# MATH 6540 Homotopy Theory

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# Contents

# Introduction/Review

Homotopy theory means many different things in different times and places. We will work on some specific examples and general constructions.

In this course, all maps are continuous.

**Definition 1.** A *homotopy* between two map $f, g : X \to Y$ is a map $H : X \times I \to Y$ such that $H|_{X\times\{0\}} = f$ and $H|_{X\times\{1\}} = g$. If a homotopy between $f, g$ exists, then $f$ is homotopic to $g$ and write $f \sim g$.

**Example 1.** Let $[X, Y]$ be the set of homotopy classes of maps $X \to Y$.

- $[*, X]$ = path connected components of $X$.
- $\left[S^1, X\right]_{1\mapsto x_0} = \pi_1 (X, x_0)$ is a group.
- $[S^n, X]_{1\mapsto x_0} = \pi_n (X, x_0), n \geq 2$ are abelian groups.

**Fact 1.** *$H_1$ is the abelianization of $\pi_1$, but $H_i \neq \pi_i, i \geq 2$.*

There is a long exact sequence of reduced homology associated with $A \hookrightarrow X \to X/A$,

$$\cdots \to \tilde{H}_i (A) \to \tilde{H}_i (X) \to \tilde{H}_i (X/A) \cong \tilde{H}_i (X, A) \to \tilde{H}_{i-1} (A) \to \cdots$$

However, excision fails for homotopy theory.

There is a long exact sequence in homotopy.

**Definition 2.** A *fibre bundle* $E \to B$ is a map such that for any $b \in B$ there exists an open neighborhood $U \ni b$ such that

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "p"] \arrow[d, "\cong"'] & U \\ U \times F \arrow[ur, "\mathrm{pr}_1"'] & \end{tikzcd}$$

This shows that fibre bundle is locally like a product, but not union.

**Example 2.** There are several examples that we've seen before.

- Covering spaces: $F$ discrete set.
- Möbius strip: $F = I$ interval.

**Theorem 1.** *Let $p : E \to B$ be a fibre bundle or fibration. $b_0 \in B$. Let $F = p^{-1}(b_0)$ and $e_0 \in F$, then there exists a long exact sequence*

$$\cdots \to \pi_i(F, e_0) \to \pi_i(E, e_0) \to \pi_i(B, b_0) \to \pi_{i-1}(F, e_o) \to \pi_{i-1}(E, e_0) \to \pi_{i-1}(B, b_0) \to \cdots \to \pi_1(B, b_0).$$

In most cases when the space is connected enough, the choice of basepoint does not matter. We will not worry about basepoint in most cases.

**Example 3.** Let $p : \tilde{X} \to X$ be a covering space with $\tilde{X}$ connected. Since $\pi_i(F) = 0, i > 1$, we have

$$0 \to \pi_i\left(\tilde{X}\right) \to \pi_i(X) \to 0, i > 1$$
$$0 \to \pi_1\left(\tilde{X}\right) \to \pi_1(X) \to \pi_0(F) \to \pi_0\left(\tilde{X}\right) = 0$$

For instance, the covering space $\mathbb{R} \to S^1$ gives $\pi_i(\mathbb{R}) = \pi_i(S) = 0, i > 1$ and $\pi_1\left(S^1\right) = \mathbb{Z}$.

We've know all the homotopy groups of $S^1$ and it's natural to ask whether we can compute the homotopy groups of $S^n$ for any arbitrary $n$. However, this is still an unsolved problem.

Smash product for $S^1$ works similarly as Cartesian product of $\mathbb{R}$. However smash product works only for cohomology but not homotopy.

**Example 4.** Let $\eta : \mathbb{C}^2 \to \mathbb{C}\times\mathbb{R}, (z_0, z_1) \mapsto \left(2z_0\bar{z}_1, |z_1|^2 - |z_0|^2\right)$, the for any $(z, x) \in \mathbb{C}\times\mathbb{R}$, $\eta^{-1}(z, x) \cong S^1$. In particular $\eta$ restricts to

$$\eta : S^3 \to S^2$$

which is surjective. This is called a Hopf fibration. We get

$$\cdots \to \pi_i\left(S^1\right) \to \pi_i\left(S^3\right) \to \pi_i\left(S^2\right) \to \pi_{i-1}\left(S^1\right) \to \pi_{i-1}\left(S^3\right) \to \pi_{i-1}\left(S^2\right) \to \cdots, i > 2$$

so $\pi_i\left(S^3\right) = \pi_i\left(S^2\right), i > 2$.

Since $\pi_n(S^n) = \mathbb{Z}, n > 0$, we have $\pi_3\left(S^2\right) = \pi_3\left(S^3\right) = \mathbb{Z}$ where $\eta$ gives the generator of $\pi_3\left(S^2\right)$.

**Theorem 2.** *(Whitehead) Let $X, Y$ be connected CW-complexes. Given $f : X \to Y$, and suppose that $\pi_n f : \pi_n X \to \pi_n Y$ is an isomorphism for all $n$, then $f$ is a homotopy equivalence.*

Note this is not true for homology groups.

*Remark* 1. This theorem shows that homotopy groups are useful, but we cannot compute them. In Hatcher, Chapter 4 appendix, there are some tools compute homotopy, e.g. EHP sequence , Gysn sequence and UnASS, etc.

**Outline for this course:**

- Develop tools that are good for homotopy theory.
  - Simplicial sets
  - Show why homotopy theorists like categories
- Brave New Algebra

  Topological spaces gives rise to ring structure via cohomology

  $$\textbf{Top} \underset{\leftarrow\,-\,-}{\xrightarrow{H^*}} \textbf{Rings}$$

  where we can study quotient, localization and completion of rings, therefore it's appealing to lift these methods to topological spaces up to homotopy. This is derived algebraic geometry.
- Develop an analog for homological algebra (Model Categories)

  **Theorem.** *Let $f_\bullet : D_\bullet \to D'_\bullet$ be a map of chain complexes. If $D_\bullet, D'_\bullet$ are pointwise projective, then $f$ is a chain homotopy equivalence if and only if $H_* f$ is an isomorphism.*

# Part I

# Simplicial Methods

## 1 Simplicial Sets

To start with, we recall two constructions in algebraic topology.

**Simplicial Complexes:**

a simplicial complex $K$ contains the data of $n$-simplices $K_n$ (subset of a vertex set).

If all vertices are order, we can define the chain complex $C_*(K)$ where $C_n(K)$ is the free abelian group generated by $n$-simplices in $K$ and

$$\partial x = \sum_{i=0}^{n} (-1)^i \left[\{a_0, \cdots, \hat{a}_i, \cdots, a_n\}\right], x = \{a_0, \cdots, a_n\}$$

However, chain complexes does not contain enough information we want.

**CW Complexes:**

[Cellular Approximation, [HA02] Section 4.8] Every map $f : X \to Y$ of CW complexes is homotopic to a cellular map.

CW complexes are good, but they contains too much information than what we need.

It turns out that simplicial sets are the right objects for us to work with.

**Simplicial Sets:**

a simplicial set contains two parts of information that we are interested, the gluing data and the context where gluing applies to.

For instance, if we consider an $n$-simplex with $n+1$ faces, we need to know about how an $(n-1)$-simplex glues into the $n$-simplex. On the other hand, degeneracies give the information of how to collapse an $n$-simplex into an $(n-1)$-simplex. In order to describe such maps, we would like to have all the vertices ordered. However, after a second thought we would see that we do not really need all the vertices to be ordered, we only need to require that for every $n$-simplex, there should be an order for the vertices involved, and there is even no need to require a partial order.

**Definition 3.** The category $\Delta$ has its objects nonempty totally ordered finite sets $[n] = \{0 < 1 < \cdots < n\}$ and morphisms are order-preserving maps.

**Fact 2.** *Morphisms in $\Delta$ are generated by $d^i : [n-1] \to [n]$ which skips the $i$-th vertex and $s^j : [n+1] \to [n]$ which repeats the $j$-th element.*

**Definition 4.** A *simplicial set* is a functor $K : \Delta^{op} \to \mathbf{Set}$. $K[n]$ is a set called $n$-simplices of $K$, usually written as $K_n$. The *face maps* are $d_i = K\left(d^i\right) : K_n \to K_{n-1}$ and the *degeneracy maps* are $s_j = K\left(s^j\right) : K_n \to K_{n+1}$. A simplex $x \in K_n$ is called *degenerate* if there exists $y \in K_{n-1}$ such that $x = s_i y$, otherwise it is non-degenerate.

**Fact 3.** *Every simplex is equal to a degeneracy ( composition of $s_j$'s ) applied to a unique non-degeneracy simplex.*

*Remark* 2. Only non-degenerate simplices contributes to geometric and topological properties that we are interested in, and degeneracy maps does not matter. However, if we work with semi-simplicial objects everything will stop to work.

Consider the geometric $n$-simplex $\Delta^n = \left\{(t_0, \cdots, t_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} t_i = 1, t_i \geq 0\right\}$, then $\delta_i : \Delta^{n-1} \to \Delta^n$ is the inclusion of the $i$-th face and $\sigma_i : \Delta^{n+1} \to \Delta^n$ compress .

**Definition 5.** [Milnor] Let $K$ be a simplicial set, define

$$\overline{K} = \bigsqcup_{n \geq 0} K_n \times \Delta^n,$$

then the *geometric realization* of $K$ is $|K| = \overline{K}/\sim$ where $(d_i k_n, x) \sim (k_n, \delta_i x)$ and $(s_i k_n, x) \sim (k_n, \sigma_i x)$.

Observation: every point in $|K|$ has a unique representation $(k_n, x)$ with $k_n$ nondegenerate and $x \in \mathrm{int}\Delta^n$ (i.e. no repeat in $x$) via the equivalence relation defined above.

**Definition 6.** The chain complex associated to $K$ is given by $C_n(k)$, the free abelian group generated by non-singular simplices $x \in K_n$, and $\partial x = \sum_{i=0}^{n} (-1)^i d_i x$.

**Example 5.** The standard $n$-simplex $\Delta^n$ is the simplicial set represented by $[n]$, i.e. $\Delta^n \simeq \mathrm{Hom}_\Delta(-, [n])$. More explicitly, $(\Delta^n)_k$ = non-decreasing sequence of length $k+1$ in $\{0, \cdots, n\}$.

0-simplices: $0, 1, \cdots, n$.

1-simplices: $\{0,0\}, \{0,1\}, \cdots, \{n,n\}$, where $\{i,j\}$ is nondegenerate if $i < j$.

2-simplices: $\{i,j,k\}_{0 \leq i \leq j \leq k \leq n}$, nondegenerate if $i < j < k$.

$\delta_i : \Delta^{n-1} \to \Delta^n$ repeats $t_i$ and $\sigma_i : \Delta^{n+1} \to \Delta^n$ skips $t_{i+1}$.

The geometric realization of $\Delta^n$ is the geometric $n$-simplex $\Delta^n$.

**Example 6.** Let $K$ be an ordered simplicial complex with vertex set $V$. We can construct a simplicial set $\tilde{K}$ by setting $\tilde{K}[n]$ = multi-subset of size $n+1$ in $V$ such that the image in $V$ is an $n$-simplex in $K$, and $\tilde{K}(d^i)$ is deleting the $i$-th term, and $\tilde{K}(s_j)$ is repeating the $j$-th term. Then $\left|\tilde{K}\right| = |K|$.

*Remark* 3. Geometric realization is not representable, but it's the homotopy colimit of representable functors.

**Example 7.** Let $X$ be a topological space, define $\operatorname{Sing}(X)_n := \operatorname{Hom}(\Delta^n, X)$, and $d_i$ is precomposing with $\Delta^{n-1} \xrightarrow{d_i} \Delta^n$, and $s_j$ is precomposing with $\Delta^{n+1} \xrightarrow{s_j} \Delta^n$.

**Theorem 3.** *We have an adjoint pair of functors* $|-| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : Sing$ .

**Definition 7.** Let $\mathcal{C}$ be a small category, the *nerve* of $\mathcal{C}$ is a simplicial set given by $(\mathcal{N}\mathcal{C})_n = \mathbf{Fun}([n], \mathcal{C})$ where we consider $[n]$ as a category with objects $\{0, \cdots, n\}$ and orders as the unique morphisms between any two objects. Then the 0-simplices are the objects, 1-simplices are morphisms, 2-simplices are pairs of composable morphisms, and etc. The face maps are given by composing morphisms, and degeneracy maps are given by adding identity morphism.

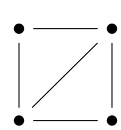
**Example 8.** Let $\mathcal{P}$ be a poset, then $\mathcal{N}\mathcal{P}$ is the maximal simplicial complex on $\mathcal{P}$.

**Example 9.** Let $G$ be a discrete group, then $|\mathcal{N}G| = BG$.

**Example 10.** Let $K, L$ be simplicial sets, the product of $K$ and $L$ is defined as $(K \times L)_n = K_n \times L_n$ with component-wise face maps and degeneracy maps.

Consider $K = L = \Delta^1$, the 0-simplices are $\{0, 1\}$, 1-simplices $\{00, 01, 11\}$, 2-simplices $\{000, 001, 011, 111\}$ and etc. The only non-degenerate simplices are $\{0, 1, 01\}$.

Then for $\Delta^1 \times \Delta^1$, the 0-simplices are $\{(0,0), (0,1), (1,0), (1,1)\}$, corresponding to the four vertices in the following graph. 1-simplices are $\{(00,00), (00,11), (11,00), (11,11), (00,01), (01,00), (01,01), (10,00), (11,01)\}$ where the last five are non-degenerate, and they corresponds to the five edges in the graph. There are 16 2-simplices, 2 of which are non-degenerate, corresponding to the two filling triangles.



**Definition 8.** $\operatorname{Hom}(K, L)_n := \operatorname{Hom}(K \times \Delta^n, L)$.

**Theorem 4.** *Let $K, L$ be simplicial sets, then $|K \times L| = |K| \times |L|$.*

*Proof.* We have a map

$$\begin{array}{rccc} \eta: & |K \times L| & \longrightarrow & |K| \times |L| \\ & (k_n \times l_n, x) & \longmapsto & (k_n, x) \times (l_n, x) \end{array}$$

by the universal property of product.

Observe that for any $n$-simplex $x \in K_n$, there is a unique way of writing it as $x = s_{i_1} \cdots s_{i_m} \tilde{x}$ where $x \in K_{n-m}$ is nondegenerate and $i_1 > \cdots > i_m$. To see this, for $x : \Delta^n \to K$, an $m$-simplex in $\Delta^n$ is $a = \{a_0, \cdots, a_m\}$ a nondecreasing $m$-tuples ranging from 0 to $n$. To get $\tilde{a}$ we need to remove all repeats, and to get all $s$'s we need to count the repeated elements. For example, if $a = 000112$, then $\tilde{a} = 012$ and $a = s_4 s_1 s_0 \tilde{a}$. So we can write $k_n = s_{i_1} \cdots s_{i_m} k_{n-m}, i_1 > \cdots > i_m$ and $l_n = s_{j_1} \cdots s_{j_p} l_{n-p}, j_1 > \cdots > j_p$.

The inverse map is given as follows. Given $(k_a, x) \times (l_b, y) \in |K| \times |L|$ with $k_a, l_b$ nondegenerate and $x, y$ no repeats, write

$$\begin{aligned} x &= (u_0, \cdots, u_{a+1}) \\ y &= (v_0, \cdots, v_{b+1}) \\ z &= (w_0, \cdots, w_{n+1}) \end{aligned}$$

where $z$ is the sorted list of $u$'s and $v$'s. Suppose $\mu_1 < \cdots < \mu_{n-a}$ gives the indices which are not $v$'s and $\gamma_1 < \cdots < \gamma_{n-b}$ are the indices which are not $u$'s. Then we can write $x = \sigma_{\mu_1} \cdots \sigma_{\mu_{n-a}} z$ and $y = \sigma_{\gamma_1} \cdots \sigma_{\gamma_{n-b}} z$. Define

$$\eta^{-1}\left((k_a, x) \times (l_b, y)\right) = \left(s_{\mu_{n-a}} \cdots s_{\mu_1} k_a \times s_{\gamma_{n-b}} \cdots s_{\gamma_1} l_b, z\right).$$

□

*Remark* 4. Left adjoints do not commute with products in general.

*Remark* 5. This is not true in **Top**. Products of CW complexes are not necessarily CW complexes (with the product topology).

*Remark* 6. In **sSet**, we have the adjoint pairs

$$(A \times B \to C) \dashrightarrow \left(A \to C^B\right)$$

since this is pure combinatorial. This is not true in **Top**. However, if we restrict to compactly generated Hausdorff spaces, it works.

**Other Constructions with Simplicial Sets**

- Base point $\mathbf{sSet}_*$. $* \in K_0$ has a unique image in every $K_n, \forall n$. Picking a base point in a simplicial set is equivalent to defining a functor $\Delta^{op} \to \mathbf{Set}_*$.

- Let $K, L \in Ob\left(\mathbf{sSet}_*\right)$, define $K \vee L = K \sqcup L / *_K \sim *_L$ gluing along the base point. Then $\left(K \vee L\right)_n = K_n \vee L_n$ and $|K \vee L| \cong |K| \vee |L|$.
- Let $K, L$ be two simplicial sets with pointwise inclusion $K \hookrightarrow L$ . Then $\left(L/K\right)_n = L_n/K_n$ and $|L/K| \cong |L| \,/\, |K|$. (Quotient is a left functor.)
- Above constructions gives $\left(K \wedge L\right)_n = K_n \wedge L_n = K_n \times L_n / \left(K_n \vee L_n\right)$. Then $|K \wedge L| = |K \times L / K \vee L| \cong |K \times L| \,/\, |K \vee L| \cong |K| \times |L| \,/\, |K| \vee |L| = |K| \wedge |L|$.

**$S^n$ in sSet**

1. $\Delta^n/\partial\Delta^n$ where $\partial\Delta^n$ is the maximal simplicial subset of $\Delta^n$ with no nondegenerate simplices of dimension $n$.
2. $S^n \cong \underbrace{S^1 \wedge S^1 \wedge \cdots \wedge S^1}_{n}$. There is a natural $\sigma_n$-action on $S^n$ induced by permutation in $n$-elements.

## 1.1 Kan Complexes

The reference for this section is [EC71] Chapter 1.8-1.16.

**Notation:** The standard $n$-simplex is denoted by $\Delta\left[n\right]$ in [EC71] and $\Delta^n$ in [GJ99]. The $k$-th horn is denoted by $\Lambda^k\left(n\right)$ in in [EC71] and $\Lambda^n_k$ in [GJ99].

**Definition 9.** A *simplicial map* $f : K \to L$ is a natural transformation of functors (preserving the relevant structures). In other words, $f = \{f_n : K_n \to L_n\}_{n \geq 0}$ such that

$$\begin{cases} d_i f_n = f_{n-1} d_i \\ s_j f_n = f_{n+1} s_j \end{cases}$$

**Definition 10.** A simplicial homotopy from $f : K \to L$ to $g : K \to L$ is a map $H : K \times \Delta^1 \to L$ such that

$$\begin{tikzcd} K \times \Delta^0 \arrow[r, "\mathrm{Id}\times d^0"] \arrow[dr, "f"'] & K \times \Delta^1 \arrow[d, "H"] & K \times \Delta^0 \arrow[l, "\mathrm{Id}\times d^1"'] \arrow[dl, "g"] \\ & L & \end{tikzcd}$$

commutes.

*Remark* 7. The structure maps $d_i$ in a simplicial set induced by $d^i : [n-1] \to [n]$ in $\Delta$ defines a functor

$$\begin{array}{cccc} \Delta^* : & \Delta & \longrightarrow & \mathbf{Top} \\ & [n] & \longmapsto & \Delta^n \\ & \left(d^i : [n-1] \to [n]\right) & \longmapsto & \left(d_i : \Delta^{n-1} \to \Delta^n\right) \end{array}$$

Note that this is not equivalence relation, and even if we consider the equivalence relation generated by simplicial homotopies, it is not convenient enough if we want to describe equivalent maps (using zigzags of homotopies).

**Example 11.** Let $K = \Delta^0$ and $f = d^0 : \Delta^0 \to \Delta^1$ (source) and $g = d^1 : \Delta^0 \to \Delta^1$ (target),

$$f\bullet \longrightarrow \bullet g$$

then we have

$$\begin{tikzcd} \Delta^0 \times \Delta^0 \arrow[r, "\mathrm{Id}\times d^0"] \arrow[dr, "f"'] & \Delta^0 \times \Delta^1 \arrow[d, "\mathrm{pr}_2"] & \Delta^0 \times \Delta^0 \arrow[l, "\mathrm{Id}\times d^1"'] \arrow[dl, "g"] \\ & \Delta^1 & \end{tikzcd}$$

the homotopy from $f$ to $g$ is given by $H = \mathrm{pr}_2 : \Delta^0 \times \Delta^1 \to \Delta^1$, but there is no homotopy from $g$ to $f$.

We want to work with simplicial sets such that the above homotopy is an equivalence relation. It turns out that Kan complexes makes things work.

**Definition 11.** The *k-th horn* $\Lambda_k^n$ is the simplicial set containing all faces of $\Delta^n$ except $d_k\left(\Delta^{n-1}\right)$.

**Example 12.** $\Lambda_0^0 = \emptyset$, $\Lambda_0^1 = \Lambda_1^1 = \{*\}$.

$$\Lambda_0[2] : \begin{tikzcd} & \bullet \arrow[dl] \arrow[dr] & \\ \bullet & & \bullet \end{tikzcd} \qquad \Lambda_1[2] : \begin{tikzcd} & \bullet \arrow[dl] & \\ \bullet \arrow[rr] & & \bullet \end{tikzcd} \qquad \Lambda_2[2] : \begin{tikzcd} & \bullet \arrow[dr] & \\ \bullet \arrow[rr] & & \bullet \end{tikzcd}$$

$$\partial\Delta[2]_* : \begin{tikzcd} & \bullet \arrow[dl] \arrow[dr] & \\ \bullet \arrow[rr] & & \bullet \end{tikzcd}$$

**Definition 12.** A Kan complex is a simplicial set $X$ such that for any $n \geq 0$ and any $0 \leq k \leq n$, there exists an extension

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\text{extension}"'] & \end{tikzcd}$$

*Remark* 8. Kan complexes are good to map into.

**Aside.** Given maps $f_0, \cdots, \hat{f}_i, \cdots, f_n : \Delta^{n-1} \to K$, these assemble to

$$\bigcup_{i \neq k} d_i \left(\Delta^{n-1}\right) = \Lambda^n_k \to K$$

if $d_i f_j = d_{j-1} f_i, \forall i < j, i \neq k, j \neq k$.

**Proposition 1.** *[[EC71] Cor 1.16] Let $L, K$ be simplicial sets. If $K$ is a Kan complex then homotopy is an equivalence relation on maps $L \to K$.*

*Proof.* **Reflexitivity**: $f \sim f$. $H_f = f \times \mathrm{pr}_1 = f \times s_0$.

$$\begin{tikzcd} L \times \Delta^0 \arrow[r, "\mathrm{Id} \times d_0"] \arrow[dr, "f"'] & L \times \Delta^1 \arrow[d, "f \times \mathrm{pr}_1"] & L \times \Delta^0 \arrow[l, "\mathrm{Id} \times d_1"'] \arrow[dl, "f"] \\ & K & \end{tikzcd}$$

**Symmetry**: $f \sim g \implies g \sim f$. Suppose we have a homotopy $H : L \times \Delta^1 \to K$ from $f$ to $g$. We also have a homotopy $H_f : L \times \Delta^1 \to K$ from $f$ to $f$. Note that $f = d_0 s_0 f = d_1 s_0 f$, so we obtain a map $\hat{H} : L \times \Lambda^2_0 \to K$. Proposition 2 tells us that this map can be extended to $\tilde{H} : L \times \Delta^2 \to K$ and the composition

$$H' : L \times \Delta^1 \xrightarrow{\mathrm{Id} \times s_0} L \times \Delta^2 \xrightarrow{\tilde{H}} K$$

gives a homotopy from $g$ to $f$.

**Transitivity**: $f \sim g, g \sim h \implies f \sim h$. Consider the standard 2-simplex $\Delta^2$, if we have $f, g, h : L \to K$ with two homotopies $H_1, H_2 : L \times \Delta^1 \to K$ where $H_1$ is a homotopy from $f$ to $g$ and $H_2$ is a homotopy from $g$ to $h$, then we can glue them together to form a map $H : L \times \Lambda^2_1 \to K$. Proposition 2 tells us that this map can be extended to $\tilde{H} : L \times \Delta^2 \to K$ and the composition

$$H' : L \times \Delta^1 \xrightarrow{\mathrm{Id} \times s_1} L \times \Delta^2 \xrightarrow{\tilde{H}} K$$

gives a homotopy from $f$ to $h$. □

**Definition 13.** The $n$-th skeleton of $L$, $L^{(n)}$ is the smallest simplicial subset of $L$ containing all nondegenerate simplices of dimension $\leq n$.

$$L = \operatorname*{colim}_n L^{(n)} = \bigcup_n L^{(n)}.$$

**Proposition 2.** *For any $m, k$, if $K$ is a Kan complex, given $L \times \Lambda^m_k \to K$ there exists an extension*

$$\begin{tikzcd}
L \times \Lambda^m_k \arrow[r] \arrow[d, hook] & X \\
L \times \Delta^m \arrow[ur, dashed, "extension"'] &
\end{tikzcd}$$

*Proof.* We can prove Proposition 2 by induction on the skeletons $L^{(n)}$ of $L$.

$n = 0$. $L^{(0)}$ is the set of vertices in $L$, so the result is trivial.

Assume that this is true for $n-1$. Consider the following diagram

$$\begin{tikzcd}
L^{(n-1)} \times \Lambda^m_k \arrow[rr] \arrow[d, hook] \arrow[dr] & & K \\
L^{(n-1)} \times \Delta^m \arrow[urr, dashed] \arrow[dr] & L^{(n)} \times \Lambda^m_k \arrow[ur] \arrow[d, hook] & \\
& L^{(n)} \times \Delta^m \arrow[uur, dashed] &
\end{tikzcd}$$

If $X \in L^{(n)}$ is degenerate, then $X \in L^{(n-1)}$ (in which case the map is defined on $X$) or $X = s_{i_1} \cdots s_{i_k} X', X' \in L^{(n-k)}$. Note that for any simplicial map $f : X \to Y$, if $x = s_i y$ then $f(x) = s_i f(y)$, so value of $f$ depends only on its value on nondegenerate simplices. Hence if we want to extend the above diagram, we only need to worry nondegenerate simplices in $L^{(n)} \times \Delta^m$ that are not in $\left(L^{(n-1)} \times \Delta^m\right) \cup \left(L^{(n)} \times \Lambda^m_k\right)$. We can do this for one simplex at a time. We want to show given (1)(2)(3), there exists an extension (4).

$$\begin{tikzcd}
\partial\Delta^n \times \Lambda^m_k \arrow[rr, "(1)"] \arrow[d, hook] \arrow[dr] & & K \\
\partial\Delta^n \times \Delta^m \arrow[urr, dashed, "(2)"] \arrow[dr] & L^{(n)} \times \Lambda^m_k \arrow[ur, "(3)"] \arrow[d, hook] & \\
& L^{(n)} \times \Delta^m \arrow[uur, dashed, "(4)"] &
\end{tikzcd}$$

This follows from the following lemma. $\square$

**Lemma 1.** *[[EC71] 1.14] Let $A \subset B$ denote any of the following pairs:*

$$\Delta^n \times \Lambda_k^m \subseteq \Delta^n \times \Delta^m$$
$$(\Delta^n \times \Lambda_k^m) \cup (\partial\Delta^n \times \Delta^m) \subseteq \Delta^n \times \Delta^m$$

*then for any Kan complex $K$ and simplicial map $f : A \to K$, $f$ can be extended to a map $g : B \to K$.*

*Proof.* For such a pair $A \subset B$, $B$ can be obtained from $A$ by iteratively attaching simplex and one of its faces, while the other faces are already in $A$. Hence this follows from applying Kan extension condition repeatedly. The explicit construction is as follows.

Note that for any simplicial map $f$, its value only depends only on the value on nondegenerate simplices. Hence we only need to extend $f$ by defining on nondegenerate simplices in $B$ that is not in $A$.

There are only finite nondegenerate simplices in $B$. For each nondegenerate $j$-simplex $(x, y)$ in $B$ but not in $A$, we define $g_j$ on $(x, y)$ as follows.

If $i \neq k$, $(d_i x, d_i y) \in (\Delta^n)_{j-1} \times (\Lambda_k^m)_{j-1} \subseteq A_{j-1}$, so we can define $z_i = f_{j-1}(d_i x, d_i y) \in K_{j-1}$. Furthermore

$$d_l z_i = d_l f_{j-1}(d_i x, d_i y) = f_{j-2}(d_l d_i x, d_l d_i y) = f_{j-2}(d_{i-1} d_l x, d_{i-1} d_l y) = d_{i-1} f_{j-1}(d_l x, d_l y) = d_{i-1} z_l =, \forall l < i, i \neq k, l \neq k$$

so by Kan extension property, there exists a simplex $z \in K_j$ such that $d_i z = z_i, i \neq k$. So we can extend $f_j : A_j \to K_j$ to $(x, y)$ by defining $(x, y) \mapsto z$, then

$$d_i g_j(x, y) = d_i z = z_i = f_{j-1}(d_i x, d_i y) = g_{j-1}(d_i x, d_i y).$$

If $i = k$, $(d_k x, d_k y) \in (\partial\Delta^n \times \Delta^m)$, the construction is similar to above one. □

**Corollary 1.** *[[EC71] 1.17] If $K$ is a Kan complex, so is $K^L := \left\{ \left(K^L\right)_n = Hom(L \times \Delta^n, K) \right\}_{n \geq 0}$.*

## 1.2 Simplicial Homotopy Groups

### Examples of Kan Complexes

**Example 13.** If $X : \Delta^{op} \to \mathbf{Grp} \xrightarrow{U} \mathbf{Set}$ is a simplicial group, then $X$ is a Kan complex.

*Claim* 1. Let $Y$ be a topological space then $\mathrm{Sing}(Y)$ is a Kan complex.

*Proof.* Given

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "f"] \arrow[d, hook] & \mathrm{Sing}(Y) \\ \Delta^n \arrow[ur, dashed, "\gamma_g"'] & \end{tikzcd}$$

we want to show the existence of $\gamma_g$.

The existence of $f : \Lambda^n_k \to \mathrm{Sing}(Y)$ is equivalent to $\left(f_0, \cdots, \hat{f}_k, \cdots, f_n\right)$ such that $d_i f_j = d_{j-1} f_i$ for $i < j$ and $i, j \neq k$.

$$\begin{tikzcd} & \Delta^{n-1} \arrow[dr, "f_i"] & \\ \Delta^{n-2} \arrow[ur, "d_{j-1}"] \arrow[dr, "d_j"'] & & \mathrm{Sing}(Y) \\ & \Delta^{n-1} \arrow[ur, "f_j"'] & \end{tikzcd}$$

Consider the geometric realization

$$\begin{tikzcd} \left|\Lambda^n_k\right| \arrow[r, "f"] \arrow[d, hook, shift left] & Y \\ \Delta^n \arrow[u, dashed, shift left] \arrow[ur, dashed, "\gamma_g"'] & \end{tikzcd}$$

we can define $\gamma_g : \Delta^n \to Y$ by pushing into the $k$-th face.

$\square$

**Definition 14.** For any $y \in X_n$, we can get $\gamma_y : \Delta^n \to X$ which maps the unique nondegenerated $n$-simplex to $y$.

**Definition 15.** [GJ99] called Kan complexes "fibrant simplicial sets".

**Lemma 2.** *The functor $|-|$ and Sing are well-defined on homotopy classes of maps.*

*Proof.* Since geometric realization commutes with products,

$$\begin{tikzcd} X \times \Delta^0 \arrow[r, "\mathrm{Id} \times d^0"] \arrow[dr, "f"'] & X \times \Delta^1 \arrow[d, "H"] & X \times \Delta^0 \arrow[l, "\mathrm{Id} \times d^1"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

becomes

$$\begin{tikzcd} |X| \times \{*\} \arrow[r] \arrow[dr, "f"'] & |X| \times I \arrow[d, "H"] & |X| \times \{*\} \arrow[l] \arrow[dl, "g"] \\ & |Y| & \end{tikzcd}$$

For Sing, the diagram needs to be modified since $\mathrm{Sing}(I)$ is much bigger than $\Delta^1$.

$$\mathrm{Sing}(X)\times\Delta^1 \xrightarrow{\mathrm{Id}\times\eta_{\Delta^1}} \mathrm{Sing}(X)\times\mathrm{Sing}\left(\left|\Delta^1\right|\right)\cong\mathrm{Sing}(X)\times\mathrm{Sing}(I)\cong\mathrm{Sing}(X\times I)\xrightarrow{\mathrm{Sing}(H)}\mathrm{Sing}(Y)$$

where $\eta_{\Delta^1}:\Delta^1\to\mathrm{Sing}\left(\left|\Delta^1\right|\right)$ is given by the unit of adjunction. □

*Remark* 9. Homotopy into $Y$ is not necessary an equivalence relation if $Y$ is not a Kan complex. Although we can consider the equivalence relation generated by homotopies, this is not good enough.

**Definition 16.** [[EC71] 1.18] Let $K$ be a Kan complex. Given two simplices $x,y\in K_n$, $x\simeq y$ if $\gamma_y$ is homotopic to $\gamma_x$ rel to their boundaries.

**Proposition 3.** *Let Let $K$ be a Kan complex. Given two simplices $x,y\in K_n$, $x\simeq y$ if and only if $d_ix=d_iy,\forall i$ and for some $0\le k\le n$ there exists $w\in K_{n+1}$ such that $d_kw=x,d_{k+1}w=y$ and $d_iw=d_is_kx=d_is_ky,k\ne i\ne k+1$.*

**Definition 17.** Let $K$ be a Kan complex. $*\in K$ The simplicial homotopy group $\pi_n(K,*)=$ equivalence classes of $n$-simplices $x\in K_n$ such that $d_ix=*,\forall i$.

**Part II**

# Model Categories

# Part III

# Spectrum

## 2 Homotopy Groups and Stable Homotopy Groups

### 2.1 Homotopy Groups

**Question.** Where does spectra come from?

In the study of homotopy groups, we define the $n$-th homotopy group of a topological space $X$ as the set of homotopy classes

$$\pi_n(X) = [\mathbb{S}^n, X]_*, n \geq 0$$

which is a group when $n \geq 1$ and abelian when $n \geq 2$.

The group structure follows from

$$[\mathbb{S}^n, X]_* \times [\mathbb{S}^n, X]_* \cong [\mathbb{S}^n \vee \mathbb{S}^n, X]_*$$

and the map

$$\mathbb{S}^n \to \mathbb{S}^n \vee \mathbb{S}^n$$

given by collapsing equator.

Note that even if we have $\pi_n(\mathbb{S}^n) = \mathbb{Z} = H_n(\mathbb{S}^n)$, homotopy groups and homology groups are not the same, for instance, $\pi_3(\mathbb{S}^2) = \mathbb{Z}$ but $H_3(\mathbb{S}^2) = 0$.

Nevertheless, there are still similarities between them.

In homology theory, if we have a good sequence (excision)

$$A \hookrightarrow X \twoheadrightarrow X/A$$

where $X/A$ is the pushout

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & X \arrow[d] \\ \bullet \arrow[r] & X/A \end{tikzcd}$$

then it induces a long exact sequence in homology groups $H_*$. The proof, roughly speaking, contains three steps.

1. construct the relative homology group $H_*(X,A)$ and prove the long exact sequence for relative homology group.
2. prove that $H_*(X,A) \cong H_*(X/A)$.
3. show excision, if $X = A \bigcup B$ then $H_*(A, A\cap B) \cong H_*(X,B)$.

Similarly, in homotopy groups, if we have a fibre sequence

$$F \longrightarrow E \longrightarrow B$$

then it induces a long exact sequence in homotopy groups $\pi_*$.

**Definition 18.** A topological space $X$ is called *$n$-connected* if $\pi_i(X) = 0, \forall i \leq n$.

If a CW pair $(X,A)$ is $n$-connected, then $\pi_i(X,A) = 0, \forall i \leq n$, consider the long exact sequence

$$\cdots \longrightarrow \pi_i(X) \longrightarrow \pi_i(X,A) \longrightarrow \pi_{i-1}(A) \longrightarrow \cdots$$

we have the following result.

**Theorem 5.** *[Blakers-Massey] [BM51] Let $X$ be a CW complex such that $X = A\bigcup B$ and $A\cap B$ is nonempty and connected. If $(A, A\cap B)$ is $m$-connected and $(B, A\cap B)$ is $n$-connected, then*

$$\pi_i(A, A\cap B) \longrightarrow \pi_i(X,B)$$

*is an isomorphism for $i < m+n$ and surjective for $i = m+n$.*

## 2.2 Stable Homotopy Group

In general homotopy groups are difficult to compute.

**Moral.** Suspension makes it better.

**Conclusion.** Suspend infinitely many times.

***Naive Stable Homotopy Group***

**Definition 19.** The *naive stable homotopy group* of a topological space$X$ is $\pi_i^S(X) := \operatorname{colim}_n \left(\mathbb{S}^{n+i}, \Sigma^n X\right)$.

$$\pi_i(X) \longrightarrow \pi_{i+1}(\Sigma X) \longrightarrow \pi_{i+2}\left(\Sigma^2 X\right) \longrightarrow \cdots$$

We have the following result for CW complexes.

**Theorem 6.** *If $X$ is a finite CW complex, then this colimit stabilizes ($n \geq 2\dim X + 2$, which is not a minimal bound).*

**Open question [Ker].** The *Kervaire invariant* is an invariant of a framed $(4k+2)$-dimensional manifold that measures whether the manifold could be surgically converted into a sphere. This invariant evaluates to 0 if the manifold can be converted to a sphere, and 1 otherwise. The Kervaire invariant problem is the problem of determining in which dimensions the Kervaire invariant can be nonzero. For differentiable manifolds, this can happen in dimensions $2, 6, 14, 30, 62$, and possibly 126, and in no other dimensions. The final case of dimension 126 remains open.

***Stable Homotopy Group***

The intuition comes from the following theorem.

**Theorem 7.** *(Brown Representability) For every cohomology theory $\{h^n : \mathbf{Top}^{op} \to \mathbf{Ab}\}_{n\in\mathbb{Z}}$ there exists a sequence of spaces*

$$E \coloneqq \{\cdots, E_{-2}, E_{-1}, E_0, E_1, E_2, \cdots\}$$

*such that $h^n(X) = [X, E_n]$ and*

$$E_n \simeq \Omega E_{n+1}, \quad n \geq 0$$
$$E_{-n} = \Omega^n E_0, \quad n > 0$$

A map $f : E \to F$ is a sequence of maps $\{f_n : E_n \to F_n\}_{n\in\mathbb{Z}}$ such that

$$\begin{tikzcd} E_n \arrow[r, "f_n"] \arrow[d] & F_n \arrow[d] \\ \Omega E_{n+1} \arrow[r, "\Omega f_{n+1}"] & F_{n+1} \end{tikzcd}$$

**Warning.** There exists phantom maps $E \xrightarrow{f} F$ such that $[-, E] \xrightarrow{f_*} [-, F]$ is zero, but $f \not\simeq$ const, i.e. $h^* : \mathbf{Top}^{op} \to \mathbf{Ch}_{\mathbb{Z}}$ is not a faithful functor, but it is full.

Note that $X \to \Omega Y$ corresponds to $\Sigma X \to Y$, so given a sequence of spaces $\{X_0, X_1, \cdots, X_n, \cdots\}$ and maps

$\Sigma X_n \to X_{n+1}$, we can define the *stable homotopy groups* as the colimit of the following diagram

$$\begin{tikzcd} \Sigma S^m \arrow[r, "\Sigma f"] \arrow[d] & \Sigma X_n \arrow[d] \\ S^{m+1} \arrow[r, "f"] & X_{n+1} \end{tikzcd}$$

**Definition 20.** A *spectrum* is a sequence of spaces $X = \{X_0, X_1, \cdots, X_n, \cdots\}$ together with maps $\sigma = \{\sigma_n : \Sigma X_n \to X_{n+1}\}_{n\geq 0}$. A *map between spectra* is a sequence of maps $f = \{f_n : X_n \to Y_n\}_{n\geq 0}$ such that

$$\begin{tikzcd} \Sigma X_n \arrow[r, "\Sigma f_n"] \arrow[d, "\sigma_n"'] & \Sigma Y_n \arrow[d, "\sigma'_n"] \\ X_{n+1} \arrow[r, "f_{n+1}"] & Y_{n+1} \end{tikzcd}$$

commutes. The *n-th stable homotopy* of a spectrum $X$ is $\pi_n(X) \coloneqq \mathrm{colim}_k \left[\mathbb{S}^{n+k}, X_k\right], n \in \mathbb{Z}$.

Note when $n < 0$, the colimit is taking from $k = -n$.

Write **Sp** the category of spectra.

A map $f : X \to Y$ is a *stable equivalence* if $\pi_n f$ is an isomorphism for any $n \in \mathbb{Z}$.

A spectrum is a *suspension spectrum* if $\sigma_n : \Sigma X_n \xrightarrow{\sim} X_{n+1}$ is a weak equivalence for any $n \in \mathbb{Z}$.

An $\Omega$-*spectrum* is a spectrum $X$ such that $\sigma'_n : X_n \to \Omega X_{n+1}$ is a weak equivalence for any $n \in \mathbb{Z}$.

**Observation.** There is a Quillen pair

$$\Sigma^\infty : \mathbf{Top}_* \rightleftarrows \mathbf{Sp} : \Omega^\infty$$

where

$$\begin{aligned} \Sigma^\infty X &= \left\{X, \Sigma X, \Sigma^2 X, \cdots\right\} \\ \Omega^\infty Y &= \mathrm{colim}_n \Omega^n Y_n \end{aligned}$$

# 3 Symmetric Spectra

The main reference for this part ARE [Sch12] and [EKMM97], the second of which is quite technical.

The space are always compactly generated weak Hausdorff spaces.

## 3.1 Smash Product

**Observation.** In the category of pointed spaces, we have the functors $\pi_k : \mathbf{Top}_* \to \mathbf{Gr}$, and we work with the reduced (co)homology $\tilde{H}^*$ and $\tilde{H}_*$.

In **Top**, we have the adjoint pair $K \times \dashv (-)^K$.

In $\mathbf{Top}_*$, we need to adjust this pair a little bit so that it works.

**Definition 21.** The smash product of two pointed spaces $(X, *)$ and $(Y, \bullet)$ is

$$X \wedge Y := \frac{X \times Y}{X \times \{\bullet\} \cup \{*\} \times Y}$$

a pointed space.

**Theorem 8.** *The reduced homology of $X \wedge Y$ satisfies*

$$\tilde{H}^*(X \wedge Y; R) \cong \tilde{H}^*(X; R) \otimes \tilde{H}^*(Y; R).$$

**Example 14.**

1. $\mathbb{S}^n \wedge \mathbb{S}^m = \mathbb{S}^{n+m}$.

2. $\mathbb{S}^1 \wedge X = \Sigma X$.

## 3.2 Symmetric Spectra

**Definition 22.** A *symmetric spectrum* consists of the following data:

- a sequence of pointed spaces $X = \{X_0, X_1, \cdots, X_n, \cdots\}$
- a basepoint preserving continuous left action of the symmetric group $\Sigma_n$ on $X_n$, for each $n \geq 0$
- based maps $\sigma_n : X_n \wedge \mathbb{S}^1 \to X_{n+1}$ which is equivariant with respect to $\Sigma_n$-action, for $n \geq 0$.

satisfying that for any $n, m \geq 0$, the composite

$$X_n \wedge \mathbb{S}^m \xrightarrow{\sigma_n \wedge 1} X_{n+1} \wedge \mathbb{S}^{m-1} \xrightarrow{\sigma_{n+1} \wedge 1} \cdots \xrightarrow{\sigma_{n+m-2} \wedge 1} X_{n+m-1} \wedge \mathbb{S}^1 \xrightarrow{\sigma_{n+m-1}} X_{n+m}$$

is $\Sigma_n \times \Sigma_m$-equivariant.

Here the symmetric group $\Sigma_m$ acts by permuting the coordinates of $\mathbb{S}^m$, and $\Sigma_n \times \Sigma_m$ acts by restriction of the $\Sigma_{n+m}$-action.

We write $\sigma_n^m : X_n \wedge \mathbb{S}^m \to X_{n+m}$ for the map that applies the structure map $\sigma_n : R_n \wedge \mathbb{S}^1 \to R_{n+1}$ $m$ times.

A morphism $f : X \to Y$ between symmetric spectra consists of $\Sigma_n$-equivariant based maps $\{f_n : X_n \to Y_n\}_{n\geq 0}$ which are compatible with the structure maps in the sense that the following diagram

$$\begin{tikzcd} X_n \wedge \mathbb{S}^1 \arrow[r, "f_n \wedge 1"] \arrow[d, "\sigma_n"'] & Y_n \wedge \mathbb{S}^1 \arrow[d, "\sigma_n"] \\ X_{n+1} \arrow[r, "f_{n+1}"] & Y_{n+1} \end{tikzcd}$$

commutes.

**Definition 23.** A *symmetric ring spectrum* $R$ is a symmetric spectrum with

- $\Sigma_n \times \Sigma_m$-equivariant multiplication maps $\mu_{n,m} : R_n \wedge R_m \to R_{n+m}$ for $n, m \geq 0$, and
- unit maps $\iota_0 : \mathbb{S}^0 \to R_0$ and $\iota_1 : \mathbb{S}^1 \to R_1$.

satisfying

- associativity: the square

$$\begin{tikzcd} R_n \wedge R_m \wedge R_p \arrow[r, "1\wedge\mu_{m,p}"] \arrow[d, "\mu_{n,m}\wedge 1"'] & R_n \wedge R_{m+p} \arrow[d, "\mu_{n,m+p}"] \\ R_{n+m} \wedge R_p \arrow[r, "\mu_{n+m,p}"] & R_{n+m+p} \end{tikzcd}$$

  commutes for all $n, m, p \geq 0$.

- unit: the two compositions

$$R_n \xrightarrow{\cong} R_n \wedge \mathbb{S}^0 \xrightarrow{\mathrm{Id}\wedge\iota_0} R_n \wedge R_0 \xrightarrow{\mu_{n,0}} R_n$$

$$R_n \xrightarrow{\cong} \mathbb{S}^0 \wedge R_n \xrightarrow{\iota_0\wedge 1} R_0 \wedge R_0 \xrightarrow{\mu_{0,n}} R_n$$

  are identities for all $n \geq 0$.

- centrality: the diagram

$$\begin{tikzcd} R_n \wedge \mathbb{S}^1 \arrow[r, "1\wedge\iota_1"] \arrow[d, "\text{twist}"'] & R_n \wedge R_1 \arrow[r, "\mu_{n,1}"] & R_{n+1} \arrow[d, "\chi_{n,1}"] \\ \mathbb{S}^1 \wedge R_n \arrow[r, "\iota_1\wedge 1"] & R_1 \wedge R_n \arrow[r, "\mu_{1,n}"] & R_{1+n} \end{tikzcd}$$

  commutes for all $n, m \geq 0$. Here $\chi_{n,m} \in \Sigma_{n+m}$ denotes the shuffle permutation which moves the first $n$ elements past the last $m$ elements, keeping each of the two blocks in order.

A morphism $f : R \to S$ of symmetric ring spectra consists of $\Sigma_n$-equivariant based maps $\{f_n : R_n \to S_n\}_{n\geq 0}$, which are compatible with the multiplication and unit maps in the sense that

$$\begin{tikzcd} R_n \wedge R_m \arrow[r, "f_n\wedge f_m"] \arrow[d, "\mu_{n,m}"'] & S_n \wedge S_m \arrow[d, "\mu_{n,m}"] \\ R_{n+m} \arrow[r, "f_{n+m}"] & S_{n+m} \end{tikzcd}$$

for any $n, m \geq 0$, and the two diagrams

$$\begin{tikzcd} \mathbb{S}^0 \arrow[r, "\iota_0"] \arrow[dr, "\iota_0"'] & R_0 \arrow[d, "f_0"] \\ & S_0 \end{tikzcd} \qquad \begin{tikzcd} \mathbb{S}^1 \arrow[r, "\iota_1"] \arrow[dr, "\iota_1"'] & R_1 \arrow[d, "f_1"] \\ & S_1 \end{tikzcd}$$

commute.

**Definition 24.** A *right module* $M$ over a symmetric ring spectrum $R$ is a spectrum $M = \{M_0, M_1, \cdots, M_n, \cdots\}$ with $\Sigma_n \times \Sigma_m$-equivariant action maps $\alpha_{n,m} : M_n \wedge R_m \to M_{n+m}$ such that the diagrams

$$\begin{tikzcd} M_n \wedge R_m \wedge R_p \arrow[r, "1\wedge\mu_{m,p}"] \arrow[d, "\alpha_{n,m}\wedge \mathrm{Id}"'] & M_n \wedge R_{m+p} \arrow[d, "\alpha_{n,m+p}"] \\ M_{n+m} \wedge R_p \arrow[r, "\alpha_{n+m,p}"] & M_{n+m+p} \end{tikzcd} \qquad \begin{tikzcd} M_n \cong M_n \wedge \mathbb{S}^0 \arrow[r, "1\wedge\iota_0"] \arrow[dr, equal] & M_n \wedge R_0 \arrow[d, "\alpha_{n,0}"] \\ & M_n \end{tikzcd}$$

commute.

We will give an important example of a ring spectrum now.

**Definition 25.** The ring spectrum $\mathbb{S}$ is $\mathbb{S} = \left\{\mathbb{S}^0, \mathbb{S}^1, \mathbb{S}^2, \cdots\right\}$ with $\mu_{m,1} : \mathbb{S}^m \wedge \mathbb{S}^1 \xrightarrow{\cong} \mathbb{S}^{m+1}$.

We will show in next lecture that all symmetric spectra are $\mathbb{S}$-modules.

**Naive Homotopy Group**

**Definition 26.** The $n$-naive homotopy group of a symmetric spectrum $X$ is

$$\hat{\pi}_n(X) := \text{colim}_k \pi_{n+k} X_k, n \in \mathbb{Z}.$$

Note when $n < 0$, the colimit is taken from $k > -n$.

**Example 15.** The $n$-naive homotopy group of spheres is $\hat{\pi}_n \mathbb{S} \text{colim}_k \left(\pi_{k+n} \mathbb{S}^k\right)$, and is denoted $\pi_n^s$.

$$\begin{aligned} \hat{\pi}_0^s &= \text{colim}_k \left(\pi_k \mathbb{S}^k\right) = \mathbb{Z} \\ \hat{\pi}_n^s &= \text{colim}_k \left(\pi_{k+n} \mathbb{S}^k\right) = 0, \quad n < 0. \end{aligned}$$

Since $\mathbb{S}^n$ is $(n-1)$-connected, the group $\pi_n^s$ is trivial for negative values of $n$. The degree of a self-map of a sphere provides an isomorphism $\pi_0^s = \mathbb{Z}$. For $n \geq 1$, the homotopy group $\pi_n^s$ is finite. This is a direct consequence of Serre's calculation of the homotopy groups of spheres modulo torsion, and Freudenthal's suspension theorem.

**Theorem 9.** *[Serre] For $m > n \geq 1$,*

$$\hat{\pi}_m \mathbb{S}^n = \begin{cases} P_{m,n} \oplus \mathbb{Z} & n = 2k, m = 2n-1, \\ P_{m,n} & o.w. \end{cases}$$

*where $P_{m,n}$ is some finite group for any fixed $m, n$.*

**Exercise 1.** $\pi_1^s = \mathbb{Z}/2$.

*Proof.* See [Sch12] P12-13. □

**Example 16.** (Suspension spectra). Every pointed space $K$ gives rise to a suspension spectrum $\Sigma^\infty K$ via

$$\left(\Sigma^\infty K\right)_n = K \wedge \mathbb{S}^n$$

with structure maps given by the canonical homeomorphism

$$(K \wedge \mathbb{S}^n) \wedge \mathbb{S}^1 \xrightarrow{\cong} K \wedge \mathbb{S}^{n+1}.$$

For example, the sphere spectrum $\mathbb{S}$ is isomorphic to the suspension spectrum $\Sigma^\infty \mathbb{S}^0$.

The naive homotopy group

$$\pi_k^s K = \hat{\pi}_k\left(\Sigma^\infty K\right) = \mathrm{colim}_n\left(\pi_{k+m} K \wedge \mathbb{S}^n\right)$$

is called the *k-th stable homotopy group* of $K$. Since $K \wedge \mathbb{S}^n$ is $(n-1)$-connected, the suspension spectrum $\Sigma^\infty K$ is connective, i.e., all homotopy groups in negative dimensions are trivial. The Freudenthal suspension theorem implies that for every suspension spectrum, the colimit system for a specific homotopy group always stabilizes. A symmetric spectrum $X$ is isomorphic to a suspension spectrum (necessarily that of its zeroth space X0) if and only if every structure map $\sigma_n : X_n \wedge \mathbb{S}^1 \longrightarrow X_{n+1}$ is a homeomorphism.

**Example 17.** (Eilenberg-Mac Lane spectra). Let $A$ be discrete abelian group, the Eilenberg-Mac Lane spectrum $HA$ is defined as

$$HA \coloneqq \left\{A, K\left(A,1\right), K\left(A,2\right), \cdots\right\}$$

where

$$\pi_k\left(K\left(A,n\right)\right) = \begin{cases} \pi_n\left(A\right) & k = n, \\ 0 & \text{o.w.} \end{cases}$$

We have a weak equivalence

$$K\left(A, n-1\right) \xrightarrow{\simeq} \Omega K\left(A,n\right)$$

taking adjoint of this map to be the structure map, then

$$\hat{\pi}_n\left(HA\right) = \begin{cases} \mathrm{colim}_k \pi_{n+k}\left(K\left(A,k\right)\right) = 0 & n \neq 0, \\ \mathrm{colim}_k \pi_k\left(K\left(A,k\right)\right) = A & n = 0 \end{cases}$$

**Any symmetric spectrum is an $\mathbb{S}$-module**

In order to show that any symmetric spectrum is an $\mathbb{S}$-module, we need to show that $\mathbb{S}$ is a ring spectrum first.

*Remark* 10. We want to define model structure on spectrum such that weak equivalences are the stable homotopy equivalences. The difficulty is, even if we define homotopy, two homotopic spectra does not necessarily remain homotopic after smashing with other (arbitrary) spectrum (unless that spectrum is cofibrant). This, however, works for the sphere spectrum $\mathbb{S}$ because it is cofibrant. In simplicial sets, this also works because everything is cofibrant.

*Claim* 2. $\mathbb{S}$ is a ring spectrum.

*Proof.* The symmetric group $\Sigma_n$ acts on $\mathbb{S}^n$ via permutation by identifying $\mathbb{S}^n \cong \left(\mathbb{S}^1\right)^{\wedge n}$, so the order does not matter. The structure map is is the isomorphism $\sigma_n : \mathbb{S}^n \wedge \mathbb{S}^1 \to \mathbb{S}^{n+1}$.

To see that $\mathbb{S}$ is a ring spectrum, we consider the multiplication $\mu_{n,m} : \mathbb{S}^n \wedge \mathbb{S}^m \to \mathbb{S}^{n+m}$ on $\mathbb{S}$. It is associative, and compatible with the structure maps. □

*Claim* 3. The naive homotopy group $\hat{\pi}_* X$ is a $\pi_*^s$-module, so is the (true) homotopy group $\pi_* X$.

*Proof.* We first define the action of $\pi_*^s$ on the naive homotopy groups $\hat{\pi}_*(X)$ of a symmetric spectrum $X$.

Suppose $f : \mathbb{S}^{k+n} \to X_n$ and $g : \mathbb{S}^{l+m} \to \mathbb{S}^m$ represents classes in $\hat{\pi}_k(X)$ respectively $\pi_l^s$. We denote by $f \cdot g$ the composition

$$f \cdot g : \mathbb{S}^{k+n+l+m} \xrightarrow{\cong} \mathbb{S}^{k+n} \wedge \mathbb{S}^{l+m} \xrightarrow{f \wedge g} X_n \wedge \mathbb{S}^m \xrightarrow{\sigma_n^m} X_{n+m}$$

then define

$$[f] \cdot [g] \coloneqq (-1)^{nl} [f \cdot g]. \tag{1}$$

The sign can be explained by the principle that all natural number must occur in the 'natural order'. In $f \cdot g$ the dimension of the sphere of origin is naturally $(k+n)+(l+m)$, but in order to represent an element of $\hat{\pi}_{k+l} X$ the numbers should occur in the order $(k+l)+(n+m)$. Hence a shuffle permutation is to be expected, and it enters in the disguise of the sign $(-1)^{nl}$.

We need to check that the multiplication is well-defined. Note this is well-defined up to homotopy class of $f$ and $g$.

If we replace $g$ by $g \wedge 1_{\mathbb{S}^1}$,

$$f \cdot (g \wedge 1_{\mathbb{S}^1}) = \sigma_n^{m+1} (f \wedge g \wedge 1_{\mathbb{S}^1}) = \sigma_{n+m} \circ (\sigma_n^m \wedge 1_{\mathbb{S}^1}) \circ (f \wedge g \wedge 1_{\mathbb{S}^1}) = \sigma_{n+m} \circ ((f \cdot g) \wedge 1_{\mathbb{S}^1}).$$

$$\begin{tikzcd}
\mathbb{S}^{k+n+l+m+1} \arrow[r, "\cong"] & \mathbb{S}^{k+n} \wedge \mathbb{S}^{l+m+1} \arrow[r, "f\wedge g\wedge 1_{\mathbb{S}^1}"] & X_n \wedge \mathbb{S}^{m+1} \arrow[r, "\sigma_n^{m+1}"] & X_{n+m+1} \\
\mathbb{S}^{k+n+l+m} \wedge \mathbb{S}^1 \arrow[r, "\cong"] \arrow[u, equal, "\cong"] & \mathbb{S}^{k+n} \wedge \mathbb{S}^{l+m} \wedge \mathbb{S}^1 \arrow[r, "f\wedge g\wedge 1_{\mathbb{S}^1}"] \arrow[u, equal, "\cong"] & X_n \wedge \mathbb{S}^m \wedge \mathbb{S}^1 \arrow[r, "\sigma_n^m\wedge 1_{\mathbb{S}^1}"] \arrow[u, "\cong"] & X_{n+m} \wedge \mathbb{S}^1 \arrow[u, "\sigma_{n+m}"']
\end{tikzcd}$$

Since the sign in the formula 1 does not change, the resulting stable class is independent of the representative $g$ of the stable class in $\pi_l^s$.

If we replace $f$ by $\sigma_n \circ (f \wedge 1_{\mathbb{S}^1}) : \mathbb{S}^{k+n+1} \to X_{n+1}$, then we have

$$\begin{tikzcd}
\mathbb{S}^{k+n+l+m+1} \arrow[r, "\cong"] \arrow[d, "1_{\mathbb{S}^{k+n}} \wedge \chi_{l+m,1}"'] & \mathbb{S}^{k+n} \wedge \mathbb{S}^{l+m+1} \arrow[r, "f \wedge g \wedge 1_{\mathbb{S}^1}"] \arrow[d, equal, "\cong"'] & X_n \wedge \mathbb{S}^{m+1} \arrow[r, "\sigma_n^{m+1}"] \arrow[d, "1_{X_n} \wedge \chi_{m,1}"] & X_{n+m+1} \arrow[d, "1 \wedge \chi_{m,1}"] \\
\mathbb{S}^{k+n+1+l+m} \arrow[r, "\cong"] & \mathbb{S}^{k+n} \wedge \mathbb{S}^1 \wedge \mathbb{S}^{l+m} \arrow[r, "f \wedge 1_{\mathbb{S}^1} \wedge g"] & X_n \wedge \mathbb{S}^{1+m} \arrow[r, "\sigma_n^{1+m}"] & X_{n+1+m}
\end{tikzcd}$$

By lemma 3, the map $1 \wedge \chi_{m,1}$ induces multiplication by $(-1)^m$ on homotopy groups after one suspension. This cancels part of the sign $(-1)^{l+m}$ that is the effect of precomposition with the shuffle permutation $\chi_{l+m,1}$ on the left. So in the colimit $\hat{\pi}_{k+1} X$ we have

$$[\sigma_n \circ (f \wedge 1_{\mathbb{S}^1}) \cdot g] = (-1)^l \left[\sigma_n^{m+1} \circ (f \wedge g \wedge 1_{\mathbb{S}^1})\right] = (-1)^l [f \cdot g].$$

Hence the multiplication is independent of the representative of the stable class $[f]$.

Now we verify biadditivity. We only show the relation $(x + x') \cdot y = x \cdot y + x' \cdot y$, and additivity in $y$ is similar. Suppose $f, f' : \mathbb{S}^{k+n} \to X_n$ and $g : \mathbb{S}^{l+m} \to \mathbb{S}^m$ represents classes in $\hat{\pi}_k(X)$ respectively $\pi_l^s$, then the sum $f + f'$ in $\pi_{k+n}(X_n)$ is given by the composite

$$f + f' : \mathbb{S}^{k+n} \xrightarrow{\text{pinch}} \mathbb{S}^{k+n} \vee \mathbb{S}^{k+n} \xrightarrow{f \vee f'} X_n$$

Then we have

$$\begin{tikzcd}
\mathbb{S}^{k+l+n+m} \arrow[r, "1 \wedge \chi_{n,1} \wedge 1"] \arrow[dd, "\text{pinch}"'] & \mathbb{S}^{k+n+l+m} \arrow[d, "\text{pinch} \wedge 1"'] \arrow[dr, "(f+f') \wedge g"] & & \\
& \left(\mathbb{S}^{k+n} \vee \mathbb{S}^{k+n}\right) \wedge \mathbb{S}^{l+m} \arrow[r, "(f \vee f') \wedge g"] & X_n \wedge \mathbb{S}^m \arrow[r, "\sigma_n^m"] & X_{n+m} \\
\mathbb{S}^{k+l+n+m} \wedge \mathbb{S}^{k+l+n+m} \arrow[r, "(1 \wedge \chi_{n,1} \wedge 1) \wedge (1 \wedge \chi_{n,1} \wedge 1)"] & \mathbb{S}^{k+n+l+m} \wedge \mathbb{S}^{k+n+l+m} \arrow[u, "\cong"] \arrow[ur, "(f \vee g) \wedge (f' \vee g)"'] & &
\end{tikzcd}$$

where the right part commutes and the left square commutes up to homotopy. The composite around the top of the diagram gives $(f + f') \cdot g$, whereas the composite around the bottom represents $f \cdot g + f' \cdot g$. This proves additivity of the dot product in the left variable.

If we specialize to $X = \mathbb{S}$ then the product provides a biadditive graded pairing

$$\cdot : \pi_k^s \times \pi_l^s \to \pi_{k+l}^s$$

of the stable homotopy groups of spheres. We claim that for every symmetric spectrum $X$ the diagram

$$\begin{tikzcd} \hat{\pi}_k X \times \pi_l^s \times \pi_j^s \arrow[r, "\cdot \times 1"] \arrow[d, "1 \times \cdot"'] & \hat{\pi}_{k+l} X \times \pi_j^s \arrow[d, "\cdot"] \\ \hat{\pi}_k X \times \pi_{l+j}^s \arrow[r, "\cdot"] & \hat{\pi}_{k+l+j} X \end{tikzcd}$$

commutes, so the product on the stable stems and the action on the homotopy groups of a symmetric spectrum are associative. After unraveling all the definitions, this associativity ultimately boils down to the equality

$$(-1)^{ln} \cdot (-1)^{j(n+m)} = (-1)^{jm} \cdot (-1)^{(l+j)n}$$

and commutativity of the square

$$\begin{tikzcd} X_n \wedge \mathbb{S}^m \wedge \mathbb{S}^q \arrow[r, "\sigma_n^m \wedge 1"] \arrow[d, "1 \wedge \cong"'] & X_{n+m} \wedge \mathbb{S}^q \arrow[d, "\sigma_{n+m}^q"] \\ X_n \wedge \mathbb{S}^{m+q} \arrow[r, "\sigma_n^{m+q}"] & X_{n+m+q} \end{tikzcd}$$

Finally, the multiplication in the homotopy groups of spheres is commutative in the graded sense. Indeed, for representing maps $f : \mathbb{S}^{k+n} \to \mathbb{S}^n$ and $g : \mathbb{S}^{l+m} \to \mathbb{S}^m$ the square

$$\begin{tikzcd} \mathbb{S}^{k+n+l+m} \arrow[r, "f \wedge g"] \arrow[d, "\chi_{k+n,l+m}"'] & \mathbb{S}^{n+m} \arrow[d, "\chi_{n,m}"] \\ \mathbb{S}^{l+m+k+n} \arrow[r, "g \wedge f"] & \mathbb{S}^{m+n} \end{tikzcd}$$

commutes. The two vertical coordinate permutations induce the signs $(-1)^{(k+n)(l+m)}$ respectively $(-1)^{nm}$ (after one suspension) on homotopy groups. So in the stable group we have

$$[f] \cdot [g] = (-1)^{nl} [f \cdot g] = (-1)^{kl+km} (g \cdot f) = (-1)^{kl} [g] \cdot [f] .$$

□

**Lemma 3.** *Let $Y$ be a pointed space, $m \geq 0$ and $f : \mathbb{S}^m \to \mathbb{S}^m$ a based map of degree $k$. Then for every homotopy class $x \in \pi_n (Y \wedge \mathbb{S}^m)$ the classes $(Id_Y \wedge f)_* (x)$ and $k \cdot x$ become equal in $\pi_{n+1} \left(Y \wedge \mathbb{S}^{m+1}\right)$ after one suspension.*

The conclusion of this lemma does not in general hold without the extra suspension, i.e., $(\mathrm{Id}_Y \wedge f)_* (x)$

need not equal $k \cdot x$ in $\pi_n(Y \wedge \mathbb{S}^m)$: consider the Hopf map

$$\eta : \mathbb{S}^3 \subseteq \mathbb{C}^2 \setminus \{0\} \xrightarrow{\text{proj}} \mathbb{C}P^1 \cong \mathbb{S}^2$$

which is a locally trivial fiber bundle with fiber $\mathbb{S}^1$. It gives rise to a long exact sequence of homotopy groups. Since the fiber $\mathbb{S}^1$ has no homotopy above dimension one, the group $\pi_3(\mathbb{S}^2)$ is free abelian of rank one, generated by the class of $\eta$, $(-1) \circ \eta$ is homotopic to $\eta$, which is not homotopic to $-\eta$ since $\eta$ generates the infinite cyclic group $\pi_3(\mathbb{S}^2)$.

**Exercise 2.** Given representing maps $f : \mathbb{S}^{k+n} \to \mathbb{S}^n$ and $g : \mathbb{S}^{l+m} \to \mathbb{S}^m$ where $n = l + m$, show that $[f \cdot g] = [g \circ f]$ if $X = \mathbb{S}$.

*Proof.* The composite

$$\mathbb{S}^{l+m+k} \xrightarrow{f} \mathbb{S}^{l+m} \xrightarrow{g} \mathbb{S}^m$$

is homotopic to (after one suspension)

$$\mathbb{S}^{k+l+m+l+m} \xrightarrow{f \wedge 1} \mathbb{S}^{l+m+l+m} \xrightarrow{g \wedge 1} \mathbb{S}^{m+m+l} \xrightarrow{\chi_{m+m,l}} \mathbb{S}^{l+m+m}$$

because the last shuffle has sign $(-1)^{2ml} = 1$, while $f \cdot g$ can be decomposed as

$$\mathbb{S}^{k+l+m+l+m} \xrightarrow{f \wedge 1} \mathbb{S}^{l+m+l+m} \xrightarrow{1 \wedge g} \mathbb{S}^{l+m+m}$$

which is homotopic to

$$\mathbb{S}^{k+l+m+l+m} \xrightarrow{f \wedge 1} \mathbb{S}^{l+m+l+m} \xrightarrow{g \wedge 1} \mathbb{S}^{m+m+l}.$$

□

## 3.3 Mapping cone and homotopy fiber

The (reduced) mapping cone $C(f)$ of a morphism of based spaces $f : A \to B$ is defined by

$$C(f) = ([0,1] \wedge A) \cup_f B$$

where the unit interval $[0,1]$ is pointed by $0 \in [0,1]$, so that $[0,1] \wedge A$ is the reduced cone of $A$. The mapping cone comes with an inclusion

$$i : B \hookrightarrow C(f)$$

and a projection

$$p : C(f) \twoheadrightarrow \mathbb{S}^1 \wedge A$$

the projection sends $B$ to the basepoint and is given on $[0,1] \wedge A$ by $p(x \wedge a) = t(x) \wedge a$ where we define

$$\begin{array}{rccc} t: & [0,1] & \to & \mathbb{S}^1 \\ & x & \mapsto & \frac{2x-1}{x(1-x)} \end{array}$$

What is relevant about the map $t$ is not the precise formula, but that it passes to a homeomorphism between the quotient space $[0,1] / \{0,1\}$ and the circle $\mathbb{S}^1$, and that it satisfies $t(1-x) = -t(x)$.

We observe that an iteration of the mapping cone construction yields the suspension of $A$, up to homotopy, which induces a long exact sequence in homotopy groups.

We want to get a similar result in spectra.

**Definition 27.** The *mapping cone* $C(f)$ of a morphism of symmetric spectra $f : X \to Y$ is defined by the reduced mapping cone of $f_n : X_n \to Y_n$,

$$C(f)_n = C(f_n) = ([0,1] \wedge X_n) \cup_f Y_n$$

where the symmetric group $\Sigma_n$ acts on $C(f)_n$ through the given action on $X_n$ and $Y_n$ and trivially on the interval.

The inclusions $i_n : Y_n \to C(f)_n$ and projections $p_n : C(f)_n \to \mathbb{S}^1 \wedge X_n$ assemble into morphisms of symmetric spectra $i : Y \to C(f)$ and $p : C(f) \to \mathbb{S}^1 \wedge X$.

We define the connecting homomorphism $\delta : \hat{\pi}_{k+1} C(f) \to \hat{\pi}_k X$ as the composite

$$\hat{\pi}_{k+1} C(f) \xrightarrow{p_*} \hat{\pi}_{k+1}\left(\mathbb{S}^1 \wedge X\right) \xrightarrow{\mathbb{S}^{-1} \wedge -} \hat{\pi}_k(X)$$

where the second map is the inverse of the suspension isomorphism

$$\mathbb{S}^1 \wedge - : \hat{\pi}_k X \longrightarrow \hat{\pi}_{k+1}\left(\mathbb{S}^1 \wedge X\right).$$

The map $\delta$ sends the class represented by a based map $\varphi : \mathbb{S}^{1+k+n} \to C(f)_n$ to $(-1)^{k+n}$ times the class of the composite

$$\mathbb{S}^{1+k+n} \xrightarrow{\varphi} C(f)_n \xrightarrow{p_n} \mathbb{S}^1 \wedge X_n \xrightarrow{\text{twist}} X_n \wedge \mathbb{S}^1 \xrightarrow{\sigma_n} X_{n+1}.$$

**Proposition 4.** *For every morphism $f : X \to Y$ of symmetric spectra the long sequence of abelian groups*

$$\cdots \longrightarrow \hat{\pi}_k X \xrightarrow{f_*} \hat{\pi}_k Y \xrightarrow{i_*} \hat{\pi}_k C(f) \xrightarrow{\delta} \hat{\pi}_{k-1} X \longrightarrow \cdots$$

*is exact.*

A continuous map $f : A \to B$ spaces is an h-cofibration if it has the homotopy extension property, i.e., given a continuous map $\varphi : B \to X$ and a homotopy $H : [0,1] \times A \to X$ such that $H(0,-) = \varphi f$, there is a homotopy $\tilde{H} : [0,1] \times B \to X$ such that $\tilde{H} \circ ([0,1] \times f) = H$and $\tilde{H}(0,-) = \varphi$.

An equivalent condition is that the map $([0,1] \times A) \cup_{0 \times f} B \to [0,1] \times B$ has a retraction.

**Proposition 5.** *For every h-cofibration the map $C(f) \to B/A$ which collapses the cone of $A$ to a point is a based homotopy equivalence.*

*Proof.* See Corollary 2.2 of Appendix A in [Sch12]. □

Let $f : X \to Y$ be a morphism of symmetric spectra that is level-wise an h-cofibration. Then by the above proposition, the morphism $c : C(f) \to Y/X$ that collapses the cone of $X$ is a level equivalence, and so it induces an isomorphism of homotopy groups. We can thus define another connecting map

$$\delta : \hat{\pi}_k(Y/X) \longrightarrow \hat{\pi}_{k-1} X$$

as the composite of the inverse of the isomorphism $c_* : \hat{\pi}_k C(f) \to \hat{\pi}_k(Y/X)$ and the connecting homomorphism $\hat{\pi}_k C(f) \to \hat{\pi}_{k-1} X$ as we defined before.

**Corollary 2.** *Let $f : X \to Y$ be a morphism of symmetric spectra that is level-wise an h-cofibration and denote by $q : Y \to Y/X$ the quotient map. Then the long sequence of naive homotopy groups*

$$\cdots \longrightarrow \hat{\pi}_k X \xrightarrow{f_*} \hat{\pi}_k Y \xrightarrow{q_*} \hat{\pi}_k(Y/X) \xrightarrow{\delta} \hat{\pi}_{k-1} X \longrightarrow \cdots$$

*is exact.*

As a consequence, we have

**Proposition 6.** *For any two spectra $X, Y$, we have a natural map $X \bigvee Y \to X \times Y$ where*

$$\begin{aligned} (X \bigvee Y)_n &= X_n \bigvee Y_n \\ (X \times Y)_n &= X_n \times Y_n \end{aligned}$$

*for every $n \geq 0$.*

1. *For every integer $k$ the canonical map*

$$\hat{\pi}_k X \bigoplus \hat{\pi}_k Y \longrightarrow \hat{\pi}_k \left( X \bigvee Y \right)$$

*is an isomorphism of abelian groups.*

2. *For every integer $k$ the canonical map*

$$\hat{\pi}_k \left( X \times Y \right) \longrightarrow \hat{\pi}_k \left( X \right) \times \hat{\pi}_k \left( Y \right)$$

*is an isomorphism of abelian groups.*

3. *The canonical morphism from the wedge to the weak product is a $\hat{\pi}_*$-isomorphism. In particular, for every finite family of symmetric spectra the canonical morphism from the wedge to the product is a $\hat{\pi}_*$-isomorphism.*

*Proof.*

1. the wedge inclusion $i_A : A \to A \bigvee B$ has a retraction. So the associated long exact homotopy group sequence of splits into short exact sequences

$$0 \longrightarrow \hat{\pi}_k A \xrightarrow{i_{A*}} \hat{\pi}_k \left( A \bigvee B \right) \xrightarrow{i_*} \hat{\pi}_k C \left( i_A \right) \xrightarrow{\delta} 0$$

The mapping cone $C \left( i_A \right)$ is isomorphic to $\left( CA \right) \bigvee B$ and thus homotopy equivalent to $B$. So we can replace $\hat{\pi}_k C \left( i_A \right)$ with $\hat{\pi}_k B$ and conclude that $\hat{\pi}_k \left( A \bigvee B \right)$ splits as the sum of $\hat{\pi}_k A$ and $\hat{\pi}_k B$, via the canonical map.

2. Unstable homotopy groups commute with products, which for finite indexing sets are also sums, which commute with filtered colimits.

3. This is a direct consequence of 1 and 2. More precisely, the composite map

$$\begin{tikzcd}
\hat{\pi}_k X \bigoplus \hat{\pi}_k Y \arrow[r] \arrow[d, "\cong"'] & \hat{\pi}_k \left( X \bigvee Y \right) \arrow[d] \\
\hat{\pi}_k \left( X \right) \times \hat{\pi}_k \left( Y \right) & \hat{\pi}_k \left( X \times Y \right) \arrow[l]
\end{tikzcd}$$

is an isomorphism as coproducts and products are the same in abelian groups. The upper and lower maps are canonical isomorphism, hence so is the right map.

□

## 3.4 Smash product

**Construction of the Smash Product**

One of the main features which distinguishes symmetric spectra from the more classical spectra without symmetric group actions is the internal smash product. The smash product of symmetric spectra is very much like the tensor product of a monoidal category.

Recall in any monoidal category (e.g. $R$-**Mod** where $R$ is a commutative ring), if we have $X \hookrightarrow X'$ and $Y \hookrightarrow Y'$ then we have a commutative diagram

$$\begin{tikzcd}
X \bigotimes X' \arrow[rr, hook] \arrow[dd, hook] & & X \bigotimes Y' \arrow[dl, dotted] \arrow[dd, hook] \\
& Z \arrow[dr, hook] & \\
X' \bigotimes Y \arrow[rr, hook] \arrow[ur, dotted] & & X' \bigotimes Y'
\end{tikzcd}$$

where $Z$ is the pushout, and this should be compatible with respect to the monoidal structure.

To stress that analogy, we recall three different ways to look at the classical tensor product and then give analogies involving the smash product of symmetric spectra.

In the following, $R$ is a commutative ring and $M, N, W \in Ob\,(R\text{-}\mathbf{Mod})$.

**Via bilinear maps.** A bilinear map $b : M \times N \to W$ is $R$-linear in both $M$ and $N$. The tensor product $M \otimes_R N$ is an $R$-module that represents the functor

$$\begin{array}{ccc} R\text{-}\mathbf{Mod} & \longrightarrow & \mathbf{Set} \\ W & \longmapsto & \mathrm{Bil}_R\,(M \times N, W) \end{array}$$

so there is a unique bilinear map $i : M \times N \to M \otimes_R N$ such that

$$\begin{array}{ccc} \mathrm{Hom}_R\,(M \otimes_R N, W) & \longrightarrow & \mathrm{Bil}_R\,(M \times N, W) \\ f & \mapsto & f \circ i \end{array}$$

is bijective.

**Definition 28.** A bimorphism $b : (X, Y) \to Z$ is a collection of morphisms $\Sigma_p \times \Sigma_q$-equivariant maps (of pointed spaces or simplicial sets)

$$b_{p,q} : X_p \wedge Y_q \longrightarrow Z_{p+q}$$

for $p, q \geq 0$ such that the bilinearity diagram

$$\begin{tikzcd}
 & X_p \wedge Y_q \wedge \mathbb{S}^1 \arrow[r, "X_p \wedge \tau"] \arrow[dl, "1\wedge\sigma_q"'] \arrow[d, "b_{p,q}\wedge 1"] & X_p \wedge \mathbb{S}^1 \wedge Y_q \arrow[d, "\sigma_p \wedge 1"] \\
X_p \wedge Y_{q+1} \arrow[dr, "b_{p,q+1}"'] & Z_{p+q} \wedge \mathbb{S}^1 \arrow[d, "\sigma_{p+q}"] & X_{p+1} \wedge Y_q \arrow[d, "b_{p+1,q}"] \\
 & X_{p+q+1} & X_{p+1+q} \arrow[l, "1\wedge\chi_{1,q}"]
\end{tikzcd}$$

commutes for all $p, q \geq 0$, where $\tau$ is the twist map.

We can then define a smash product of $X$ and $Y$ as a pair $(X \wedge Y, i)$ consisting of a symmetric spectrum $X \wedge Y$ and a universal bimorphism $i : (X, Y) \to X \wedge Y$ , i.e., a bimorphism such that for every symmetric spectrum $Z$ the map

$$\begin{array}{ccc} \mathrm{Sp}\,(X \wedge Y, Z) & \longrightarrow & \mathrm{Bimor}\,((X, Y)\,, Z) \\ f & \longmapsto & fi = \{f_{p+q} \circ i_{p.q}\}_{p,q\geq 0} \end{array}$$

is bijective.

**Adjoint to internal Hom.** In $\mathbf{Mod}\,(R)$ we define the internal hom

$$W^N := \mathrm{Hom}_R\,(N, W)$$

which has a natural $R$-module structure, thus we have a functor

$$(-)^N = \mathrm{Hom}_R\,(N, -) : \mathbf{Mod}\,(R) \to \mathbf{Mod}\,(R)\,.$$

We have an adjoint pair

$$- \otimes_R N : \mathbf{Mod}\,(R) \rightleftarrows \mathbf{Mod}\,(R) : (-)^N$$

which gives a bijective map

$$\mathrm{Hom}_R\,(M \otimes_R N, W) \xrightarrow{\cong} \mathrm{Hom}_R\left(M, W^N\right)\,.$$

For any two symmetric spectra $Y, Z$, the internal hom spectra $Z^Y$ is defined as

$$\left(Z^Y\right)_n = \mathbf{Map}\left(Y, \mathrm{sh}^n Z\right)$$

where $\left(\mathrm{sh}^n Z\right)_k = Z_{n+k}$ and the $\Sigma_k$-action is induced by the inclusion $\Sigma_k \hookrightarrow \Sigma_{n+k}$. The structure map

$$\sigma_n : \left(Z^Y\right)_n \wedge \mathbb{S}^1 \to \left(Z^Y\right)_{1+n}$$

is the composite

$$\mathbf{Map}\left(Y, \mathrm{sh}^n Z\right) \wedge \mathbb{S}^1 \xrightarrow{a} \mathbf{Map}\left(Y, \mathbb{S}^1 \wedge \mathrm{sh}^n Z\right) \xrightarrow{\lambda_*} \mathbf{Map}\left(Y, \mathrm{sh}^{1+n} Z\right)$$

where

$$\begin{array}{rccc} a: & \mathbf{Map}\left(Y, \mathrm{sh}^n Z\right) \wedge \mathbb{S}^1 & \longrightarrow & \mathbf{Map}\left(Y, \mathbb{S}^1 \wedge \mathrm{sh}^n Z\right) \\ & f \wedge t & \longmapsto & \left(\begin{array}{ccc} Y & \to & \mathbb{S}^1 \wedge \mathrm{sh}^n Z \\ y & \mapsto & t \wedge f(y) \end{array}\right) \end{array}$$

and $\lambda : \mathbb{S}^1 \wedge \mathrm{sh}^n Z \to \mathrm{sh}^{1+n} Z$ is defined level-wise as $\lambda_k : \mathbb{S}^1 \wedge Z_{n+k} \to Z_{1+n+k}$ by the composite

$$\mathbb{S}^1 \wedge Z_{n+k} \xrightarrow[\tau]{\cong} Z_{n+k} \wedge \mathbb{S}^1 \xrightarrow{\sigma_n} Z_{n+k+1} \xrightarrow{\chi_{n+k,1}} Z_{1+n+k} ,$$

and $\lambda_* : \mathbf{Map}\left(X, \mathbb{S}^1 \wedge \mathrm{sh}^n Y\right) \to \mathbf{Map}\left(X, \mathrm{sh}^{1+n} Y\right)$ is given by postcomposition with $\lambda$.

Then the morphism from $Y$ to $Z$ are (in natural bijection with) the points (respectively vertices) of the 0th level of $Y^Z$.

We claim that for fixed symmetric spectra $X$ and $Y$, the set-valued functor $\mathrm{Sp}\left(X, (-)^Y\right)$ is representable; the smash product $X \wedge Y$ can then be defined as a representing symmetric spectrum. This point of view can be reduced to the first perspective since the sets $\mathrm{Sp}\left(X, Z^Y\right)$ and $\mathrm{Bimor}\left((X, Y), Z\right)$ are in natural bijection. In particular, since the functor $\mathrm{Bimor}\left((X, Y), -\right)$ is representable, so is the functor $\mathrm{Sp}\left(X, (-)^Y\right)$.

**Smash product as a construction.** In $\mathbf{Ch}_R$ we can define the tensor product of two chain complexes as the chain complex

$$\left(C_\bullet \otimes D_\bullet\right)_n := \bigoplus_{p+q=n} C_p \otimes C_q$$

with $d_{C \otimes D} = d_C \otimes \mathrm{Id} + \mathrm{Id} \otimes d_D$.

This construction is given by the composite of functors

$$\mathbf{Ch}_R \times \mathbf{Ch}_R \longrightarrow \mathbf{DblCh}_R \xrightarrow{\text{total}} \mathbf{Ch}_R$$

where the first functor is given by $(C_\bullet, D_\bullet) \mapsto (C_p \otimes D_q)_{\bullet\bullet}$.

We can extend this idea to symmetric spectra.

We have a functor

$$\mathbf{SymSp} \times \mathbf{SymSp} \longrightarrow \mathbf{DblSymSp}$$

given by $(X, Y) \mapsto \{X_p \wedge Y_q\}_{p,q\geq 0}$. The question is how to define the "total" functor, i.e. how to internalize it?

We observe that $(X \wedge Y)_n \not\cong \bigvee\limits_{p+q=n} (X_p \wedge Y_q)$ because of the twist. More explicitly, if we consider $X = Y = \mathbb{S}$, we would like $\mathbb{S}$ to be a unit, however, we can check level-wise to see that in degree 0: $\mathbb{S}^0 \wedge \mathbb{S}^0 \cong \mathbb{S}^0$; in degree 1: $\left(\mathbb{S}^0 \wedge \mathbb{S}^1\right) \bigvee \left(\mathbb{S}^1 \wedge \mathbb{S}^0\right) \cong \mathbb{S}^1 \bigvee \mathbb{S}^1$ two copies of $\mathbb{S}^1$; and the higher degrees, this becomes a mess because of the twist. Hence we want to define a proper $\Sigma_n$-action such that the twists get killed off.

First, we construct

$$\bigvee_{p+q=n} (\Sigma_n)_+ \bigwedge_{\Sigma_p \times \Sigma_q} (X_p \wedge Y_q)$$

which has a natural $\Sigma_n$-action on the first component if we consider $(\Sigma_n)_+$ as a discrete pointed space or simplicial set in proper settings.

Second, there are two maps

$$\begin{tikzcd} \bigvee\limits_{p+q=n} (\Sigma_n)_+ \bigwedge\limits_{\Sigma_p \times \Sigma_q} \left(X_p \wedge \mathbb{S}^1 \wedge Y_q\right) \arrow[d, shift right, "\alpha_X"'] \arrow[d, shift left, "\alpha_Y"] \\ \bigvee\limits_{p+q=n} (\Sigma_n)_+ \bigwedge\limits_{\Sigma_p \times \Sigma_q} (X_p \wedge Y_q) \end{tikzcd}$$

where $\alpha_X$ is induced by the structure $X_p \wedge \mathbb{S}^1 \xrightarrow{\sigma_p} X_{p+1}$ on $X$

$$X_p \wedge \mathbb{S}^1 \wedge Y_q \xrightarrow{\sigma_p \wedge 1} X_{p+1} \wedge Y_q \ ,$$

and $\alpha_Y$ is induced by a twist of $\mathbb{S}^1$ and $Y_q$ and the structure map $Y_q \wedge \mathbb{S}^1 \xrightarrow{\sigma_p} Y_{q+1}$ on $Y$

$$X_p \wedge \mathbb{S}^1 \wedge Y_q \xrightarrow{1\wedge\tau} X_p \wedge Y_q \wedge \mathbb{S}^1 \xrightarrow{1\wedge\sigma_q} X_p \wedge Y_{q+1} \xrightarrow{1\wedge\chi_{q,1}} X_p \wedge Y_{1+q} \ .$$

We want to identify these two maps, so we define

$$(X \wedge Y)_n \coloneqq \operatorname{coeq}\left\{ \bigvee_{p+q=n} (\Sigma_n)_+ \bigwedge_{\Sigma_p \times \Sigma_q} \left(X_p \wedge \mathbb{S}^1 \wedge Y_q\right) \overset{\alpha_X}{\underset{\alpha_Y}{\rightrightarrows}} \bigvee_{p+q=n} (\Sigma_n)_+ \bigwedge_{\Sigma_p \times \Sigma_q} (X_p \wedge Y_q) \right\}$$

The structure map $(X \wedge Y)_n \wedge \mathbb{S}^1 \to (X \wedge Y)_{n+1}$ is induced on coequalizers by the wedge of the maps

$$(\Sigma_n)_+ \bigwedge_{\Sigma_p \times \Sigma_q} X_p \wedge Y_q \wedge \mathbb{S}^1 \longrightarrow (\Sigma_n)_+ \textstyle\bigwedge_{\Sigma_p \times \Sigma_q} X_p \wedge Y_{q+1}$$

induced by $1 \wedge \sigma_q : X_p \wedge Y_q \wedge \mathbb{S}^1 \to X_p \wedge Y_{q+1}$.

Equivalently we could have defined the structure map by moving the circle past $Y_q$, using the structure map of $X$ and then shuffling back with the permutation $\chi_{1,q}$; the definition of $(X \wedge Y)_{n+1}$ as a coequalizer precisely ensures that these two possible structure maps coincide, and that the collection of maps

$$X_p \wedge Y_q \longrightarrow \bigvee_{p+q=n} (\Sigma_n)_+ \bigwedge_{\Sigma_p \times \Sigma_q} (X_p \wedge Y_q) \xrightarrow{\text{proj}} (X \wedge Y)_{p+q}$$

forms a bimorphism, and in fact a universal one.

# References


[HA02] A. Hatcher, Algebraic topology. Cambridge: Cambridge University Press (2002).

[EC71] E. B. Curtis, Simplicial homotopy theory, Advances in Math. 6, 107–209 (1971). MR 0279808, https://doi.org/10.1016/0001-8708(71)90015-6.

[GJ99] P.G. Goerss, J.F. Jardine, Simplicial Homotopy Theory, Birkhäuser, Basel (1999).

[M] J. P. May, A Concise Course in Algebraic Topology, available at https://www.math.uchicago.edu/~may/CONCISE/ConciseRevised.pdf.

[R16] E. Riehl. Category theory in context. Courier Dover Publications (2016), available at http://www.math.jhu.edu/~eriehl/context.pdf.

[R] E. Riehl, A leisurely introduction to Simplicial Sets, available at math.uchicago.edu/~eriehl/.

[BM51] Albert Blakers, William Massey, The homotopy groups of a triad I , Annals of Mathematics 53: 161–204, (1951).

[Ker] Kervaire invariant, https://en.wikipedia.org/wiki/Kervaire_invariant.

[Sch12] S. Schwede, Symmetric Spectra, available at http://131.220.132.180/people/schwede/SymSpec-v3.pdf.

[EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, J. P. May, Rings, modules, and algebras in stable homotopy theory. With an appendix by M. Cole, Mathematical Surveys and Monographs, 47, American Mathematical Society, Providence, RI, (1997).