamma'(s) = \frac{d}{dt}\theta^*_{\xi,t}(\eta)|_{t=s} = \theta^*_{\xi,s}\mathcal{L}_\xi(\eta) = \theta^*_{\xi,s}([\xi,\eta]) = 0$$

so $\gamma(s) = \gamma(0)$, i.e. $\theta^*_{\xi,s}(\eta) = \eta$. So be lemma 2, $\theta_{\xi,s}\circ\theta_{\eta,t} = \theta_{\eta,t}\circ\theta_{\xi,s}$. □

Theorem 7.49. *Let $\xi_1, \xi_2, \cdots, \xi_k \in \mathcal{T}(M)$. Suppose*

1. *$[\xi_i, \xi_j] = 0$ for all i, j, and*
2. *$\xi_1, \cdots, \xi_k$ are linearly independent at $a \in M$.*

then there exists a chart (U, φ) centered at a such that $\xi_i|_U = \varphi^*(\frac{\partial}{\partial x_i})$ for $1 \le i \le k$.

Proof. Using g a preliminary chart (V, ϕ) centered at a, the vector fields

$$\eta_i = \phi^*(\xi_i|_V)$$

on $\phi(V)$ satisfying

1. $[\eta_i, \eta_j] = 0$ for all i, j, and
2. $\eta_1(0) = e_1, \cdots, \eta_k(0) = e_k$ the standard basis on $\mathbb{R}^n$.

Now assume M to be an open neighborhood of $a = 0 \in \mathbb{R}^n$, and $\xi_i(0) = e_i$ for $1 \le i \le k$. Put $\theta_i = \theta_{\xi_i}$ and

$$F(x) = F(x_1, \cdots, x_n) = \prod_{i=1}^k \theta_{i,x_i}(0, \cdots, 0, x_{k+1}, \cdots, x_n)$$

then $F(0) = 0$ and $F : W \to \mathbb{R}^n$ is well-defined, smooth on an open neighborhood of $0 \in \mathbb{R}^n$.
For $i > k$,

$$F(0, \cdots, 0, x_i, 0, \cdots, 0) = (0, \cdots, 0, x_i, 0, \cdots, 0),$$

so $DF(0)e_i = e_i$.
For $i \le k$,

$$\begin{array}{rcl} F(x_1, \cdots x_i + h, \cdots, x_n) & = & \theta_{1,x_1}\circ\cdots\circ\theta_{i,x_i+h}\circ\cdots\circ\theta_{n,x_n}(0, \cdots, 0, x_{k+1}, \cdots, x_n) \\ & = & \theta_{i,h}\circ F(x_1, \cdots, x_n) \end{array}$$

by commuting flow theorem. This says

$$F(\vartheta_{i,h}(x)) = \theta_{i,h}(F(x))$$

where $\vartheta_{i,h} = x + he_i$. i.e. ϑ_i is the flow of $\frac{\partial}{\partial x_i}$. Hence $\frac{\partial}{\partial x_i} \sim_F \xi_i$, i.e.

$$T_x F(\frac{\partial}{\partial x_i}) = (\xi_i)_{F(x)}.$$

Put $x = 0$, $T_0 F(\frac{\partial}{\partial x_i}) = (\xi_i)_0$, so $DF(0)(e_i) = e_i$.

Therefore, $DF(0) = I_n$. F is a local diffeomorphism. Hence

$$F_*(\frac{\partial}{\partial x_i}) = \xi_i.$$

Now let $\varphi = F^{-1}$, which is defined on a suitable neighborhood of $0 \in \mathbb{R}^n$. Then (U, φ) is a chart centered at a and $\varphi^*(\frac{\partial}{\partial x_i}) = \xi_i$. $\square$

8 Vector Bundle

8.1 Variations on the notion of manifolds

Let $\mathbb{M}$ be a fixed topological space. An $\mathbb{M}$-valued chart on a set M is a pair (U, φ), where $U \subseteq M$ and $\varphi : U \to \mathbb{M}$ is a bijection to an open subset of $\mathbb{M}$. A transition between two charts $(U, \varphi), (V, \phi)$ is $\phi \circ \varphi^{-1} : \varphi(U \cap V) \to \phi(U \cap V)$. These transitions might satisfy various notions of compatibility; see Example 8.1 below.

Example 8.1. Manifolds

	Model	Transition	Type of Manifold
(1)	$\mathbb{R}^n$	smooth diffeomorphisms	smooth manifolds
(2)	$\mathbb{R}^n$	C^r-diffeomorphisms ($r \in \mathbb{N} \cup \{\infty, \omega\}$)	C^r-manifolds
(3)	$\mathbb{C}^n$	biholomorphisms (complex analytic diffeomorphisms)	complex manifolds
(4)	$\mathbb{Q}_p^n$	analytic diffeomorphisms	p-adic manifolds
(5)	$\mathbb{R}^{n-1} \times [0, \infty)$	smooth diffeomorphisms	manifolds with boundary
(6)	$[0, \infty)^n$	smooth diffeomorphisms	manifolds with corners
(7)	$\mathbb{R}^n$	affine diffeomorphisms	affine manifolds
(8)	$B \times F$ B top. space F vector space	vector bundle transitions	vector bundle over B with Fibre F
(9)	$B \times F$ B manifold F vector space	smooth vector bundle transitions	smooth vector bundle

Table 1: Manifolds

Remark 8.2. Comments:

1. "C^0-diffeomorphism" means "homeomorphism". "C^ω-diffeomorphism" means "real analytic".

   A function $f : U \to \mathbb{R}$ ($U \subseteq \mathbb{R}^n$ open) is *real analytic* if for all $x \in U$, the Taylor series $T(f, x)$ at x converges to f uniformly on a neighborhood of x.

   Example. (Counterexample) Function

   $$f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

   is smooth but not real analytic. The Taylor series for f at 0 is just $T(f, 0) = 0$, which converges, but not to f , so f is not analytic at $x = 0$.

   Fact. If $1 \leq r \leq s$, every C^r-structure on M contains a C^s-structure. [M. Husch, Differential Topology]. This is false when $r = 0$.

2. An *affine map* $\mathbb{R}^n \to \mathbb{R}^n$ is a linear map composed with a transition. An *affine structure* on M is a maximal atlas in which the transitions are (restrictions of) affine map. On an affine manifold, there are well-defined notions of straight lines and affine submanifolds.

   Example. $\mathbb{R}^n$, $\mathbb{P}^n(\mathbb{F})$, $\mathbb{T}^n = (\mathbb{S}^1)^n$ is the quotient of $\mathbb{R}^n$ by a lattice.

3. Let B be a topological space and F a finitely dimensional vector space over $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. B is the base, F the fibre. Given F its natural topology.

   A vector bundle chart on a set E is a pair (O, φ) where $O \subseteq E$ and $\varphi : O \to B \times F$ is an injective map with the property that $\varphi(O) = U \times F$ satisfies U open in B.

Two vector bundle charts (O_1, φ_1) and (O_2, φ_2) are compatible if

$$\varphi_2 \circ \varphi_1^{-1} : (U_1 \cap U_2) \times F \to (U_1 \cap U_2) \times F$$

$$\varphi_1 \circ \varphi_2^{-1} : (U_1 \cap U_2) \times F \to (U_1 \cap U_2) \times F$$

are of the form $(x, v) \mapsto (x, g(x, v))$ where

$$g : (U_1 \cap U_2) \times F \to F$$

is continuous and for each $x \in U_1 \cap U_2$,

$$\begin{array}{cccc} g_x : & F & \to & F \\ & v & \mapsto & g_x(v) \end{array}$$

is $\mathbb{F}$-linear. Since $\varphi_2 \circ \varphi_1^{-1}$ is bijective, g_x is invertible for each $x \in U_1 \cap U_2$.

Suppose E has a vector bundle atlas, i.e. a collection of compatible vector bundle charts over E. Define projection $\pi = \pi_E : E \to B$ as follows: choose chart $(O, \varphi : O \to U \times F)$ at $y \in E$ and put $\pi(y) = \mathrm{pr}_1(\varphi(y))$.

$$\begin{array}{ccc} O & \xrightarrow{\varphi} & U \times F \\ \pi & \searrow & \downarrow \mathrm{pr}_1 \\ & & U \end{array}$$

This is independent of choices of chart. $\varphi : O \to U \times F$ is a bijection, so φ restricts to a bijection

$$\pi^{-1}(\pi(y)) \to \{\pi(y)\} \times F$$

We demand that $\pi : E \to B$ is surjective.

4. Assume B is a smooth manifold, Two vector bundle charts (O_1, φ_1) and (O_2, φ_2) are smoothly compatible if $\varphi_2 \circ \varphi_1^{-1}, \varphi_1 \circ \varphi_2^{-1}$ are smooth. Equivalently, g_x is a diffeomorphism.

8.2 Vector Bundle

Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. Let $r \in \mathbb{N}$, E, b be manifolds and $\pi : E \to B$ a smooth map.

A *(smooth)* $\mathbb{F}$*-vector bundle chart* on E is a pair (U, φ) with $U \subseteq B$ open and a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times \mathbb{F}^r$ with the property that

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times \mathbb{F}^r \\ \pi|_{\pi^{-1}(U)} \downarrow & \swarrow & \mathrm{pr}_1 \\ U & & \end{array}$$

commutes, i.e. $\pi|_{\pi^{-1}(U)} = \mathrm{pr}_1 \circ \varphi$. (Here we give $\mathbb{F}^r = \mathbb{R}^{r \dim_{\mathbb{R}}(\mathbb{F})}$ its usual topology and smooth structure.

For $b \in B$, let

$$\varphi_b : \pi^{-1}(b) \xrightarrow{\varphi} \{b\} \times \mathbb{F}^r \xrightarrow{\mathrm{pr}_2} \mathbb{F}$$

be a diffeomorphism.

Two vector bundle charts (U_1, φ_1) and (U_2, φ_2) are *compatible* if

$$\mathbb{F}^r \xrightarrow{\varphi_{1,b}} \pi^{-1}(b) \xrightarrow{\varphi_{2,b}} \mathbb{F}^r$$

is $\mathbb{F}$-linear for all $b \in U_1 \cap U_2$. If that is the case, the map

$$\begin{array}{cccc} g_{1,2} : & U_1 \cap U_2 & \to & GL(r, \mathbb{F}) \\ & b & \mapsto & g_{1,2}(b) \end{array}$$

is smooth.

Definition 8.3. A *vector bundle atlas* is a compatible set of vector bundle charts (U, φ) whose domains cover B. A maximal vector bundle atlas on E makes E an $\mathbb{F}$-vector bundle over B of rank (or fibre dimension) r. $\pi : E \to B$ is a smooth surjection.

E is called the ***total space*** of the vector bundle and B is called the ***base space***.

Remark 8.4. (Notation)

1. For any open $U \subseteq B$, write $E|_U = \pi^{-1}(U)$ the restriction of E to U, then $\pi : E|_U \to U$ is a vector bundle of rank r on U.

2. For $b \in B$, write $E_b = \pi^{-1}(b)$. We make E_b an r-dimensional vector space over $\mathbb{F}$ by declaring $\varphi_b : E_b \to \mathbb{F}^r$ to be an $\mathbb{F}$-linear isomorphism. This is independent of the choice of local trivialization because $g_{12}(b)$ is linear for all local trivializations (U_1, φ_1) and (U_2, φ_2) in the atlas.

Remark 8.5. $\pi : E \to B$ is a submersion, so each $\pi^{-1}(b)$ is a submanifold of E and $\varphi_b : E_b \to \mathbb{F}^r$ is a diffeomorphism.

Example 8.6. The tangent bundle of a smooth manifold M with the map $\pi : TM \to M$ of a manifold M is a real vector bundle of rank $r = \dim(M)$. For (U, φ) a chart on M, we have a tangent chart $(TU, T\varphi)$, where

$$T\varphi : TU \xrightarrow{\sim} \varphi(U) \times \mathbb{R}^n.$$

Since $\varphi(U)$ is diffeomorphic to U, composing with $\varphi^{-1} \times \mathrm{id}_{\mathbb{R}^n}$ gives

$$\bar{\varphi} : (\varphi^{-1} \times \mathrm{id}_{\mathbb{R}^n}) \circ T\varphi : TU \to U \times \mathbb{R}^n$$

Then $(U, \bar{\varphi})$ is a vector bundle chart on TM. The transition from a vector bundle chart $(U_1, \bar{\varphi_1})$ to $(U_1, \bar{\varphi_1})$ is

$$\begin{array}{cccc} g_{1,2}: & U_1 \cap U_2 & \to & GL(n, \mathbb{R}) \\ & a & \mapsto & D(\bar{\varphi}_2 \circ \bar{\varphi}_1^{-1})(\bar{\varphi}_1(a)) \end{array}.$$

Example 8.7. Let B be any manifold, and $E = B \times \mathbb{F}^r$. Then $\pi : E \to B$ is projection onto the first factor, with the vector bundle atlas generated by the global vector bundle chart $(U = B, \varphi = \mathrm{id}_E)$. This makes E into a vector bundle called the ***trivial vector bundle*** over B of rank r.

Definition 8.8. The ***cocycle*** of $\pi : E \to B$ relative to a vector bundle chart atlas $\{(U_i, \varphi_i) | i \in I\}$ is the collection of smooth maps

$$\begin{array}{cccc} g_{ij}: & U_{ij} & \to & GL(r, \mathbb{F}) \\ & b & \mapsto & \varphi_{j,b} \circ \varphi_{i,b}^{-1} : \mathbb{F}^r \to \mathbb{F}^r \end{array}.$$

Let $U_{ijk} = U_i \cap U_j \cap U_k$, then

$$\begin{cases} g_{ii} = \mathrm{id}_{U_{ii}} & x \in U_i \\ g_{jk} \circ g_{ij} = g_{ik} & x \in U_{ijk} \end{cases}$$

so $g_{ji} = g_{ij}^{-1}$ on U_{ij}.

Given cocycle $\mathfrak{g} = \{g_{ij}\}$, i.e. an open cover $\{U_i | i \in I\}$ and a collection of maps $\{g_{ij}\}$ satisfying the properties above, define

$$\tilde{E} = \coprod_{i \in I} (U_i \times \mathbb{F}^r)$$

and define $(a, u) \in U_i \times \mathbb{F}^r$ to be equivalent to $(b, v) \in U_j \times \mathbb{F}^r$ if $a = b$ and $g_{ij}(a)u = v$.

Fact 8.9. *This equivalent relation is regular and $E_{\mathfrak{g}} = \tilde{E}/\sim$ is a vector bundle over B with cocycle $\mathfrak{g}$.*

If $\mathfrak{g}$ is the cocycle of E, then there is an isomorphism of vector bundles $E \xrightarrow{\sim} E_{\mathfrak{g}}$.

Example 8.10. Let $B = \mathbb{S}^1$, with open cover given by two line sections and cocycle $g_{12} : U_{12} \to GL(1, \mathbb{R}) = \mathbb{R}^\times$.

$$g_{12} = \begin{cases} 1 & U_{12}^+ \\ -1 & U_{12}^- \end{cases}$$

This gives Mobius strip as a real line bundle over $\mathbb{S}^1$.

Definition 8.11. A section of E is a map $\sigma : B \to E$ satisfying $\pi \circ \sigma = \mathrm{id}_B$, i.e. $\pi(\sigma(b)) = b$, i.e. $\sigma(b) \in E_b$ for all $b \in B$.

Composing σ with a vector bundle chart (U, φ) gives a smooth map

$$\tilde{\sigma} = \mathrm{pr}_2 \circ \varphi \circ \sigma : U \to \mathbb{F}^r$$

$$\begin{tikzcd}
E|_U \arrow[r, "\varphi"] \arrow[d, shift left, "\pi"] & U \times \mathbb{F}^r \arrow[dl, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] \\
U \arrow[u, shift left, "\sigma"] & \mathbb{F}^r
\end{tikzcd}$$

We have

$$\varphi \circ \sigma(b) = (b, \tilde{\sigma}(b))$$

for $b \in U$. $\tilde{\sigma}$ is the *empression* for σ in the chart. σ is smooth if and only if $\tilde{\sigma}$ is smooth for all vector bundle charts.

Sections can be added (addition in E_b):

$$(\sigma_1 + \sigma_2)(b) = \sigma_1(b) + \sigma_2(b)$$

and multiplied by functions(scalar multiplication in E_b):

$$(f\sigma)(b) = f(b)\sigma(b)$$

We have

$$\begin{cases} (\widetilde{\sigma_1 + \sigma_2}) = \tilde{\sigma_1} + \tilde{\sigma_2} \\ (\widetilde{f\sigma}) = \tilde{f}\tilde{\sigma} \end{cases}$$

for all vector bundle charts.

So if $\sigma_1, \sigma_2, \sigma, f$ are smooth., then $\sigma_1 + \sigma_2$, $f\sigma$ are smooth.

Remark 8.12. (Notation) We denote by $\Gamma(E)$ the set of smooth sections of $\pi : E \to B$. For $U \subseteq B$ open, $\Gamma(U, E) = \Gamma(E|_U)$ is the smooth sections of $E|_U$. For each U, $\Gamma(U, E)$ is a $C^\infty(U)$-module.

Example 8.13. $\mathcal{T}(M) = \Gamma(TM)$.

Definition 8.14. A k-***frame*** on E is a k-tuple $(\sigma_1, \cdots, \sigma_k)$ of sections $\sigma_i \in \Gamma(E)$ such that $\sigma_1(b), \cdots, \sigma_k(b) \in E_b$ are linearly independent in the $\mathbb{F}$-vector space E_b for all $b \in B$.

If E has a k-frame, then $k \leq \mathrm{rank}(E)$.

A trivialization $\varphi : E \to B \times \mathbb{F}^r$ gives rise to an r-frame $(\sigma_1, \cdots, \sigma_r)$ given by the standard basis $e_1, \cdots, e_r$ of $\mathbb{F}^r$

$$\sigma_j(b) = \varphi^{-1}(b, e_j)$$

$$\begin{tikzcd}
E \arrow[r, shift left, "\varphi"] \arrow[d, "\pi"] & B \times \mathbb{F}^r \arrow[l, shift left, "\varphi^{-1}"] \\
B \arrow[ur] &
\end{tikzcd}$$

Conversely, given an r-frame $(\sigma_1, \cdots, \sigma_r)$ we define

$$\begin{array}{rccc} \phi : & B \times \mathbb{F}^r & \to & E \\ & (b, y) & \mapsto & \displaystyle\sum_{i=1}^{r} y_i \sigma_i(b) \end{array}$$

Lemma 8.15. *ϕ is a diffeomorphism and ϕ^{-1} is a trivialization (i.e. global vector bundle chart) on E.*

Proof. Since $\sigma_1, \cdots, \sigma_r$ are smooth, so is ϕ. Since $\sigma_1, \cdots, \sigma_r$ are linearly independent, $r = \mathrm{rank}(E)$, then ϕ is bijective. So it remains to show that $T_{(b,y)}\phi$ is bijective for all $(b, y) \in B \times \mathbb{F}^r$.

Choose a vector bundle chart (U, φ_E) at $b \in B$ and also a chart (U, φ_B) on B at b.

$$f : \varphi_B(U) \times \mathbb{F}^r \xrightarrow{\varphi_B \times \mathrm{id}_{\mathbb{F}^r}} U \times \mathbb{F}^r \xrightarrow{\phi} E|_U \xrightarrow{\varphi_E} U \times \mathbb{F}^r \xrightarrow{\varphi_B \times \mathrm{id}} \varphi_B(U) \times \mathbb{F}^r$$

f is a smooth map of the form

$$f(x,y) = (x, A_x(y))$$

where $A_x(y) = A(x,y) : \varphi_B(U) \times \mathbb{F}^r \to \mathbb{F}^r$ is smooth, $\mathbb{F}$-linear for each $x \in \varphi_B(U)$, and

$$y \mapsto A(x,y)$$

is invertible for each x. Hence

$$Df(x,y) = \begin{pmatrix} I_n & 0 \\ D^1 A(x,\cdot) & A(x,\cdot) \end{pmatrix}$$

For all x, $A(x,\cdot) \in GL(r,\mathbb{F})$ so $Df(x,y)$ is invertible, therefore $T_{(b,y)}\phi$ is invertible. $\square$

For any vector bundle, there is a one-to-one correspondence between trivializations and r-frames, and for any $U \subseteq B$ open, there is a one-to-one correspondence between vector bundle charts with domain U and r-frames on $E|_U$.

Lemma 8.16. *Let $(\sigma_1, \cdots, \sigma_k)$ be a k-frame on E and $b_0 \in B$. Then there exists an open neighborhood U of b_0 and sections $\sigma_1, \cdots, \sigma_r$ such that $(\sigma_1, \cdots, \sigma_r)$ is an k-frame over U.*

Proof. Choose a vector bundle chart (V, φ) on E at b_0. The vectors $\sigma_1(b), \cdots, \sigma_k(b) \in E_b$ are linearly independent, so $\varphi(\sigma_1(b)), \cdots \varphi(\sigma_k(b)) \in \mathbb{F}^r$ are linearly independent. So after composing φ with an invertible $r \times r$ matrix A, we may assume $\varphi(\sigma_j(b_0)) = e_j$ is the standard basis vector, with $1 \leq j \leq k$.

For $j > k$, $b \in V$, put $\sigma_j(b) = \varphi^{-1}(b, e_j)$, then $\sigma_1, \cdots, \sigma_r$ are linearly independent at b_0, and therefore independent for all b in a neighborhood U of b_0. $\square$

Remark 8.17. Let $\pi : E \to B$ be a smooth vector bundle over $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. E has a zero section

$$\begin{array}{rccc} 0_E : & B & \to & E \\ & b & \mapsto & 0_b \in E_b \end{array}$$

$0_E : B \to E$ is an embedding, and by an abuse of language, $0_E(B) \cong B$ is often called the zero section of E.

8.3 Subbundle

Let $F \subseteq E$.

Definition 8.18. A ***vector bundle chart*** (U, φ) on E is a subbundle chart for $F \subseteq E$ of rank s if

$$\pi^{-1}(U) \times F = \varphi^{-1}(U \times \mathbb{F}^s)$$

where we regard $\mathbb{F}^s$ as an $\mathbb{F}$-linear subspace of $\mathbb{F}^r$.

Definition 8.19. $F \subseteq E$ is a ***subbundle*** if for all $b \in B$, there is a vector bundle chart (U, φ) on E with $b \in U$ and

$$\pi^{-1}(U) \times F = \varphi^{-1}(U \times \mathbb{F}^s)$$

for some $s \leq r$.

A subbundle is a vector bundle in its own right. For each $b \in B$, F_b is a linear subspace of E_b, and moreover, F is a submanifold of E.

Proposition 8.20. *The following are equivalent:*

1. *$F \subseteq E$ is a subbundle of rank s.*

2. *F is a submanifold and $F_b = E_b \cap F$ is an s-dimensional $\mathbb{F}$-linear subspace of E_b for all $b \in B$.*

3. *For all $b_0 \in B$, the fibre F_{b_0} is an $\mathbb{F}$-linear subspace of E_{b_0} and there is an open neighborhood U of b_0 and a* s-frame $\sigma_1, \cdots, \sigma_s$ of E_U such that for any $b \in U$, $\sigma_1(b), \cdots, \sigma_s(b) \in E_b$ is a basis of F_b.

Proof. $3 \Rightarrow 1$: By previous lemma, after shrinking U we can extend $\sigma_1, \cdots, \sigma_s$ to a frame $\sigma_1, \cdots, \sigma_r$ of $E|_U$, then the trivialization given by $\sigma_1, \cdots, \sigma_r$ is a subbundle chart for F.

$$\varphi^{-1}(x, y) = y_1\sigma_1(x) + \cdots + y_r\sigma_r(x)$$

$$\begin{array}{ccc} E|_U & \xrightarrow{\varphi} & U \times \mathbb{F}^r \\ \sigma_i \uparrow\downarrow \pi & & \\ U & & \end{array}$$

$1 \Rightarrow 2$ immediate from definition.

$2 \Rightarrow 3$ Let $b_0 \in B$, after a preliminary choice of vector bundle chart (U, φ_E) on E at b_0 and a chart (U, φ_B) on B at b_0,

$$E|_U \xrightarrow{\varphi_E} U \times \mathbb{F}^r \xrightarrow{\varphi_B \times \mathrm{id}} \varphi_B(U) \times \mathbb{F}^r$$

we can assume $B = U = \varphi_B(U)$ is an open neighborhood of $0 \in \mathbb{F}^n$ and $E = U \times \mathbb{F}^r$ is the trivial bundle, and $F_{b_0} = F_0 = F \cap E_0 = \{0\} \times \mathbb{F}^s$. Let $\mathrm{pr} : \mathbb{F}^r \to \mathbb{F}^s$ be the projection,

$$p = \mathrm{id}_U \times \mathrm{pr} : E \to U \times \mathbb{F}^s$$

and

$$f = p|_F : F \to U \times \mathbb{F}^s$$

is smooth. For each $x \in U$, f restricts to a linear map

$$f_x : F_x \to \{x\} \times \mathbb{F}^s$$

and in particular $f_0 = \mathrm{id}_{\mathbb{F}^s}$. So after shrinking U we may assume that f_x is a linear isomorphism for all $x \in U$.

F is a submanifold of E containing $U \times \{0\}$ and $\{0\} \times \mathbb{F}^s$, so

$$T_0F = T_0U \oplus \mathbb{F}^s = T_0(U \times \mathbb{F}^s)$$

(why?) and

$$T_0f : T_0F \xrightarrow{\mathrm{id}} T_0(U \times \mathbb{F}^s).$$

By the inverse function theorem, there is a neighborhood of $0 \in \mathbb{R}^n$ (which we may assume to be U) and a neighborhood V of $0 \in \mathbb{F}^s$ such that f has a local inverse $g : U \times V \to F$.

Choose s vectors $v_1, \cdots, v_s \in V$ which spans $\mathbb{F}^s$. For $x \in U$, put $\sigma_j(x) = g(x, v_j)$. Then $\sigma_1, \cdots, \sigma_s$ are smooth, $\sigma_j(x) \in F_x$ and $\sigma_1(x), \cdots, \sigma_s(x)$ span F_x. □

Example 8.21. Let $B = \mathbb{P}(V)$, the set of lines through the origin in an $\mathbb{F}$-vector space V. Let $E = \mathbb{P}(V) \times V$ the trivial bundle, with fibers V. Let F be determined from E by the incidence relation of a point lying on a line.

$$F = \tilde{V} = \{(l, v) \in E | v \in l\}$$

Then F is a submanifold of E and for $l \in \mathbb{P}(V)$,

$$F_l = E_l \cap F = \{l\} \times l \subseteq \{l\} \times V = E_l$$

so F is a subbundle of E, called the ***tautological bundle*** over $\mathbb{P}(V)$.

$$\begin{array}{ccccc} & & \tilde{V} & & \\ & \swarrow_{\pi} & & {}_{\beta}\searrow & \\ \mathbb{P}(V) & & & & V \end{array}$$

where π is the $\mathbb{F}$-linear bundle projection and β is the blow-up map.

9 Foliation

9.1 Tangent Subbundle

Definition 9.1. Let M be a manifold. A ***tangent subbundle*** or ***distribution*** over M is a subbundle of TM.

Example 9.2. Tangent subbundles.

1. Let $\xi_1, \cdots, \xi_r$ be an r-frame on M, with $r \le n = \dim(M)$. Let E be the span of $\{\xi_1, \cdots, \xi_r\}$, that is ,

$$E_x = \operatorname{span}\{\xi_1(x), \cdots, \xi_r(x)\} \subseteq T_x M$$

for any $x \in M$. By the proposition, E is a distribution on M.

2. Let $f : M \to N$ be a subimmersion ($\operatorname{rank}(f) = r$). Let $E = \ker(Tf)$, that is,

$$E_x = \ker T_x f$$

for $x \in M$. For all $x \in M$,the subspace $E_x \subseteq T_x M$ has dimension $\dim(M) - r$. Choosing charts (U, φ) on M, (V, ϕ) on N such that

$$\tilde{f}(x) = (x_1, \cdots, x_r, 0, \cdots 0)$$

we see that the tangent chart $(TU, T\varphi)$ is a subbundle chart for E, so E is a tangent subbundle.

Let $\operatorname{Gr}(r, \mathbb{R}^n)$ be the Grassmannian of r-planes in $\mathbb{R}^n$.

Definition 9.3. Let E be a subbundle of rank r of the trivial rank n bundle $B \times \mathbb{R}^n$. The ***Gauss map*** of E is the map $f : B \to \operatorname{Gr}(n, r)$ defined by $f(b) = E_b$.

Definition 9.4. $\operatorname{Fr}(r, \mathbb{R}^n)$ is the ***Stiefel manifold*** of all r-frames in $\mathbb{R}^n$.

Lemma 9.5. *f is smooth.*

Proof. We can cover B with open subsets U for which there exists an r-frame $\sigma_1, \cdots, \sigma_r \in \Gamma(U, \mathbb{R}^n)$ such that $\sigma_1, \cdots, \sigma_r$ span $E|_U$. Define

$$\begin{array}{rccc} \bar{f}: & U & \to & \operatorname{Fr}(r, \mathbb{R}^n) \subseteq M_{n\times r}(\mathbb{R}) \\ & b & \mapsto & (\sigma_1(b), \cdots, \sigma_r(b)) \end{array}$$

$\bar{f}$ is smooth, and $f = p \circ \bar{f}$, where

$$\begin{array}{rccc} p: & \operatorname{Fr}(\mathbb{R}^n) & \to & \operatorname{Gr}(n, r) \\ & (x_1, \cdots, x_r) & \mapsto & \operatorname{span}\{x_1, \cdots, x_r\} \end{array}$$

(here the smooth structure of $\operatorname{Gr}(n, r)$ is compatible with the one defined in homework 1).

p is smooth because it is the quotient map for the action θ of $\operatorname{GL}(r, \mathbb{R})$ on $\operatorname{Fr}(r, \mathbb{R}^n)$ defined by

$$\theta(g, (x_1, \cdots, x_r)) = (x_1, \cdots, x_n) g^{-1}$$

This action is free and proper (why?), so p is a smooth submersion. □

Example 9.6. Let $0 \le r \le n$. Let V_1 be an open neighborhood of $0 \in \mathbb{R}^r$ and V_2 an open neighborhood of $0 \in \mathbb{R}^{n-r}$, and $V = V_1 \times V_2 \subseteq \mathbb{R}^r \times \mathbb{R}^{n-r} \cong \mathbb{R}^n$. Let $f : V \to M_{(n-r)\times r}(\mathbb{R})$ be smooth. For $(x, y) \in V$, let

$$E^{f(x,y)} = \{(\overrightarrow{a}, f(x, y)\overrightarrow{a}) | (x, y) \in V, \overrightarrow{a} \in \mathbb{R}^r\} \subseteq \mathbb{R}^n.$$

Let

$$E^f = \bigcup_{(x,y)\in V} \{(x, y)\} \times E^{f(x,y)} \subseteq V \times \mathbb{R}^n = TV.$$

This is the tangent subbundle on V of rank r. E^f has a global frame: let $\partial_i = \frac{\partial}{\partial x_i}$ be the standard vector fields corresponding to the standard basis vectors $e_1, \cdots, e_n \in \mathbb{R}^n$. So a basis of $E^{f(x,y)}$ is $(e_i, f(x, y)e_i)$, $i = 1, \cdots, r$. So the vector fields

$$\xi_i(x, y) = (\partial_i, f(x, y)\partial_i)$$

form an r-frame that span E^f.

Let $\pi_i : V \to V_i$ be the projection for $i = 1, 2$. The first component of ξ_i is ∂_i, that is, $\xi_i \sim_{\pi_1} \partial_i$. Therefore

$$[\xi_i, \xi_j] \sim_{\pi_1} [\partial_i, \partial_j] = 0,$$

that is,

$$T_{(x,y)}\pi_1[\xi_i, \xi_j](x, y) = 0$$

Also if $f(0, 0) = 0$, then $E^{f(0,0)} = \mathbb{R}^r$ and $\xi_i(0, 0) = (\partial_i, 0)$ is the first r standard basis vectors of $\mathbb{R}^n$.

This example is important because any distribution (tangent subbundle) locally looks like the one above.

Proposition 9.7. *Let E be a tangent subbundle of rank r over M. For every $a \in M$, there is a chart $(U, \varphi : U \to \mathbb{R}^n)$ centered at a and a smooth $f : \varphi(U) \to M_{(n-r)\times r}(\mathbb{R})$ such that $f(0) = 0$ and $E|_U = (T\varphi)^{-1}(E^f)$.*

Proof. Choose a basis $v_1, \cdots, v_n \in T_aM$ such that $E_a = \mathrm{Span}\{v_1, \cdots, v_r\}$. Let $\alpha_1, \cdots, \alpha_n \in T_a^*M$ be the dual basis. Choose chart (U', φ) centered at a such that $d_a\varphi_i = \alpha_i$. Then $d_a\varphi : T_aM \to \mathbb{R}^n$ maps E_a to $\mathbb{R}^r \subseteq \mathbb{R}^n$.

Let $F = (T\varphi)(E|_U)$ be a subbundle of $TV' = V' \times \mathbb{R}^n$ where $V' = \varphi(U')$. Let $f' : V' \to \mathrm{Gr}(r, \mathbb{R}^n)$ be the Gauss map of F. On $\mathrm{Gr}(r, \mathbb{R}^n)$, we have a chart (O, φ), with $\mathbb{R}^r \subseteq O$, with $\varphi : O \to M_{(n-r)\times r}$. Let $V = (f')^{-1}(0)$ and $U = \varphi^{-1}(V)$, and $f = f'|_V$. Then $F = E^f$. □

9.2 Integral Manifold

Definition 9.8. An ***integral manifold*** of a distribution $E \subseteq TM$ is an immersed submanifold $A \subseteq M$ such that $T_aA = E_a$ for all $a \in A$.

Example 9.9. $E = \mathrm{span}\{\xi\}$ where ξ is a 1-frame, i.e. a nowhere vanishing vector field. Trajectories $\gamma : I \to M$ of ξ are integral manifolds of E.

Example 9.10. $E = \ker(f)$ where $f : M \to N$ is a subimmersion. The fibres $f^{-1}(y)$ are integral manifolds.

Example 9.11. $E = E^f$, where $f : V \to M_{(n-r)\times r}(\mathbb{R})$ is smooth. $V = V_1 \times V_2 \subseteq \mathbb{R}^{n-r} \times \mathbb{R}^r$ is a product of open sets $V_1 \subseteq \mathbb{R}^r$ and $V_2 \subseteq \mathbb{R}^{n-r}$. E^f is the tangent subbundle over Vgiven by

$$E^f_{(x,y)} = \mathrm{graph}(f(x, y) : \mathbb{R}^r \to \mathbb{R}^{n-r}) \subseteq \mathbb{R}^n.$$

For every $(x_0, y_0) \in V_1 \times V_2$ and every integral manifold A of E containing (x_0, y_0), by the implicit function theorem, there are open $U_i \subseteq V_i$ $(i = 1, 2)$, $U = U_1 \times U_2$, such that $A \cap U = \mathrm{graph}(u)$ for a unique smooth function $u : U_1 \to U_2$.

Conversely, suppose that $u : V_1 \to V_2$ is a smooth map, then $A_u = \mathrm{graph}(u) \subseteq V_i \times V_2 = V$ is an n-dimensional submanifold of V.

When is A_u an integral submanifold?

A_u is the image of an embedding

$$\begin{array}{rccc} \tilde{u}: & V_1 & \to & V \\ & x & \mapsto & \begin{pmatrix} x \\ u(x) \end{pmatrix} \end{array}.$$

Let $(x, y) \in A_u$, that is, $y = u(x)$. A basis of $T_{(x,0)}A_u$ is

$$D\tilde{u}(x)e_i = \begin{pmatrix} I_r \\ Du(x) \end{pmatrix} e_i = \begin{pmatrix} e_i \\ Du(x)e_i \end{pmatrix}$$

and a basis of $E^f_{(x,y)}$ are the columns of

$$\begin{pmatrix} I_r \\ f(x, y) \end{pmatrix}, i = 1, \cdots, r.$$

that is, the vectors

$$\begin{pmatrix} e_i \\ f(x, y)e_i \end{pmatrix}, i = 1, \cdots, r.$$

We want $T_{(x,y)}A_u = E^f_{(x,)}$ to make A_u an integral submanifold, so

$$Du(x)e_i = f(x,y)e_i, i = 1, \cdots, r.$$

Or more precisely,

$$Du(x) = f(x,y).$$

In fact, we have the following result.

Proposition 9.12. *A_u is an integral submanifold if and only if u satisfies the system of first-order partial differential equations*

$$Du(x) = f(x, u(x)). \tag{3}$$

The next theorem tells us that we cannot have integral submanifolds that meet in a point or anything like that.

Theorem 9.13. *Let A_1, A_2 be integral manifolds of E. Then $A_1 \cap A_2$ is an open submanifold of A_1 and A_2.*

Proof. Let $a \in A_1 \cap A_2$. Without loss of generality, $M = V = V_1 \times V_2$, where $V_1 \subseteq \mathbb{R}^r$, $V_2 \subseteq \mathbb{R}^{n-r}$ are open neighborhoods of 0, $a = (0,0) \in \mathbb{R}^r \times \mathbb{R}^{n-r} = \mathbb{R}^n$, $E = E^f$ for some smooth

$$f : V \to M_{(n-r)\times r}(\mathbb{R}),$$

and $A_1 = \mathrm{graph}(u_1)$, $A_2 = \mathrm{graph}(u_2)$ for two solutions $u_1, u_2 : V_1 \to V_2$ of the PDE (3).

We have that $u_1(0) = 0 = u_2(0)$. Without loss of generality, we may assume that V_1 is connected. Let $x \in V_1$, choose a smooth path

$$\gamma : [0,1] \to V_1 \overset{[}{u_2}]u_1 \rightrightarrows V_2$$

with $\gamma(0) = 0 \in \mathbb{R}^r$, $\gamma(1) = x$. Put $v_i = u_i \circ \gamma$ for $i = 1, 2$. Then v_i satisfies an ODE

$$v_i'(t) = Du_i(\gamma(t))\gamma'(t) = f(\gamma(t), u_i(\gamma(t)))\gamma'(t) = f(\gamma(t), v_i(t))\gamma'(t)$$

for $0 \leq t \leq 1$. We have $v_1(0) = 0 = v_2(0)$. Hence, by uniqueness of ODE solutions, $v_1 = v_2$. In particular, $v_1(1) = v_2(1)$, which means $u_1(x) = u_2(x)$ for all $x \in V_1$. Therefore, $u_1 = u_2$, and it follows that $A_1 = A_2$. ☐

Corollary 9.14. *Let $\{A_i | i \in I\}$ be a family of integral manifolds of E. Then $\bigcup_{i\in I} A_i$ is an integral manifold as well. Moreover, if I is finite, then $\bigcap_{i\in I} A_i$ is an integral manifold of E.*

9.3 Integrability

Definition 9.15. E is ***integrable*** if for every $a \in M$, there is an integral manifold of E containing a.

The union of all connected integral manifolds containing a is the unique largest connected integral manifold of E containing a, and is called the ***leaf*** of a.

So M is a disjoint union of leaves. Each leaf is an r-dimensional immersed submanifold.

Remark 9.16. (Caution) Topology on $A = \bigcup_{i\in I} A_i$ may be finer than subspace topology inherited from M.

Example 9.17. Let $M = \mathbb{R}^3$ with coordinates (x_1, x_2, y). E is spanned by

$$\begin{aligned} \xi &= \frac{\partial}{\partial x_1} + f(x_1,x_2)\frac{\partial}{\partial y} \\ \eta &= \frac{\partial}{\partial x_2} + g(x_1,x_2)\frac{\partial}{\partial y} \end{aligned}$$

where $f, g : \mathbb{R}^2 \to \mathbb{R}$ is smooth. That is, for $x \in \mathbb{R}^3$,

$$E_x = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ f(x_1,x_2) \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ g(x_1,x_2) \end{pmatrix} \right\}.$$

This is the graph of a linear map $\mathbb{R}^2 \to \mathbb{R}$ with matrix

$$(f(x_1, x_2), g(x_1, x_2)).$$

An integral manifold A through x in $\mathbb{R}^3$ is, near x, the graph of a function $u : U \to \mathbb{R}$, where U is an open neighborhood of $(x_1, x_2) \in \mathbb{R}^2$. Let

$$\tilde{u}(x_1, x_2) = \begin{pmatrix} x_1 \\ x_2 \\ u(x_1, x_2) \end{pmatrix},$$

then A is the image of $\tilde{u} : U \to \mathbb{R}^3$, so $T_x A$ is the image of $D\tilde{u}(x_1, x_2)$, which is the column space of the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} \end{pmatrix}$$

so we must have that

$$\begin{aligned} \tfrac{\partial u}{\partial x_1} &= f \\ \tfrac{\partial u}{\partial x_2} &= g \end{aligned}$$

The integrability condition is that

$$\frac{\partial f}{\partial x_2} = \frac{\partial g}{\partial x_1}.$$

Let E be a distribution on M. For each open $U \subseteq M$, $\Gamma(U, E)$ is a subspace of $\Gamma(U, TM) = \mathcal{T}(U)$.

Definition 9.18. E is ***involutive*** if $\Gamma(U, E)$ is a Lie subalgebra of $\mathcal{T}(U)$ for all open U.

Lemma 9.19. *If E is integrable, then E is involutive.*

Proof. Let $\xi, \eta \in \Gamma(U, E)$. For every $a \in U$, ξ, η are tangent to an integral manifold A containing a. Hence $[\xi, \eta]$ is tangent to A, i.e. $[\xi, \eta] \in \Gamma(U, E)$. □

Theorem 9.20. *(Frobenius Integrability Theorem). The following are equivalent:*

1. *E is integrable.*
2. *E is involutive .*
3. *For all $a_0 \in M$, there is a chart (U, φ) centered at a_0 such that $E|_U$ is spanned by the frame $\varphi^*(\partial_1), \cdots, \varphi^*(\partial_r)$.*

Proof. $1 \Rightarrow 2$ proved in the above lemma.

$2 \Rightarrow 3$ Let $a_0 \in M$. Without lose of generality, assume $a = 0 \in \mathbb{R}^n$, $M = V = V_1 \times V_2$, where V_1 is an open neighborhood of $0 \in \mathbb{R}^r$ and V_2 an open neighborhood of $0 \in \mathbb{R}^{n-r}$. $E = E^f$ for some smooth

$$f : V \to M_{(n-r)\times r}(\mathbb{R}),$$

i.e.

$$E^f_{(x,y)} = \mathrm{graph}(f(x, y)) \subseteq \mathbb{R}^r \times \mathbb{R}^{n-r} \cong \mathbb{R}^n.$$

E^f has a global frame

$$\xi_i = \begin{pmatrix} \partial \\ f(x, y)\partial_i \end{pmatrix}, i = 1, 2. \cdots, r$$

and $\xi_i \sim_{\pi_1} \partial_i$, where $\pi_1 : \mathbb{R}^n \to \mathbb{R}^r$, i.e. $T\pi_1(\xi_i) = \partial_i$. so

$$T\pi_1([\xi_i, \xi_j]) = [\partial_i, \partial_j] = 0.$$

Since $\Gamma(E)$ is a Lie subalgebra of $\mathcal{T}(M)$, $[\xi_i, \xi_j] \in \Gamma(E)$, which means

$$[\xi_i, \xi_j](x, y) \in E_{(x,y)} = E^f_{(x,y)}$$

for all $(x, y) \in V_1 \times V_2$. Now

$$T\pi_1 : E^f_{(x,y)} \xrightarrow{\cong} T_x V_1 \cong \mathbb{R}^r$$

, therefore

$$[\xi_i, \xi_j](x, y) = 0, \forall (x, y) \in V$$

which means $[\xi_i, \xi_j] = 0$. By a previous theorem, there is a chart (U, φ) centered at a_0 such with $\xi_i = \varphi^*(\partial_i)$. $\square$

Remark 9.21. The submanifolds

$$\begin{array}{ccc} \varphi_{r+1}(x) & = & c_1 \\ & \cdots & \\ \varphi_n(x) & = & c_{n-r} \end{array}$$

are integral submanifolds of $E|_U$.

Fact 9.22. *If M is paracompact, then the leaves of an integrable distribution are also paracompact.*

9.4 Distributions on Lie Groups

Let $M = G$ be a Lie group. A vector field $\xi \in \mathcal{T}(G)$ is ***left-invariant*** if $T_1 L_g(\xi_1) = \xi_g$ for all $g \in G$. Write $\mathcal{T}(G)_L$ for the space of all left-invariant vector fields.

Recall left trivialization

$$\begin{array}{rccc} \varphi_L : & G \times \mathfrak{g} & \to & TG \\ & (g, \xi) & \mapsto & T_1 L_g(\xi) \end{array}$$

where $\mathfrak{g} = T_1 L_g$, is a diffeomorphism. For $\xi \in \mathfrak{g}$, let $\xi_{L,g} = \varphi_L(g, \xi)$, then $\xi_L \in T(G)_L$.

Lemma 9.23. *$\mathcal{T}(G)_L$ is a Lie subalgebra of $\mathcal{T}(G)$.*

Proof. If $\xi, \eta \in \mathcal{T}(G)_L$, then $L_g^*([\xi, \eta]) = [L_g^*(\xi), L_g^*(\eta)] = [\xi, \eta]$ for all $g \in G$. $\square$

Via the isomorphism $\varphi_L : \mathfrak{g} \to \mathcal{T}(G)_L$, this makes $\mathfrak{g} = T_1 G$ a Lie algebra, called the ***Lie algebra*** of G.

Remark 9.24. If we had done this with right-invariant vector fields instead, we would end up with the opposite Lie algebra ,$\mathfrak{g}^{\mathrm{op}}$ which is the same except the bracket has a negative sign.

10 Differential forms

Let $\pi : E \to B$ be a smooth vector bundle over $\mathbb{F} = \mathbb{R}, \mathbb{C}$. We want to perform algebraic operations on fibres of E.

Example 10.1. Let E^* be the dual of E, defined by

$$E^* = \coprod_{b \in B} (E_b)^*$$

where $(E_b)^* = \mathrm{Hom}_{\mathbb{F}}(E_b, \mathbb{F})$ is the dual vector space of $\mathbb{F}$-linear maps. Define

$$\pi = \pi_{E^*} : \begin{array}{ccc} E^* & \to & B \\ (E_b)^* & \mapsto & b \end{array}$$

To get a vector bundle chart on E^*, take a chart (U, φ) for E:

$$\begin{array}{ccc} E|_U & \xrightarrow{\varphi} & U \times \mathbb{F}^r \\ \downarrow & \swarrow & \\ U & & \end{array}$$

we have $\mathbb{F}$-linear isomorphisms

$$\varphi_b : E_b \xrightarrow{\varphi} \{b\} \times \mathbb{F}^r \to \mathbb{F}^r$$

for $b \in B$, and the transpose map

$$\varphi_b^T : (\mathbb{F}^r)^* \xrightarrow{\cong} (E_b)^*.$$

Define a chart (U, φ^*) on E^*:

$$\begin{array}{ccc} E^*|_U & \xrightarrow{\varphi^*} & U \times (\mathbb{F}^r)^* \\ \downarrow & \swarrow & \\ U & & \end{array}$$

by letting the restriction of φ^* to $(E_b)^*$ be the inverse transpose map

$$\varphi_b^* : (E_b)^* \xrightarrow{(\varphi_b^T)^{-1}} (\mathbb{F}^r)^* \xrightarrow{\cong} \{b\} \times (\mathbb{F}^r)^*$$

To verify that this is an atlas, we need to check compatibility. Given two charts (U_1, φ_1) and (U_2, φ_2) on E with transition map

$$g_{12} : \begin{array}{ccc} U_1 \cap U_2 & \to & \mathrm{GL}(r, \mathbb{F}) \\ b & \mapsto & \varphi_{2,b} \circ \varphi_{1,b}^{-1} \end{array}$$

then the transition map for (U_1, φ_1^*) and (U_2, φ_2^*) is

$$g_{12}^* : \begin{array}{ccc} U_1 \cap U_2 & \to & \mathrm{GL}(r, \mathbb{F}) \\ b & \mapsto & (\varphi_{2,b}^T)^{-1} \circ \varphi_{1,b}^T = (g_{12}(b)^T)^{-1} \end{array}$$

i.e. the following diagram commutes,

$$\begin{array}{ccc} U_1, \cap U_2 & \xrightarrow{g_{12}} & \mathrm{GL}(r, \mathbb{F}) \\ g_{12}^* \searrow & & \swarrow f \\ & \mathrm{GL}(r, \mathbb{F}) & \end{array}$$

$g_{12}^* = f \circ g_{12}$, where $f(A) = (A^T)^{-1}$. f is a Lie group homomorphism, so g_{12}^* is smooth and E^* is a vector bundle.

If $\sigma_1, \cdots, \sigma_r$ is an r-frame on $E|_U$ corresponding to a chart (U, φ) on E, then the r-frame on $E^*|_U$ corresponding to (U, φ^*) is the ***dual frame*** $\sigma_1^*, \cdots, \sigma_r^* \in \Gamma(U, E^*)$ characterized by

$$\sigma_i^*(\sigma_j) = \delta_{ij}$$

that is,

$$\sigma_i^*(b)(\sigma_j(b)) = \delta_{ij}$$

for all $b \in U$.

Example 10.2. (Special case) Let $E = TM$ be the tangent bundle, where M is a manifold, then $E^* = T^*(M)$ is the cotangent bundle. A chart (U, φ) on M gives rise to an n-frame

$$\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}$$

of $E|_U = TU$. The dual frame on $E^*|_U = T^*U$ is denoted by

$$dx_1, \cdots, dx_n.$$

Sections of T^*M are differential forms of of degree 1, or 1-frame.
A general 1-frame on U looks like

$$\alpha = \sum_{i=1}^{n} f_i dx_i$$

with $f_i \in C^\infty(U)$.

10.1 General Construction

Let $\mathcal{C}$ be the category of finite dimensional vector spaces over $\mathbb{F}$. Let $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ be a functor.
For a vector bundle $\pi : E \to B$, define

$$\mathcal{F}(E) = \coprod_{b \in B} \mathcal{F}(E_b)$$

as a set, and define

$$\begin{array}{llll} \pi: & \mathcal{F}(E) & \to & B \\ & \mathcal{F}(E_b) & \mapsto & b \end{array}$$

A chart (U, φ) on E gives rise to a chart $(U, \mathcal{F}(\varphi))$ on $\mathcal{F}(E)$ as follows,

$$\begin{array}{ccc} \mathcal{F}(E)|_U & \xrightarrow{\mathcal{F}(\varphi)} & U \times \mathcal{F}(\mathbb{F}^r) \\ \downarrow & \swarrow & \\ U & & \end{array}$$

for each $b \in U$, we have $\varphi_b : E_b \xrightarrow{\cong} \mathbb{F}^r$.

1. $\mathcal{F}$ is covariant, then apply $\mathcal{F}$:

$$\mathcal{F}(\varphi_b) : \mathcal{F}(E_b) \xrightarrow{\cong} \mathcal{F}(\mathbb{F}^r).$$

   Define the restriction of $\mathcal{F}(\varphi)$ to $\mathcal{F}(E_b)$ to be the map

$$\mathcal{F}(E_b) \xrightarrow{\mathcal{F}(\varphi_b)} \mathcal{F}(\mathbb{F}^r) \to \{b\} \times \mathcal{F}(\mathbb{F}^r).$$

2. $\mathcal{F}$ is contravariant, then apply $\mathcal{F}$ to φ_b to get

$$\mathcal{F}(\varphi_b) : \mathcal{F}(\mathbb{F}^r) \xrightarrow{\cong} \mathcal{F}(E_b).$$

   Define the restriction of $\mathcal{F}(\varphi)$ to $\mathcal{F}(E_b)$ to be the map

$$\mathcal{F}(E_b) \xrightarrow{\mathcal{F}(\varphi_b)^{-1}} \mathcal{F}(\mathbb{F}^r) \to \{b\} \times \mathcal{F}(\mathbb{F}^r).$$

To check compatibility of the charts, suppose (U_1, φ_1) and (U_2, φ_2) are charts on E with transition

$$g_{12} : U_1 \cap U_2 \to \mathrm{GL}(r, \mathbb{F})$$

then the charts $(U_1, \mathcal{F}(\varphi_1))$ and $(U_2, \mathcal{F}(\varphi_2))$ have transition map

$$\mathcal{F}(g_{12}) : U_1 \cap U_2 \to \mathrm{GL}(r, \mathbb{F})$$

given by either

$$\mathcal{F}(g_{12})(b) = \mathcal{F}(g_{12}(b))$$

in the covariant case, or

$$\mathcal{F}(g_{12})(b) = \mathcal{F}(g_{12}(b))^{-1}$$

in the contravariant case.

We need this to be smooth for the charts to be compatible. But $\mathcal{F}$ defines a map

$$\mathrm{Hom}(\mathbb{F}^r, \mathbb{F}^r) \xrightarrow{\mathcal{F}} \mathrm{Hom}(\mathcal{F}(\mathbb{F}^r), \mathcal{F}(\mathbb{F}^r)),$$

which restricts to a map

$$\mathrm{GL}(r, \mathbb{F}) \xrightarrow{\mathcal{F}} \mathrm{GL}(\mathcal{F}(\mathbb{F}^r)), \tag{4}$$

which is a homomorphism if $\mathcal{F}$ is covariant and a antihomomorphism if $\mathcal{F}$ is contravariant.

$$\begin{array}{ccc} U_{12} & \xrightarrow{g_{12}} & \mathrm{GL}(\mathbb{F}^r) \\ {\scriptstyle \mathcal{F}(g_{12})} & \searrow & \downarrow_{\mathcal{F}/\tau \circ \mathcal{F}} \\ & & \mathrm{GL}(\mathcal{F}(\mathbb{F}^r)) \end{array} \tag{5}$$

where τ is the inversion map.

Theorem 10.3. *Suppose $\mathcal{F}$ has the property that* (4) *is smooth, then $\mathcal{F}(E)$ is a smooth vector bundle over B. If $\{g_{ij} : U_{ij} \to GL(\mathbb{F}^r)\}$ is a cocycle representing E, then a cocycle for $\mathcal{F}(E)$ is $\{\mathcal{F}(g_{ij}) : \mathcal{F}(U_{ij}) \to GL(\mathcal{F}(\mathbb{F}^r)\}$ as defined in* (5).

Example 10.4. Let $\mathcal{F}$ be the dual functor, $\mathcal{F}(E) = E^*$, we obtain the dual vector bundle. Similarly, we could let $\mathcal{F}$ be the tensor algebra, $\mathcal{F}(E) = T(E)$, symmetric algebra, $\mathcal{F}(E) = S(E)$, alternating algebra $\mathcal{F}(E) = A(E)$, or Clifford algebra, $\mathcal{F}(E) = Cl(E)$, over a vector space V, giving other bundles.

Remark 10.5. A functor $\mathcal{F}$ is smooth if for all objects $V, W \in \mathcal{C}$, the category of finite dimensional vector spaces and linear maps over $\mathbb{F} = \mathbb{R}, \mathbb{C}$,

$$\begin{cases} \mathrm{Hom}(V, W) & \xrightarrow{\mathcal{F}} & \mathrm{Hom}(\mathcal{F}(V), \mathcal{F}(W)) & \mathcal{F} \text{ covariant} \\ \mathrm{Hom}(V, W) & \xrightarrow{\mathcal{F}} & \mathrm{Hom}(\mathcal{F}(W), \mathcal{F}(V)) & \mathcal{F} \text{ contravariant} \end{cases}$$

is smooth.

A smooth funtor can be applied to the fibres of a vector bundle $E \to B$, to yield a new vector bundle $\mathcal{F}(E) \to B$.

Example 10.6. Let $\mathcal{F}$ be the dual functor, $\mathcal{F}(V) = V^*$, $\mathcal{F}(f) = f^T : W^* \to V^*$. The map

$$\mathrm{Hom}(V, W) \xrightarrow{\mathcal{F}} \mathrm{Hom}(W^*, V^*)$$

is smooth because it's linear.

10.2 Alternating (or exterior) algebras

Let k be a commutative ring and V a k-module. For example, $k = \mathbb{R}, V = T_x M$, or $k = C^\infty(M), V = \mathcal{T}(M)$.

Definition 10.7. The ***alternating algebra*** of V is the k-module $A(V)$ spanned by all symbols

$$u_1 \wedge u_2 \wedge \cdots \wedge u_n$$

for $n \in \mathbb{N}$, $u_1, \cdots, u_n \in V$, subject to all relations of the form:

- MULT (multlinearity):

$$u_1 \wedge \cdots \wedge (au_i + a'u_i') \wedge \cdots \wedge u_n = a(u_1 \wedge \cdots \wedge u_i \wedge \cdots \wedge u_n) + a'(u_1 \wedge \cdots \wedge u_i' \wedge \cdots \wedge u_n$$

  for $a, a' \in k$, $u_1, \cdots, u_i, u_i', \cdots, u_n \in V$.

- ALT (alternating property):
$$u_1 \wedge \cdots \wedge u_i \wedge \cdots \wedge u_i \wedge \cdots \wedge u_n = 0$$
if $u_i = u_j$ for some $i \neq j$.

If $n = 0$, we interpret $u_1 \wedge u_2 \wedge \cdots \wedge u_n$ to be $1 \in k$.

Example 10.8. $5 + 3v + v \wedge w \in A(V)$ for any $u, v, w \in V$.

Example 10.9. $u \wedge v = -v \wedge u$ for all $u, v \in V$, because

$$(u + v) \wedge (u + v) = 0$$

by the alternating property.

$A(V)$ is an associative algebra over k with unit and multiplication defined on generators by

$$(u_1 \wedge \cdots \wedge u_m) \wedge (v_1 \wedge \cdots \wedge v_n) = u_1 \wedge \cdots \wedge u_m \wedge v_1 \wedge \cdots \wedge v_n$$

Definition 10.10. Some variants.

1. If we omit ALT, the resulting algebra is the ***tensor algebra*** $T(V)$ with multiplication denoted by $\otimes$.
2. If we replace ALT with SYM (symmetry):
$$u_1 u_2 \cdots u_i \cdots u_j \cdots u_n = u_1 u_2 \cdots u_j \cdots u_i \cdots u_n,$$
we obtain the ***symmetric algebra*** $S(V)$ of V with multiplication denoted by juxtaposition, or " " (empty symbol).
3. If we replace ALT with CL (Clifford Axiom):
$$uu = qu,$$
where $q : V \to k$ is a quadratic form, we get the ***Clifford algebra*** $Cl(V, q)$ of the pair (V, q).

10.2.1 Degree

Definition 10.11. An element of $A(V)$ has ***degree*** n if it's a linear combination of generators $u_1 \wedge u_2 \wedge \cdots \wedge u_m$, with $u_i \in V$. The elements of degree n form a k-submodule, the n-th alternating power $A^n(V)$ of V.

We have that
$$A(V) = \bigoplus_{n=0}^{\infty} A^n(V)$$
where $A^0(V) = k$, $A^1(V) = V$. Therefore $A(V)$ is a graded algebra,

$$A^i(V) \cdot A^j(V) \subseteq A^{i+j}(V).$$

If $x \in A^i(V)$ and $y \in A^j(V)$, then $y \wedge x = (-1)^{ij}(x \wedge y)$.

Definition 10.12. The ***graded commutator*** is given on basis elements $x \in A^i(V)$, $y \in A^j(V)$ by

$$[x, y] = x \wedge y - (-1)^{ij} y \wedge x$$

and extended by linearity. This is also called the ***Koszul sign rule***.

$A(V)$ is graded commutative since $[x, y] = 0$ for all $x, y \in A(V)$.
The alternating algebra construction $V \mapsto A(V)$ defines a functor

$$A : \mathbf{Mod_k} \to \mathbf{Alg_k}$$

where $\mathbf{Mod_k}$ is the category of $\mathbf{k}$-modules and $\mathbf{Alg_k}$ is the category of $\mathbf{k}$-algebras. If V, W are modules and $f : V \to W$ is $\mathbf{k}$-linear, then define

$$A(f) : A(V) \to A(W)$$

on generators by

$$A(f)(u_1 \wedge u_2 \wedge \cdots \wedge u_n) = f(u_1) \wedge \cdots \wedge f(u_n)$$

and extend by k-linearity.

This is well-defined, k-linear and moreover multiplicative (which means that $A(f)(x \wedge y) = A(f)(x) \wedge A(f)(y)$ for all x, y). We also say that $A(f)$ is of ***degree zero***, that is,

$$\deg(A(f)(x)) = \deg(x).$$

We say $A(f)$ is a homomorphism of graded algebras.

10.2.2 Basis

Some questions we may ask are: what does a basis of $A(V)$ look like? What's the matrix of a linear transformation $A(f)$ relative to the matrix for f ?

Let $V^* = \mathrm{Hom}_{\mathbf{k}}(V, \mathbf{k})$ be the k-dual of V. Define a pairing

$$\begin{array}{ccc} A^l(V^*) \times A^l(V) & \to & \mathbf{k} \\ (\varphi_1 \wedge \cdots \wedge \varphi_l)(u_1 \wedge \cdots \wedge u_l) & \mapsto & \det(\varphi_i(u_j))_{l \times l} \end{array}$$

for $\varphi_i \in V^*, u_j \in V$. V. This is well-defined and k-bilinear.

Now assume that V is a free k-module of finite rank n. Choose a basis $b_1, \cdots, b_n$ of V, with dual basis $b_1^*, \cdots, b_n^*$ such that $b_i^*(b_j) = \delta_{ij}$.

Let $I = (i_1, \cdots, i_r)$ be a multi-index with $1 \leq i_l \leq n$. Write $b_I = b_{i_1} \wedge \cdots \wedge b_{i_r}$ and $b_I^* = b_{i_1}^* \wedge \cdots \wedge b_{i_r}^*$. We say that I is increasing if $1 \leq i_1 < \cdots < i_r \leq n$. Let $\mathcal{I}_r^n$ be the set of such multi-indices.

Lemma 10.13. *The elements b_I (respectively, b_I^*), $I \in \mathscr{I}_k^n$, span $A^k(V)$ (respectively, $A^k(V)^*$).*

Proof. Let $u_1, \cdots, u_k \in V$, enough to show $u_1 \wedge \cdots \wedge u_k$ is a linear combination of the b_I. Expressing $u_i = \sum_{j=1}^{n} c_{ij} b_j$ with $c_{ij} \in \mathbf{k}$ and using MULT we see

$$u_1 \wedge \cdots \wedge u_k = \sum_{j_1=1}^{k} \cdots \sum_{j_k=1}^{k} c_{1j_1} \cdots c_{kj_k} b_{j_1} \wedge \cdots \wedge b_{j_k}.$$

Using ALT, rearrange this in the form

$$u_1 \wedge \cdots u_k = \sum_{I \in \mathscr{I}_k^n} c_I b_I.$$

□

Lemma 10.14. *$b_I^*(b_J) = \delta_{IJ}$ for all $I, J \in \mathscr{I}_k^n$.*

Proof. See

$$\begin{array}{rcl} b_I^*(b_J) & = & (b_{i_1}^* \wedge \cdots \wedge b_{i_k}^*)(b_{j_1} \wedge \cdots \wedge b_{j_k}) \\ & = & \det(b_{i_t}^*(b_{j_s})) \\ & = & \det(\delta_{i_t j_s}) \\ & = & \begin{cases} \det(I_k) = 1 & I = J \\ 0 & I \neq J \end{cases} \end{array}$$

when $I \neq J$, there exist zero rows in $(\delta_{i_t j_s})_{k \times k}$. □

Theorem 10.15. *Suppose V is a free $\mathbf{k}$-module of rank n with basis $b_1, \cdots, b_n$.*

1. *The collections* $\{b_I | I \in \mathscr{I}_k^n\}$, $\{b_I^* | I \in \mathscr{I}_k^n\}$ *are basis of the* $\mathbf{k}$*-module* $A^k(V)$, *respectively* $A^k(V)^*$. *In particular,* $A^k(V)$ *and* $A^k(V)^*$ *are free of rank* $|\mathscr{I}_k^n| = \binom{n}{k}$. *Also* $A^k(V) = A^k(V)^* = 0$ *for* $k > n$.

2. *The pairing* $A^k(V)^* \times A^k(V) \to \mathbf{k}$ *is nondegenerate in the sense that the associated map*

$$A^k(V^*) \xrightarrow{\cong} A^k(V)^*$$

is an isomorphism.

3. *As a* $\mathbf{k}$*-module,* $A(V)$ *is free of rank* $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Now let V, W be free $\mathbf{k}$-modules and $f : V \to W$ a linear map. Choose bases $b_1, \cdots, b_n$ of V and $c_1, \cdots, c_m$ of W. Let $(f_{ij}) \in M_{m\times n}(\mathbf{R})$ be the matrix of f, i.e. $f_{ij} = c_i^*(f(b_j)) \in \mathbf{k}$.

Recall $f : V \to W$ induces a functor

$$\begin{array}{cccc} A^k(f): & A^k(V) & \to & A^k(W) \\ & u_1 \wedge \cdots \wedge u_k & \mapsto & f(u_1) \wedge \cdots \wedge f(u_k) \end{array}.$$

Theorem 10.16. *The matrix element of* $A^k(f)$ *relative to the bases* $\{b_J | J \in \mathscr{J}_k^n\}$ *and* $\{c_I | I \in \mathscr{I}_k^m\}$ *are* $\det(f_{IJ})$ *where* $f_{IJ} \in M(k, \mathbf{k})$ *is the* $k \times k$ *minor of the matrix* (f_{ij}) *with rows* $i_1, \cdots, i_k$ *and columns* $j_1, \cdots, j_k$.

Proof. For $b_J \in A^k(V)$ and $c_I \in A^k(W)$,

$$\begin{array}{rcl} c_I^*(A^k(f)(b_J) & = & (c_{i_1}^* \wedge \cdots \wedge c_{i_k}^*)(f(b_{j_1}) \wedge \cdots \wedge f(b_{j_k})) \\ & = & \det(c_{i_p}^*(f(b_{j_q}))_{k\times k} \\ & = & \det(f_{IJ}). \end{array}$$

∎

Now let $\mathbf{k} = \mathbf{R}$, $\mathrm{Vec}_{\mathbf{R}}$ is the category of finite dimensional real vector spaces.

Corollary 10.17. *The funtors* $A^k : Vec_{\mathbf{R}} \to Vec_{\mathbf{R}}$ *are smooth.*

Proof. For all $V, W \in \mathrm{Vec}_{\mathbf{R}}$,

$$\begin{array}{cccc} A^k: & \mathrm{Hom}_{\mathbf{R}}(V, W) & \to & \mathrm{Hom}_{\mathbf{R}}(A^k(V), A^k(W)) \\ & \cong\downarrow & & \cong\downarrow \\ & M_{m\times n}(\mathbf{R}) & & M_{\binom{m}{k}\times\binom{n}{k}}(\mathbf{R}) \end{array}$$

is a smooth (in fact multilinear) map. ∎

So for every smooth vector bundle $\pi : E \to B$ with cocycles $\{g_{ij}\}$, we get new vector bundles $A^k(E) \to B$ and $A^k(E)^* \to B$ with cocycles $\{A^k(g_{ij})\}$, respectively $\{A^k(g_{ij}^*)\}$.

Also have alternating algebra bundle $A(E) = \bigoplus_{k=1}^{n} A^k(E)$.

We have isomorphisms $A^k(E^*) \to A^k(E)^*$.

Example 10.18. $E = TM$, where M is a manifold.

$\mathfrak{X}(M)$ and $\Omega(M)$ are algebras over $C^\infty(M)$.

Remark 10.19. Interpretation of $A^k(V)^* = \mathrm{Hom}(A^k(V), \mathbf{k})$.

Let $V^k = V \times \cdots \times V$ (k times), a map $\mu : V^k \to \mathbf{k}$ is ***multilinear*** if

$$\mu(u_1, \cdots, au_i + a'u_i', u_k) = a\mu(u_1, \cdots, u_i, \cdots, u_k) + a'\mu(u_1, \cdots, u_i', \cdots, u_k),$$

and ***alternating*** if

$$\mu(u_1, \cdots, u_k) = 0, \text{ if } u_i = u_j \text{ for some } i \neq j.$$

We put $\mathrm{Alt}^k(V) = \{\mu : V^k \to \mathbf{k} | \mu \text{ is multilinear and alternating}\}$.

Bundles	Sections	Notations
$A(TM)$	multivector fields	$\mathfrak{X}(M) = \bigoplus_{k=0}^{\infty} \mathfrak{X}^k(M)$
$A(T^*M)$	differential forms	$\Omega(M) = \bigoplus_{k=0}^{\infty} \Omega^k(M)$

Table 2:

Example 10.20. $V = \mathbf{k}^n$, then $\det \in \mathrm{Alt}^k(V)$.

Fact 10.21. $Alt^k(V) \cong A^k(V)^*$.