MATH7350 Topics in Homological Algebra

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Contents

Introduction

This is a graduate course in homological algebra, which is a continuation of the course MATH6350 Homological algebra in Spring 2018. There are several notes from previous years, which are also notes for courses taught by Professor Yuri Berest in recent years.

The outline for this course are as follows. The first two parts are determined with certainty, but the rest may be changed according to audience's preferences.

Part.I Classical Homological Algebra

1. Dold-Kan Correspondence.

   This correspondence builds between homological algebra and homotopical algebra. More explicitly, the (simplicial) Dold-Kan functor

   $$\mathcal{N} : \mathbf{sMod}\,(R) \xrightarrow{\simeq} \mathbf{Com}_+\,(R)$$

   gives an equivalence between the category of simplicial R-modules and nonnegatively graded chain complexes of R-modules. The left hand side has a generalization to simplicial model category $\mathbf{s}\mathcal{C}$.

   There is a dual version of the Dold-Kan correspondence. We have a cosimplicial Dold-Kan functor

   $$\mathcal{N} : \mathbf{csMod}\,(R) \xrightarrow{\simeq} \mathbf{Com}^+\,(R)$$

   which gives an equivalence between the category of cosimplicial R-modules and nonnegatively graded cochain complexes.

2. Variations.

   (a) "Differential form" version of DK correspondence. See [MK97] for details.

   (b) "Duplical" categories version. See Dwyer-Kan's DK normalization.

   (c) Monoidal versions.

      i. Let $R = k$ be a commutative ring. We have a Quillen equivalence $\mathbf{sAlg}_k \simeq_Q \mathbf{DGA}_k$ which, however, is not derived from the above functor. See [SS03] for more details.

      ii. Beilinson's Theorem from number theory.

      In general, the cosimplicial functor does not restricts to commutative algebras, but if we revise a little to a "small" version, we have

$$\mathcal{N}:\mathbf{csCommAlg}_k^{\text{small}}\xrightarrow{\simeq}\left(\mathbf{CDGA}_k^+\right)^{\text{small}}$$

where the second "small" can be explained very explicitly: the differential graded commutative algebras are generated by A^0 and A^1, the degree 0 and degree 1 components.

Application: Beilinson's conjecture on regulator maps for arbitrary schemes, which generalizes Borel's regulator map at one point (up to a factor $\frac{1}{2}$). See [BG02] for details.

iii. Categorical versions

$$\mathbf{sMod}\,(k)\,\text{-}\mathbf{Cat}\simeq_Q \mathbf{DGCat}_k^+$$

where $\mathbf{DGCat}_+\,(k)$ contains the small additive categories enriched over $\mathbf{Com}_+\,(k)$, and $\mathbf{sMod}\,(k)\,\text{-}\mathbf{Cat}$ contains the small categories enriched in simplicial modules. In Dwyer-Kan's model structure, the weak equivalences are quasi-isomorphisms.

Application: Rational homotopy theory. $f:X_*\to Y_*$ is called a rational homotopy equivalence of simply connected spaces if $\pi_i f:\pi_i\,(X,*)\otimes_{\mathbb{Z}}\mathbb{Q}\xrightarrow{\sim}\pi_i\,(Y,*)\otimes_{\mathbb{Z}}\mathbb{Q}$ as $\mathbb{Q}$-vector spaces for any $i>0$.

$$\begin{array}{c}\text{rational homotopy types of simply connected spaces}\\ \text{Quillen '69}\ \big\updownarrow\ 1-1\\ \text{homotopy types of reduced DG Lie algebra over }\mathbb{Q}\\ \text{Sullivan '71}\ \big\updownarrow\ \text{Koszul duality}\\ \text{homotopy types of }\mathbf{CDGA}_{\mathbb{Q}}^+\end{array}$$

$\mathbb{Q}$ can be replaced by any field k of characteristic 0.

Example. Let X be a 1-connected real C^∞-manifold, then $\Omega^\bullet\,(X)\in H_0\left(\mathbf{CDGA}_{\mathbb{R}}^+\right)$.

Problem. What algebraic objects can be taken as algebraic models for general spaces?

Solution 1. Look at "closed tensor" differential graded categories, or "commutative" differential graded categories. See Katzarkov–Pantev–Toën 2008 [KPT08], Pridham 2008 [P08], Moriya 2012 [Mor12].

Part.II Simplicial Methods

1. Simplicial groups **sGr**.

Theorem. *[Kan's Loop Group Theorem] There is a classical Quillen equivalence between the category* *of pointed connected topological spaces and the category of reduced simplicial sets. Furthermore, the Kan loop functor gives a Quillen equivalence between the category of reduced simplicial sets and the category of simplicial groups.*

$$\mathbf{Top}_{0,*} \underset{|-|}{\overset{\simeq_Q}{\longrightarrow}} \mathbf{sSet}_0 \underset{W}{\overset{\mathbb{G}}{\rightleftarrows}} \mathbf{sGr}$$

This theorem is a generalization of the classifying space $B\Gamma$ of a group Γ.

$$H_0\left(\mathbf{Top}_{0,*}\right) \xrightarrow{\cong} H_0\left(\mathbf{sGr}\right) \hookleftarrow \mathbf{Gr}$$

$$B\Gamma \longleftarrow\!\shortmid \Gamma$$

Game: Take your favorite group theoretic construction (or property of a group) and extend it (up to homotopy) to all spaces.

- Homology of spaces is an extension of abelianization of groups.
- Homotopy normal maps is an extension of normal maps $N \hookrightarrow G$ where $N \triangleleft G$.
- Representation homology is an extension of representation variety of groups.

2. Crossed simplicial groups. (cyclic theory and cyclic homology)

3. Simplicial Chern-Weil theory

Part.III A Survey in Simplicial Presheaves in Derived Algebraic Geometry (DAG) and Homotopical Algebraic Geometry (HAG)

Part I

Simplicial and Cosimplicial Objects

1 Simplicial Objects

In this section, we will discuss about

- definitions and examples of simplicial objects
- formal (categorical) properties: simplicial adjunctions

1.1 The (Co)simplicial Category Δ

Definition 1.1. Δ is the category whose objects are the finite totally ordered sets $[n] = \{0 \leq 1 \leq 2 \leq \cdots \leq n\}, n \geq 0$, and the morphisms are order-preserving maps

$$\mathrm{Hom}_{\Delta}([m],[n]) \coloneqq \{f \in \mathrm{Hom}_{\mathbf{Set}}([m],[n]) \,|\, f(i) \leq f(j), \forall 0 \leq i \leq j \leq m\}$$

Remark 1.1. (trivial observation)

1. $\mathrm{Aut}_{\Delta}([n]) = \{\mathrm{Id}\}, \forall n \geq 0$.
2. Δ has terminal object $[0]$ but no initial object.

There are two classes of morphisms in Δ.

- the *coface* maps $d^i : [n-1] \hookrightarrow [n], 0 \leq i \leq n, n \geq 1$ defined by the property that d^i is injective and contains no "i" in its image, i.e.

$$d^i(k) = \begin{cases} k & k < i \\ k-1 & k \geq i \end{cases}$$

- the *codegeneracy* maps $s^j : [n+1] \twoheadrightarrow [n], 0 \leq j \leq n, n \geq 0$ defined by the property that s^j is surjective and takes the value "j" twice, i.e.

$$s^j(k) = \begin{cases} k & k \leq i \\ k-1 & k > i \end{cases}$$

Convention. We will write $f : [m] \to [n]$ for cosimplicial case and $f : [n] \to [m]$ for simplicial ones.

Lemma 1.1. *Every* $f \in Hom_\Delta([n],[m])$ *can be decomposed (in a unique way) into the composition* $f = d^{i_1} d^{i_2} \cdots d^{i_r} s^{j_1} s^{j_2} \cdots s^{j_s}$ *where* $m = n + r - s$*,* $i_1 < i_2 < \cdots i_r$ *and* $j_1 < j_2 < \cdots < j_s$*.*

Proof. See [GZ67] Lemma 2.2. □

Example 1.1. Consider the unique morphism $f : [3] \to [1]$

$$\begin{tikzcd}
{[3]} & 0 \arrow[d] & 1 \arrow[d] & 2 \arrow[d] & 3 \arrow[dl] \\
{[2]} & 0 \arrow[d] & 1 \arrow[dl] & 2 \arrow[dl] & \\
{[1]} & 0 & 1 & &
\end{tikzcd}$$

f can be factored as $f = s^0 \circ s^2$.

The morphisms d^i and s^j satisfy the cosimplicial relation

$$\begin{cases} d^j d^i = d^i d^{j-1} & i < j \\ s^j s^i = s^i s^{j+1} & i \leq j \\ s^j d^i = \begin{cases} d^i s^{j-1} & i < j \\ \mathrm{Id} & i = j, j+1 \\ d^{i-1} s^j & i > j+1 \end{cases} \end{cases} \tag{1}$$

Remark 1.2. Δ can be defined abstractly as the category with objects $\{[n]\}_{n \geq 0}$ and morphisms generated by $d^i : [n-1] \to [n]$ and $s^j : [n+1] \to [n]$ satisfying the cosimplicial relation 1.

(Co)simplicial Objects

Definition 1.2. Let $\mathscr{C}$ be a category.

1. a *cosimplicial object* in $\mathcal{C}$ is a functor $X^* : \Delta \to \mathscr{C}, [n] \mapsto X^n \coloneqq X^*[n]$.

2. a *simplicial object* in $\mathcal{C}$ is a functor $X_* : \Delta^{op} \to \mathscr{C}, [n] \mapsto X_n \coloneqq X_*[n]$.

The category of simplicial ans cosimplicial objects in $\mathscr{C}$ are denoted $\mathbf{s}\mathscr{C} = \mathscr{C}^{\Delta^{op}} = \mathbf{Fun}(\Delta^{op}, \mathscr{C}) = \mathscr{C}_\Delta$ and $\mathbf{cs}\mathscr{C} = \mathscr{C}^\Delta = \mathbf{Fun}(\Delta, \mathscr{C})$. In both cases, morphisms are natural transformations of functors.

Example 1.2. $\mathscr{C} = \mathbf{Set}, \mathbf{s}\mathscr{C} = \mathbf{sSet}$ is the category of simplicial sets.

The dual category Δ^{op} has the presentation $Ob(\Delta^{op}) = Ob(\Delta) = \{[n], n \geq 0\}$, and morphisms are generated by the *face* maps $d_i : [n] \to [n-1]$ and the *degeneracy* maps $s_j : [n] \to [n+1]$ satisfying simplicial relations.

Remark 1.3. It's convenient to think of simplicial objects in $\mathcal{C}$ as right "modules" over Δ and cosimplicial objects as left "modules" over Δ.

Any simplicial object X_* can be written explicitly in the following manner.

$$X_* = \left[\begin{tikzcd} X_0 \arrow[r, "s_0", dotted] & X_1 \arrow[l, shift right=2, "d_0"'] \arrow[l, shift left=2, "d_1"] \arrow[r, dotted, shift left] \arrow[r, dotted, shift right] & \cdots \arrow[l, shift right=2] \arrow[l] \arrow[l, shift left=2] \end{tikzcd} \right]$$

where $X_n \coloneqq X_*[n]$ and $d_i \coloneqq X_*\left(d^i\right), s_j \coloneqq X_*\left(s^j\right)$.

1.2 More Terminology

Let $\mathcal{C}$ be a category. If $X_* \in Ob(\mathbf{sSet})$, then $X_n \coloneqq X_*[n]$ is called the set of n-simplices. An n-simplex $x \in X_n$ is called degenerate if $x \in \operatorname{Im}(s_j)$ for some j. Denote

$$X_n^{\deg} \coloneqq \bigcup_{j=1}^{n-1} s_j\left(X_{n-1}\right)$$

the set of degenerate simplices in X_n.

Remark 1.4. Here we abuse the notation s_j by taking it as $X_*(s_j)$ in the category $\mathcal{C}$. We'll do this a lot in the future, and the proper elements should be considered when used.

Exercise 1.1. Using simplicial identities we can express

$$X_n^{\deg} = \bigcup_{f:[n]\twoheadrightarrow[k], f\neq \mathrm{Id}} X(f)(X_k) \cong \operatorname{colim} X(f)(X_k) \tag{2}$$

Remark 1.5. Formula 2 makes sense for more general categories so we can define degenerate objects for any simplicial objects.

2 Basic Examples

1. Geometric

2. Combinatorial

3. Categorical

2.1 Geometric Examples of Simplicial Sets

Geometric Simplices

This is the motivation for simplicial homotopy theory.

The *geometric n-simplex* Δ^n is the topological space defined by

$$\Delta^n = \left\{ (x_0, \cdots, x_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} x_i = 1, x_i \geq 0, \forall 0 \leq i \leq n \right\},$$

i.e. it is the convex hull of unit vectors $\{e_i = (0, \cdots, 1, \cdots, 0)\}_{i=0}^n$ in $\mathbb{R}^{n+1}$.

Given a morphism $f : [n] \to [m]$ in Δ, we define

$$\begin{array}{rccc} \Delta^*(f): & \Delta^n \subseteq \mathbb{R}^{n+1} & \longrightarrow & \Delta^m \subseteq \mathbb{R}^{m+1} \\ & e_i & \longmapsto & e_{f(i)} \end{array}$$

which defines a functor

$$\begin{array}{rccc} \Delta^*: & \Delta & \longrightarrow & \mathbf{Top} \\ & [n] & \longmapsto & \Delta^n \end{array}$$

i.e. a cosimplicial space.

Let's compute the coface and codegeneracy maps in Δ^n.

Recall that

$$d^i(e_k) = \begin{cases} e_k, & k < i \\ e_{k+1}, & k \geq i \end{cases}$$

so it extends to $\sum_{k=0}^{n-1} x_k e_k \mapsto \sum_{k=0}^{i-1} x_k e_k + \sum_{k=i}^{n-1} x_k e_{k+1}$, and we have $d^i : \Delta^{n-1} \to \Delta^n, (x_0, \cdots, x_{n-1}) \mapsto (x_0, \cdots, x_{i-1}, 0, x_i, \cdots, x_n)$. Geometrically, in Δ^n we define i-th $(n-1)$-dimensional face to be the one opposite to e_i. Then $d^i : \Delta^{n-1} \hookrightarrow \Delta^n$ is the inclusion of i-th face into Δ^n.

Dually, $s^j : \Delta^{n+1} \to \Delta^n$ is given by $(x_0, \cdots, x_{n+1}) \mapsto (x_0, \cdots, x_{j-1}, x_j + x_{j+1}, \cdots, x_{n+1})$. Geometrically, s^j collapse the differential graded between j-th and $(j+1)$-th vertices in Δ^{n+1} to a point.

Given any subset $\emptyset \neq \sigma \subset [n]$, we define Δ_σ to be the convex hull of $\{e_i\}_{i\in\sigma} \subseteq \Delta^n$, called the σ-face of Δ. Thus we have a bijection

$$\{\text{nonempty subset of } [n]\} \longleftrightarrow \{\text{faces in } \Delta^n\}\ .$$

Definition 2.1. A finite *polyhedron* X is a topological space homeomorphic to a union of faces in Δ^n, i.e. $X \cong \bigcup_{i=1}^{r} \Delta_{\sigma_i} \subseteq \Delta^n$. The choice of such homeomorphism is called *triangulation* on X.

(Abstract) Simplicial Complexes

Definition 2.2. A *simplicial complex* X on a set V is a collection of nonempty finite subsets in V which is closed under taking subsets, i.e.

$$\forall \sigma \in X, \emptyset \neq \tau \subset \sigma \implies \tau \in X.$$

Remark 2.1. We do not assume that V is finite and that $V = \bigcup_{\sigma\in X} \sigma$.

To associate to (X, V) a space, we consider the $\mathbb{R}$-vector space $\mathbb{V} \coloneqq \operatorname{span}_{\mathbb{R}}(V)$ and define for $\sigma \in X$, $\Delta_\sigma \subseteq \mathbb{V}$ to be the convex hull of $\sigma \subset V \subseteq \mathbb{V}$. Δ_σ is a topological space equipped with topology induced from $\mathbb{V}$.

Definition 2.3. The *geometric realization* of a simplicial set X is

$$|X| \coloneqq \bigcup_{\sigma\in X} \Delta_\sigma \subseteq \mathbb{V}$$

Definition 2.4. A *polyhedron* is a topological space homeomorphic to $|X|$ for some simplicial complex.

Remark 2.2. Simplicial complexes have "bad" functorial properties. For instance, if $Y \subseteq X$ is a simplicial subcomplex, the quotient X/Y is not a simplicial complex.

Simplicial sets can be viewed as generalization of simplicial complexes.

To an (ordered) simplicial complex (X, V) we can define the simplicial set $SS_*(X)$ as follows,

$$SS_n(X) \coloneqq \left\{(v_0, \cdots, v_n) \in \mathbb{V}^{n+1} | \{v_0, \cdots, v_n\} \in X\right\}.$$

Remark 2.3. We allow $v_i = v_j$ for some $i \neq j$, and by $\{\ \}$ we mean the underlying set.

For $f:[n]\to[m]$ in Δ, we define

$$\begin{array}{rccc} SS_*(f): & SS_m(X) & \longrightarrow & SS_n(X) \\ & (v_0,\cdots,v_m) & \longmapsto & \left(v_{f(0)},\cdots,v_{f(n)}\right) \end{array}$$

Explicitly,

$$\begin{array}{rccc} d_i: & SS_n(X) & \longrightarrow & SS_{n-1}(X) \\ & (v_0,\cdots,v_n) & \longmapsto & (v_0,\cdots,v_{i-1},v_{i+1},\cdots,,v_n) \\ s_j: & SS_n(X) & \longrightarrow & SS_{n+1}(X) \\ & (v_0,\cdots,v_n) & \longmapsto & (v_0,\cdots,v_{j-1},v_j,v_j,\cdots,,v_n) \end{array}$$

Exercise 2.1. Show that

1. X can be recovered from $SS_*(X)$. More precisely, $SS_*(X)^{\text{nondeg}}\cong X$. This implies (see later)

2. for any (ordered) simplicial complex X, $|X|\cong|SS_*(X)|$ where $|X|$ is the realization of X as a simplex, and $|SS_*(X)|$is the geometric realization of simplicial set.

Proof. Nondegenerate n-simplices in $SS_*(X)$ corresponds to distinguished n-tuples in X, i.e. subsets of V of size n. □

Simplicial Sets of a Space

Definition 2.5. Given a topological space X, we define the simplicial set $S_*(X)$ by

$$S_n(X) := \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, X), n\geq 0$$

and for any $f:[n]\to[m]$,

$$\begin{array}{rccc} S_*(f): & S_m(X) & \longrightarrow & S_n(X) \\ & \varphi & \longmapsto & \varphi\circ\Delta^*(f) \end{array}$$

This defines a functor $S_*:\mathbf{Top}\to\mathbf{sSet}$, called *singular functor.*

2.2 Combinatorial Examples of Simplicial Sets

Discrete (constant) Simplicial Set

$$\begin{array}{ccc} \mathbf{Set} & \longrightarrow & \mathbf{sSet} \\ X & \longmapsto & \{X_k = X, d_i = s_j = \mathrm{Id}_X\}_{k\geq 0} \end{array}$$

Standard Simplicies

Recall that for any locally small category $\mathscr{C}$, we define the category $\hat{\mathscr{C}} = \mathbf{Fun}(\mathscr{C}^{op}, \mathbf{Set})$ the presheaves on $\mathscr{C}$, or $\mathscr{C}$-sets.

There is a canonical (Yoneda) functor

$$\begin{array}{cccc} h: & \mathscr{C} & \longrightarrow & \hat{\mathscr{C}} \\ & c & \longmapsto & h_c := \mathrm{Hom}_{\mathscr{C}}(-, c) \\ & \left(c_1 \xrightarrow{f} c_2\right) & \longmapsto & \left(\begin{array}{ccc} h_f: & h_{c_1} & \to & h_{c_2} \\ & (- \to c_1) & & \left(- \to c_1 \xrightarrow{f} c_2\right) \end{array}\right) \end{array}$$

Lemma 2.1. *The Yoneda functor h is fully faithful. More generally, for any $c \in Ob(\mathscr{C})$ and any $X \in Ob\left(\hat{\mathscr{C}}\right)$*

$$\begin{array}{ccc} Hom_{\hat{\mathscr{C}}}(h_c, X) & \xrightarrow{\cong} & X(c) \\ \varphi = \{\varphi_d : h_c(d) \to X(d)\}_{d \in Ob(\mathscr{C})} & \mapsto & \varphi_c(Id_c) \end{array}$$

Definition 2.6. A functor $F \in Ob\left(\hat{\mathscr{C}}\right)$ is called *representable* if there exists $c \in Ob(\mathscr{C})$ such that $F \simeq h_c$.

By definition, $\mathbf{sSet} = \mathbf{Fun}(\Delta^{op}, \mathbf{Set})$ is the category of presheaves on Δ.

Question. What simplicial sets are representable?

Definition 2.7. A *standard simplex* $\Delta[n]_*$ is the simplicial set of the form $h_{[n]}$ for fixed $n \geq 0$.

$$\begin{array}{cccc} \Delta[n]_* : & \Delta & \to & \mathbf{Set} \\ & [k] & \mapsto & [n]_k := \mathrm{Hom}_\Delta([k], [n]) \end{array}$$

Explicitly,

$$\Delta[n]_k = \{(j_0, \cdots, j_k) \,|\, 0 \leq j_0 \leq j_1 \leq \cdots \leq j_k \leq n\}.$$

Consider in Δ the special maps

$$\begin{cases} d^i : [k-1] \to [k] & 1 \leq i \leq k, k \geq 1 \\ s^l : [k+1] \to [k] & 0 \leq l \leq k, k \geq 0 \end{cases}$$

These maps under $h_{[n]}$ become

$$\begin{array}{rccc} d_i : & \Delta[n]_k & \to & \Delta[n]_{k-1} \\ & (j_0, \cdots, j_k) & \mapsto & \left(j_0, \cdots, \hat{j}_i, \cdots, j_k\right) \\ s_l : & \Delta[n]_j & \to & \Delta[n]_{k+1} \\ & (j_0, \cdots, j_k) & \mapsto & (j_0, \cdots, j_l, j_l, \cdots, j_k) \end{array}$$

Inspection. If we think of Δ^n as an abstract simplicial complex, then $\Delta[n]_*$ is the associated simplicial set, $\Delta[n]_* = SS_*(\Delta^n)$. This immediately implies that $|\Delta[n]_*| = \Delta^n$.

Lemma 2.2. $\Delta[n]_*$ *corepresents the functor*

$$\begin{array}{ccc} \mathbf{sSet} & \longrightarrow & \mathbf{Set} \\ X_* & \longmapsto & X_n \end{array}$$

Proof. This follows from Yoneda Lemma,

$$\mathrm{Hom}_{\hat{\Delta}}(\Delta[n]_*, X_*) = \mathrm{Hom}_{\hat{\Delta}}\left(h_{[n]}, X_*\right) \cong X_*[n] = X_n.$$

□

Explicitly, to each simplex $x \in X_n$ we can associate a unique simplicial map $\hat{x} : \Delta[n]_* \to X_*$ such that $\hat{x}\left(\mathrm{Id}_{[n]}\right) = x$. The Yoneda functor (in this case) is

$$\begin{array}{ccc} \Delta & \longrightarrow & \hat{\Delta} = \mathbf{sSet} \\ [n] & \longmapsto & h_{[n]} = \Delta[n]_* \end{array}$$

The assignment defines a cosimplicial set $\Delta[\bullet]_* \in \mathbf{csSet}$, where $\mathbf{csSet} = \mathbf{Fun}(\Delta, \mathbf{Fun}(\Delta^{op}, \mathbf{Set})) \cong \mathbf{Fun}(\Delta \times \Delta^{op}, \mathbf{Set})$.

We think of objects in **csSet** as graded-bimodules over Δ, i.e. functor $\Delta \times \Delta^{op} \to \mathbf{Set}$. This allows us to define two very important simplicial objects in $\Delta[n]_*$.

$$\partial\Delta[n]_* = \bigcup_{0 \le i \le n} d^i(\Delta[n-1]_*) \subseteq \Delta[n]_*$$

where $d^i : \Delta[n-1]_* \to \Delta[n]_*$ is the morphism of simplicial sets. $\partial\Delta[n]_*$ is called the *boundary* of $\Delta[n]_*$.

Example 2.1. $\Delta[0]_* = \{0, 00, 000, \cdots\}$ contains only one nondegenerate simplex $0 \in \Delta[0]_0$.

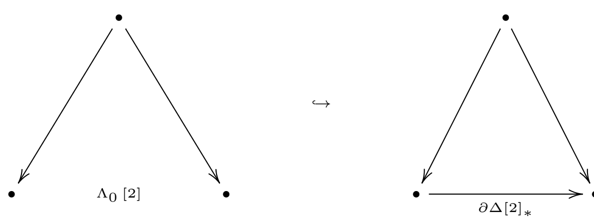
Example 2.2. $\Delta[1]_k = \{(j_0, \cdots, j_k) \,|\, 0 \le j_0 \le \cdots \le j_k \le 1\} = \left\{ \left(\underbrace{0, \cdots, 0}_{i}, \underbrace{1, \cdots, 1}_{k+1-i} \right), 0 \le i \le k+1 \right\}$.

$$\begin{array}{rccc} d^0: & \Delta[0]_* & \to & \Delta[1]_* \\ & (0, \cdots, 0) & \mapsto & (0, \cdots, 0) \\ d^1: & \Delta[0]_* & \to & \Delta[1]_* \\ & (0, \cdots, 0) & \mapsto & (1, \cdots, 1) \end{array}$$

So $\partial\Delta[1]_* = \{0, 1, 00, 11, 000, 111, \cdots\}$. The nondegenerate simplices are $0 \in \Delta[1]_0$ and $(0,1) \in \Delta[1]_1$.

Geometrically, $\partial\Delta[n]_*$ is the smallest simplicial subcomplex of $\Delta[n]_*$ generated by its $(n-1)$-dimensional faces.

Definition 2.8. The *k-th horn* of $\Delta[n]_*$ is defined as $\Lambda_k[n]_* := \bigcup_{0 \le i \le n, i \ne k} d^i(\Delta[n-1]_*) \subseteq \partial\Delta[n]_* \subseteq \Delta[n]_*$. This is a corn with vertex k.



Simplicial Spheres $\mathbb{S}^n_*$

Definition 2.9. The *simplicial n-sphere* $\mathbb{S}^n_*$ is defined as the quotient simplicial set

$$\mathbb{S}^n_* := \Delta[n]_* / \partial\Delta[n]_*$$

or equivalently, as the pushout

$$\begin{tikzcd} \partial\Delta[n]_* \arrow[r, hook] \arrow[d] & \Delta[n]_* \arrow[d] \\ \bullet \arrow[r] & \mathbb{S}^n_* \end{tikzcd}$$

Simplicial Circle $\mathbb{S}^1_*$

Let $n = 1$.

$\Delta[1]_k = \{(j_0, \cdots, j_k) \,|\, 0 \le j_0 \le \cdots \le j_k \le 1\} = \left\{ \left(\underbrace{0, \cdots, 0}_{i}, \underbrace{1, \cdots, 1}_{k+1-i} \right), 0 \le i \le k+1 \right\}$.

$\partial\Delta[1]_k = \{(0, \cdots, 0), (1, \cdots, 1)\}$.

$\mathbb{S}^1_k = \left\{ *, \left(\underbrace{0, \cdots, 0}_{i}, \underbrace{1, \cdots, 1}_{k+1-i} \right), 1 \leq i \leq k \right\}$ has $(k+1)$ elements, where $* = (0, \cdots, 0) \sim (1 \cdots, 1)$. $\{*, t = (0,1)\}$ are the only nondegenerate elements.

$\mathbb{S}^1_0 = \{*\}$.

$\mathbb{S}^1_1 = \{s_0(*), t = (0,1)\}$.

$\mathbb{S}^1_2 = \left\{ s_0^2(*), s_0(t) = (0,0,1), s_1(t) = (0,1,1) \right\}$.

$\cdots$

$\mathbb{S}^1_k = \left\{ s_0^k(*), s_{k-1}s_{k-2} \cdots \hat{s}_i \cdots s_0(t) \,|\, 0 \leq i \leq k-1 \right\}$.

Observe that there is a natural bijection $\mathbb{S}^1_k \xrightarrow{\cong} \mathbb{Z}_{k+1} = \mathbb{Z}/(k+1)\mathbb{Z}$ given by

$$s_0^k(*) \mapsto 0, s_{k-1}s_{k-2} \cdots \hat{s}_i \cdots s_0(t) \mapsto i + 1 \pmod{k+1}.$$

Then we can explicitly write

$$\mathbb{S}^1_* = \left\{ 0 \;\substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow}\; \mathbb{Z}_2 \;\substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow \\ \dashrightarrow \\ \longleftarrow}\; \mathbb{Z}_3 \cdots \right\}$$

However, the face and degeneracy maps are not simplicial group homomorphisms, so this is not a simplicial group. They actually satisfy the crossed relation, i.e.

$$f(gg') = f(g) f^g(g')$$

This is an example of a crossed simplicial group.

To explain this observation, we introduce a new category.

Cyclic Category and Cyclic Objects

Definition 2.10. The *cyclic (or Connes') category* $\Delta\mathbf{C}$ is defined by

Objects: $\{[n]\}_{n \geq 0}$.

Morphisms: generated by

$d^i : [n-1] \to [n]$	$1 \leq i \leq n, n \geq 1$	coface
$s^j : [n+1] \to [n]$	$0 \leq i \leq n, n \geq 0$	codegeneracy
$\tau_n : [n] \to [n]$	$n \geq 0$	cocyclic

satisfying

1. the usual cosimplicial relations (as in Δ) for d^i's and s^j's.

2. $\tau_n d^i = d^{i-1}\tau_{n-1}, 1 \leq i \leq n$, and $\tau_n d^0 = d^n$.

   $\tau_n s^j = s^{j-1}\tau_{n+1}, 1 \leq j \leq n$, and $\tau_n s^0 = s^n \tau_{n+1}^2$.

3. $\tau_n^{n+1} = \mathrm{Id}_{[n]}, n \geq 0$.

Remark 2.4.

1. Although $\Delta\mathbf{C}$ has the "same" objects as Δ, we do not regard $[n]$ as sets, because morphisms cannot be viewed as set maps. For example, $[0]$ is not the terminal object in $\Delta\mathbf{C}$ (see Remark 2.5). In fact, $\Delta\mathbf{C}$ has neither initial nor terminal object. We have Connes duality $\Delta\mathbf{C}^{op} \cong \Delta\mathbf{C}$.

2. The two relations $\tau_n d^0 = d^n$ and $\tau_n s^0 = s^n \tau_{n+1}^2$ are redundant and can be omitted. Indeed,

$$
\begin{aligned}
d^n &= \tau_n^{n+1} d^n = \tau_n^n (\tau_n d^n) = \tau_n^n \left(d^{n-1}\tau_{n-1}\right) = \cdots = \tau_n d^0 \tau_{n-1}^n = \tau_n d^0 \\
s^n \tau_{n+1}^2 &= \tau_n^{n+1} s^n \tau_{n+1}^2 = \tau_n^n (\tau_n s^n) \tau_{n+1}^2 = \tau_n \left(s^{n-1}\tau_{n+1}\right)\tau_{n+1}^2 = \cdots = \tau_n s^0 \tau_{n+1}^{n+2} = \tau_n s^0
\end{aligned}
$$

The structure of $\Delta\mathbf{C}$ is described by

Theorem 2.1. $\Delta\mathbf{C}$ *contains* Δ *as a subcategory (but not full) and*

1. $Aut_{\Delta\mathbf{C}}([n]) \cong \mathbb{Z}/(n+1)\mathbb{Z}, \forall n \geq 0$.

2. *every morphism* $f \in Hom_{\Delta\mathbf{C}}([n],[m])$ *can be uniquely factored as* $f = \varphi \circ \gamma$ *where* $\gamma \in Aut_{\Delta\mathbf{C}}([n])$ *and* $\varphi \in Hom_{\Delta}([n],[m])$.

We will prove this in general for any crossed simplicial group.

Remark 2.5. $[0]$ is not the terminal object in $\Delta\mathbf{C}$ because for any $n \geq 1$, and $f \in \mathrm{Hom}_{\Delta\mathbf{C}}([n],[0])$, $\mathrm{Id} \neq g \in \mathrm{Aut}_{\Delta\mathbf{C}}([n])$, there exists a unique factorization

$$
\begin{tikzcd}
{[n]} \arrow[r, "g"] \arrow[d, "\varphi"'] & {[n]} \arrow[d, "f"] \\
{[0]} \arrow[r, "\mathrm{Id}_{[0]}"] & {[0]}
\end{tikzcd}
$$

i.e. $\varphi = f \circ g \neq f$. Hence there exists more than one element in $\mathrm{Hom}_{\Delta\mathbf{C}}([n],[0])$ for any $n \geq 1$.

Definition 2.11. A *cyclic object* in a category $\mathscr{C}$ is a functor $X : \Delta\mathbf{C}^{op} \to \mathscr{C}$.

In $\Delta\mathbf{C}^{op}$ the objects are the same $\{[n]\}_{n\geq 0}$ and morphisms are

$$
\begin{array}{lll}
d_i := \left(d^i\right)^o : [n] \to [n-1] & 1 \leq i \leq n, n \geq 1 & \text{face} \\
s_j := \left(s^j\right)^o : [n] \to [n+1] & 0 \leq i \leq n, n \geq 0 & \text{degeneracy} \\
t_n := (\tau_n)^o : [n] \to [n] & n \geq 0 & \text{cyclic}
\end{array}
$$

Cyclic set C_*.

We define a functor $C_* : \Delta\mathbf{C}^{op} \to \mathbf{Set}$ as follows.

Objects: $[n] \mapsto C_n : \mathrm{Aut}_{\Delta\mathbf{C}^{op}}([n]) \cong \mathbb{Z}/(n+1)\mathbb{Z}$.

Morphisms: any $f \in \mathrm{Hom}_{\Delta\mathbf{C}}([n],[m])$ can be uniquely factored as $f = \varphi \circ \gamma$ where $\gamma \in \mathrm{Aut}_{\Delta\mathbf{C}}([n])$ and $\varphi \in \mathrm{Hom}_{\Delta}([n],[m])$. Take any $g \in \mathrm{Aut}_{\Delta\mathbf{C}}([m])$ and any $a \in \mathrm{Hom}_{\Delta\mathbf{C}}([n],[m])$ and consider $f = g \circ a$, by unique factorization, there exists a unique $\varphi = g^*(a) \in \mathrm{Hom}_{\Delta}([n],[m])$ and a unique $\gamma = a_*(g) \in \mathrm{Aut}_{\Delta\mathbf{C}}([n])$ such that $g \circ a = \varphi \circ \gamma = g^*(a) \circ a_*(g)$.

Thus for fixed $g \in \mathrm{Aut}_{\Delta\mathbf{C}}([m])$, we define a map

$$\begin{array}{rccc} g^* : & \mathrm{Hom}_{\Delta\mathbf{C}}([n],[m]) & \to & \mathrm{Hom}_{\Delta}([n],[m]) \\ & a & \mapsto & g^*(a) \end{array}$$

and for fixed $a \in \mathrm{Hom}_{\Delta\mathbf{C}}([n],[m])$, we define a map

$$\begin{array}{rccc} a_* : & \mathrm{Aut}_{\Delta\mathbf{C}}([m]) & \to & \mathrm{Aut}_{\Delta\mathbf{C}}([n]) \\ & g & \mapsto & a_*(g) \end{array}$$

Dualize everything we get

$$\begin{array}{rccc} g^* : & \mathrm{Hom}_{\Delta\mathbf{C}^{op}}([m],[n]) & \to & \mathrm{Hom}_{\Delta^{op}}([m],[n]) \\ & a & \mapsto & g^*(a) \\ a_* : & \mathrm{Aut}_{\Delta\mathbf{C}^{op}}([m]) & \to & \mathrm{Aut}_{\Delta\mathbf{C}^{op}}([n]) \\ & g & \mapsto & a_*(g) \end{array}$$

such that $a \circ g = a_*(g) \circ g^*(a)$.

The second is associative, i.e.

$$(a' \circ a)_*(g) = (a')_*(a_*(g)), \forall a \in \mathrm{Hom}_{\Delta\mathbf{C}^{op}}([m],[n]), a' \in \mathrm{Hom}_{\Delta\mathbf{C}^{op}}([n],[n'])$$

which follows from the following commutative diagram

$$\begin{tikzcd}
{[m]} \arrow[r, "g"] \arrow[d, "g^*(a)"'] & {[m]} \arrow[d, "a"] \\
{[n]} \arrow[r, "a_*(g)"] \arrow[d, "(a_*(g))_*(a')"'] & {[n]} \arrow[d, "a'"] \\
{[n']} \arrow[r, "(a')_*(a_*(g))"] & {[n']}
\end{tikzcd}$$

and unique factorization property.

Furthermore, $a_* \left(\mathrm{Id}_{[m]}\right) = \mathrm{Id}_{[n]}$ by unique factorization property.

Thus we can define the functor

$$\begin{array}{cccc}
C_* : & \Delta\mathbf{C}^{op} & \longrightarrow & \mathbf{Set} \\
 & [n] & \longmapsto & C_n \\
 & (a : [m] \to [n]) & \longmapsto & (a_* : C_m \to C_n)
\end{array}$$

Explicitly, by the formula $a \circ g = a_* (g) \circ g^* (a)$, we have

$$d_i \circ t_n = (d_i)_* (t_n) \circ (t_n)^* (d_i) .$$

On the other hand, by the relation in $\Delta\mathbf{C}^{op}$, we have

$$d_i \circ t_n = t_{n-1} \circ d_{i-1}, \quad 1 \leq i \leq n \ \cdot$$

Hence by unique factorization, we have

$$\begin{aligned}
d_i^c (t_n) &\coloneqq t_{n-1}, \quad 1 \leq i \leq n \\
d_0^c (t_n) &\coloneqq \mathrm{Id} \\
s_j^c (t_n) &\coloneqq t_{n+1}, \quad 1 \leq j \leq n \\
s_0^c (t_n) &\coloneqq t_{n+1}^2
\end{aligned}$$

Lemma 2.3. *We have the isomorphism of simplicial sets*

$$\begin{array}{rcl}
\mathbb{S}^1_* & \xrightarrow{\cong} & C_* \\
s_0^n (*) & \mapsto & (t_n)^0 = 1 \\
s_{n-1} \cdots \hat{s}_i \cdots s_0 (t) & \mapsto & t_n^{i+1}
\end{array}$$

Remark 2.6. [A. Connes, J. L. Loday] The geometric realization of simplicial circle is $\left|\mathbb{S}^1_*\right| = |B_*\mathbb{Z}| = \mathbb{S}^1$ the topological group, which is not a coincidence.

2.3 Categorical Examples

Nerve of a category

Recall a category $\mathscr{C}$ is small if its objects form a (proper) set. Associate to such a category a simplicial set $\mathcal{N}_*\mathscr{C}$ (or $\mathcal{B}_*\mathscr{C}$) defined by

$$
\begin{aligned}
\mathcal{N}_0\mathscr{C} &= \{\text{objects in } \mathscr{C}\} = Ob\,(\mathscr{C}) \\
\mathcal{N}_1\mathscr{C} &= \{\text{morphisms in } \mathscr{C}\} = Mor\,(\mathscr{C}) \\
\mathcal{N}_2\mathscr{C} &= \left\{\text{composable morphisms } c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} c_2 \text{ in } \mathscr{C}\right\} \\
&\cdots \\
\mathcal{N}_n\mathscr{C} &= \left\{n\text{-composable morphisms } c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} c_n \text{ in } \mathscr{C}\right\}
\end{aligned}
$$

$$
\mathcal{N}_2\mathscr{C}:\quad
\begin{tikzcd}
 & c_1 \arrow[dr, "f_1"] & \\
\bullet \arrow[ur, "f_0"] \arrow[rr, dashed, "f_1f_1"] & & c_2
\end{tikzcd}
$$

$$
\mathcal{N}_3\mathscr{C}:\quad
\begin{tikzcd}
 & c_1 \arrow[rr, "f_2f_1"] \arrow[dr, "f_1"] & & \\
\bullet \arrow[ur, "f_0"] \arrow[rrr, dashed, "f_2f_1f_0"] \arrow[rr, "f_1f_0"'] & & c_2 \arrow[r, "f_2"'] & c_3
\end{tikzcd}
$$

This suggests that

$$
\begin{array}{rccc}
d_i: & \mathcal{N}_n\mathscr{C} & \longrightarrow & \mathcal{N}_{n-1}\mathscr{C} \\
 & \left(c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} c_n\right) & \longmapsto & \left(c_0 \xrightarrow{f_0} \cdots \xrightarrow{f_{i-2}} c_{i-1} \xrightarrow{f_{i-1}} \hat{c}_i \xrightarrow{f_i} c_{i+1} \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_{n-1}} c_n\right) \\
 & (f_{n-1}, \cdots, f_0) & \longmapsto & \begin{cases} (f_{n-1}, \cdots, f_1) & i = 0 \\ (f_{n-1}, \cdots, f_i f_{i-1}, \cdots, f_0) & 1 \le i \le n-1 \\ (f_{n-2}, \cdots, f_0) & i = n \end{cases}
\end{array}
$$

$$\begin{array}{cccc} s_j : & \mathcal{N}_n\mathscr{C} & \longrightarrow & \mathcal{N}_{n+1}\mathscr{C} \\ & \left(c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} c_n\right) & \longmapsto & \left(c_0 \xrightarrow{f_0} \cdots \xrightarrow{f_{i-1}} c_i \xrightarrow{\mathrm{Id}} c_i \xrightarrow{f_i} c_{i+1} \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_{n-1}} c_n\right) \\ & (f_{n-1}, \cdots, f_0) & \longmapsto & (f_{n-1}, \cdots, f_i, f_i, \cdots, f_0) \end{array}$$

Another way to view this construction is the following. Let

- $s, t : Mor(\mathscr{C}) \to Ob(\mathscr{C})$ the source and target maps.
- $i : Ob(\mathscr{C}) \to Mor(\mathscr{C})$ the identity morphism map.
- $\circ : Mor(\mathscr{C}) \times_{Ob(\mathscr{C})} Mor(\mathscr{C}) \to Mor(\mathscr{C})$ where $Mor(\mathscr{C}) \times_{Ob(\mathscr{C})} Mor(\mathscr{C})$ is the fibred product

$$\begin{tikzcd} Mor(\mathscr{C}) \times_{Ob(\mathscr{C})} Mor(\mathscr{C}) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] & Mor(\mathscr{C}) \arrow[d, "t"] \\ Mor(\mathscr{C}) \arrow[r, "s"] & Ob(\mathscr{C}) \end{tikzcd}$$

Notice that the structure of $\mathscr{C}$ gives us

$$\begin{tikzcd} Ob(\mathscr{C}) \arrow[r, dashed, "s" description] & Mor(\mathscr{C}) \arrow[l, shift right=2, "i"'] \arrow[l, shift left=2, "t"] \arrow[r, dashed, shift left=1] \arrow[r, dashed, shift right=1] & Mor(\mathscr{C}) \times_{Ob(\mathscr{C})} Mor(\mathscr{C}) \arrow[l, shift right=3] \arrow[l] \arrow[l, shift left=3] \end{tikzcd} .$$

The new construction can be viewed as an "extension" of a category to a "full" simplicial set

$$\begin{tikzcd} \mathcal{N}_0(\mathscr{C}) \arrow[r, dashed, "s" description] & \mathcal{N}_1(\mathscr{C}) \arrow[l, shift right=2, "i"'] \arrow[l, shift left=2, "t"] \arrow[r, dashed, shift left=1] \arrow[r, dashed, shift right=1] & \mathcal{N}_2(\mathscr{C}) \cdots \arrow[l, shift right=3] \arrow[l] \arrow[l, shift left=3] \end{tikzcd}$$

where

$$\mathcal{N}_n(\mathscr{C}) = \underbrace{Mor(\mathscr{C}) \times_{Ob(\mathscr{C})} Mor(\mathscr{C}) \times_{Ob(\mathscr{C})} \cdots \times_{Ob(\mathscr{C})} Mor(\mathscr{C})}_{n} =: Mor_n(\mathscr{C})$$

The maps $d_i's$ and s_j's are the structure maps of iterated fibred products.

Question. $\mathcal{N}_*\mathscr{C}$ (as a simplicial set up to isomorphism) determines $\mathscr{C}$ uniquely up to isomorphism. What kind of simplicial sets do we obtain this way? This gives us a proper way to relax the definition of category, and therefore we get ∞-category or quasi-category.

Čech nerve of a covering

Let X be a topological space with a covering $\mathscr{U} = \{U_i\}_{i \in I}$, where I is not ordered. We can define a category $\mathscr{C} = X_{\mathscr{U}}$ by

Objects: the disjoint union of all U_i's, $\bigsqcup_{i \in I} U_i = \{(x, U_i) \,|\, x \in U_i \in \mathscr{U}\}$

Morphisms: Disjoint union of all intersections of U_i's and U_j's $\coprod_{(i,j)\in I^2} U_i \cap U_j$,

$$\mathrm{Hom}_{\mathscr{C}}\left((x, U_i), (y, U_j)\right) = \begin{cases} x \to y, & x = y \in U_i \cap U_j \\ \emptyset, & \text{o.w.} \end{cases}$$

$\mathcal{N}_* X_{\mathscr{U}}$ is called the Čech nerve of the covering $\mathscr{U}$. This is a simplicial space.

Remark 2.7. (Classical Fact) If $\mathscr{U}$ is contractible and "good", i.e. all U_i's are contractible spaces and all finite intersection $U_{i_1} \cap \cdots \cap U_{i_r}$ are either empty or contractible, then the geometric realization of the Čech nerve determines X up to homotopy

$$X \simeq \left|\mathcal{N}_* X_{\mathscr{U}}\right|.$$

This allows us to define homotopy types for various objects (e.g. étale homotopy types).

Question. Given an affine variety $X = \mathrm{Spec} A$ over $\mathbb{C}$ (with topology induced from the topology on $\mathbb{C}$) we can consider its (Čech nerve) Grothendieck homotopy type defined in terms of covering. Dually, we can assign homotopy type to A by viewing it as an object (resolution of A) in $\mathbf{sCommAlg}_k$ (there is a model structure on it). What is the relation between the two ways of defining these homotopy types?

The nerve of a group If G is a discrete group, there are two ways to view as a category.

1. $\mathcal{C} = \underline{G}$ is the category with single object $\{*\}$ and morphisms are the elements in G.
   $\mathcal{N}_*\underline{G} = \mathcal{B}_*\underline{G}$ is the nerve of G.
   $\mathcal{B}_n G = G^n, n \geq 0$.

$$\begin{array}{cccc} d_i : & G^n & \to & G^{n-1} \\ & (g_1, \cdots, g_n) & \to & \begin{cases} (g_2, \cdots, g_n) & i = 0 \\ (g_1, \cdots, g_i g_{i+1}, \cdots, g_n) & 1 \leq i \leq n-1 \\ (g_1, \cdots, g_{n-1}) & i = n \end{cases} \end{array}$$

$$\begin{array}{cccc} s_j : & G^n & \to & G^{n+1} \\ & (g_1, \cdots, g_n) & \mapsto & (g_1, \cdots, g_{i-1}, 1, g_i, \cdots, g_n) \end{array}$$

2. The category EG has G as its objects and

$$\mathrm{Hom}_{EG}(g_1, g_2) = \{h \in G | h g_1 = g_2\} = \left\{g_2 g_1^{-1}\right\}$$

The nerve of EG is $\mathcal{E}_*G = \mathcal{N}_*EG$ given by $\mathcal{E}_0G = G, \mathcal{E}_1G = G^2, \cdots, \mathcal{E}_nG = G^{n+1}$ and

$$\begin{array}{rccc} d_i : & \mathcal{E}_nG & \to & \mathcal{E}_{n-1}G \\ & (g_0, \cdots, g_n) & \mapsto & (g_0, \cdots, \hat{g}_i, \cdots, g_n) \\ s_j : & \mathcal{E}_nG & \to & \mathcal{E}_{n+1}G \\ & (g_0, \cdots, g_n) & \mapsto & (g_0, \cdots, g_j, g_j, \cdots, g_n) \end{array}$$

Remark 2.8. Note that we don't use group action, so this also makes sense for any set X. We can define $E_*X = \left\{E_nX = X^{n+1}\right\}$ with $d_i's$ and s_j's as above.

Relation between B_*G and E_*G

1. There is a natural projection of simplicial sets

$$\begin{array}{rccc} p : & E_*G & \longrightarrow & B_*G \\ & (g_0, \cdots, g_n) & \longmapsto & \left(g_0g_1^{-1}, g_1g_2^{-1}, \cdots, g_{n-1}g_n^{-1}\right) \end{array}$$

2. There is a right G-action of E_*G

$$\begin{array}{ccc} E_*G \times G & \longrightarrow & E_*G \\ (g_0, \cdots, g_n) \times g & \longmapsto & (g_0g, \cdots, g_ng) \end{array}$$

such that

$$\begin{tikzcd} E_*G \arrow[rr, "p"] \arrow[dr] & & B_*G \\ & E_*G/G \arrow[ur, "\cong"'] & \end{tikzcd}$$

This gives an example of simplicial principal G-bundle.

3. Note that E_*G is a simplicial group which acts on the left on B_*G

$$\begin{array}{ccc} E_*G \times B_*G & \longrightarrow & B_*G \\ (g_0, \cdots, g_n) \times (h_1, \cdots, h_n) & \longmapsto & \left(g_0h_1g_1^{-1}, g_1h_2g_2^{-1}, \cdots, g_{n-1}h_{n-1}g_n^{-1}\right) \end{array}$$

Remark 2.9. (Twisted nerve) B_*G is actually a cyclic set,

$$B_*G: \quad \begin{tikzcd} \Delta^{op} \arrow[r] \arrow[d, hook] & \mathbf{Set} \\ \Delta\mathbf{C}^{op} \arrow[ur, dashed] & \end{tikzcd}$$

called twisted nerve.

More generally, for $z \in \mathcal{Z}(G) \subseteq G$

$$\begin{array}{cccc} t_n: & G^n & \to & G^n \\ & (g_1, \cdots, g_n) & \longmapsto & \left(z(g_1 \cdots g_n)^{-1}, g_2, \cdots, g_n\right) \end{array}$$

then $t_n^{n+1}(g_1, \cdots, g_n) = \left(zg_1z^{-1}, \cdots, zg_nz^{-1}\right) = (g_1, \cdots, g_n)$, so $t_n^{n+1} = \mathrm{Id}$.

For $z = 1$, $B_*(G, 1) = B_*G$ is the canonical structure on B.

Borel construction

Let X be a set with a group G action,

$$\begin{array}{ccc} G \times X & \to & X \\ (g, x) & \mapsto & g \cdot x \end{array}$$

Define a category $\mathscr{C} = G \ltimes X$ called *action groupoid* by

Objects: $x \in X$.

Morphisms: $\mathrm{Hom}(x, y) = \{g \in G | gx = y\}$.\\

The nerve $\mathcal{B}_*(G \ltimes X) \coloneqq \mathcal{N}_*(G \ltimes X)$ is given by

$\mathcal{B}_0(G \ltimes X) = X$.

$\mathcal{B}_1(BG \ltimes X) = G \times X$.

$\cdots$

$\mathcal{B}_n(G \ltimes X) = G^n \times X, n \geq 0$.

$$\begin{array}{cccc} d_i: & \mathcal{B}_n(G \ltimes X) & \to & \mathcal{B}_{n-1}(G \ltimes X) \\ & (g_1, \cdots, g_n, x) & \mapsto & \begin{cases} (g_2, \cdots, g_n, x), & i = 0 \\ (g_1, \cdots, g_i g_{i+1}, \cdots, g_n, x) & 1 \leq i \leq n-1 \\ (g_1, \cdots, g_{n-1}, x) & i = n \end{cases} \end{array}$$

$$\begin{array}{rccc} s_j: & \mathcal{B}_n\left(G\ltimes X\right) & \to & \mathcal{B}_{n+1}\left(G\ltimes X\right) \\ & \left(g_1,\cdots,g_n,x\right) & \mapsto & \left(g_1,\cdots,g_{j-1},1,g_j,\cdots,g_n,x\right) \end{array}$$

Notice that if $X=\{*\}$, then $\mathcal{B}_*\left(G\ltimes\{*\}\right)=\mathcal{B}_*G$, and if $X=G$, then $\mathcal{B}_*\left(G\ltimes G\right)=\mathcal{E}_*G$.

In general,

$$\mathcal{B}_*\left(G\ltimes X\right)\cong\mathcal{E}_*G\times_G X\coloneqq\left\{\mathcal{E}_nG\times X/\sim\right\}_{n\geq 0}$$

where

$$\left(g_1,\cdots,g_n,x\right)\sim\left(g_1,\cdots,g_{j-1},1,g_j,\cdots,g_n,x\right).$$

Exercise 2.2. (Homotopy normal maps) How to extend the notion of a normal subgroup with the inclusion $N\hookrightarrow G$? The idea of normal group comes from the quotient group G/N. It turns out that Borel construction is what we want. We cam also extend to normal morphisms in model categories.

Bousfield-Kan construction

If we think of G as a category $\underline{G}$, then the action of G on X is a functor

$$\begin{array}{rccc} X: & \underline{G} & \to & \mathbf{Set} \\ & * & \mapsto & X \end{array}$$

Given any small diagram $X:\mathscr{C}\to\mathbf{Set}$ (or $\mathbf{Top}$), define the BK category $\mathscr{C}_X\coloneqq\mathscr{C}\ltimes X$ (or $\mathscr{C}\int X$) by

Objects: $\left\{(c,x)\,|c\in Ob\left(\mathscr{C}\right),x\in X\left(c\right)\right\}$

Morphisms: $\mathrm{Hom}\left(\left(c,x\right),\left(c',x'\right)\right)=\left\{f\in\mathrm{Hom}_{\mathscr{C}}\left(c,c'\right)|X\left(f\right)x=x'\right\}$.

Definition 2.12. $\mathcal{B}_*\left(\mathscr{C}_X\right)\coloneqq\mathcal{N}_*\left(\mathscr{C}_X\right)$ is called the *Bousfield-Kan construction.*

Remark 2.10. Notice that

1. if

$$\begin{array}{rccc} X: & \mathscr{C} & \to & \mathbf{Set} \\ & x & \mapsto & * \end{array}$$

   then $\mathcal{B}_*(\mathscr{C}_*)=\mathcal{B}_*\mathscr{C}$.

2. if $X:\underline{G}\to\mathbf{Set}$ then $\mathcal{B}_*\left(\underline{G}_X\right)=\mathcal{B}_*\left(G\ltimes X\right)$ is the Borel construction.

Example 2.3. Take $\mathscr{C}=\{0\to 1\}$ then $X:\mathscr{C}\to\mathbf{Set}$ is just a map $f:X_0\to X_1$ in $\mathbf{Set}$. $\mathscr{C}_X$ is given by

Objects: $X_0\bigsqcup X_1=\left\{\left(0,x_0\right)|x_0\in X_0\right\}\cup\left\{\left(1,x_1\right)|x_1\in X_1\right\}$

Morphisms: $Mor(\mathscr{C}_X) \cong X_0 \bigsqcup X_0 \bigsqcup X_1$ since

$$\mathrm{Hom}_{\mathscr{C}_X}((i,x_i),(j,x_j)) = \begin{cases} \mathrm{Id}_{x_i}, & i=j, x_i = x_j \\ f & i=0, j=1, f(x_0)=x_1 \\ \emptyset & \text{o.w.} \end{cases}$$

Hence the BK construction is given by

$\mathcal{N}_0\mathscr{C}_X = X_0 \bigsqcup X_1$.

$\mathcal{N}_1\mathscr{C}_X = X_0 \bigsqcup X_0 \bigsqcup X_1 \cong \bigsqcup_{x_0 \in X_0} \mathrm{Hom}(x_0, f(x_0)) \bigsqcup \{\mathrm{Id}_{x_0}\}_{x_0\in X_0} \bigsqcup \{\mathrm{Id}_{x_1}\}_{x_1 \in X_1}$.

Exercise 2.3. Show that the only non-degenerate simplices are $(\mathcal{N}_*\mathscr{C}_X)^{\mathrm{nondeg}} = \underset{\dim 0}{(X_0 \bigsqcup X_1)} \bigsqcup \underset{\dim 1}{X_0}$.

Proof. The only n-composable morphisms in $\mathscr{C}_X$ are either of the form $\mathrm{Id}^n_{x_i} = s_0^{n-1}(\mathrm{Id}_{x_i})$ or $\mathrm{Id}_{x_1} \circ \cdots \circ \mathrm{Id}_{x_1} \circ f \circ \mathrm{Id}_{x_0} \circ \cdots \circ \mathrm{Id}_{x_0} = s_{i+1}^{n-i-1} s_0^i(f)$. □

This will imply that the geometric realization of $\mathcal{N}_*\mathscr{C}_X$ is the mapping cylinder of f, $|\mathcal{N}_*\mathscr{C}_X| = \mathrm{Cyl}(f) := X_0 \bigsqcup (X_0 \times \Delta^1) \bigsqcup X_1 / \sim$ where $x_0 \sim (x_0, 0)$ and $(x_0,1) \sim f(x_0)$.

Exercise 2.4. Let $\mathscr{C} = \{1 \leftarrow 0 \rightarrow 2\}$, then $\mathscr{C}_X$ gives the double cylinder.

Grothendieck construction

Let **Cat** be the category of all small categories. Notice that we have a natural inclusion **Set** $\hookrightarrow$ **Cat** as discrete categories. It is natural to consider diagrams of categories $X : \mathscr{C} \to$ **Cat**.

Define the category $\mathscr{C} \int X$ (or $\mathscr{C} \ltimes X$) by

Objects: $\{(c,x) \,|\, c \in \mathscr{C}, x \in Ob(X(c))\}$.

Morphisms: $\mathrm{Hom}((c,x),(c',x')) = \{(f,\varphi) \,|\, f \in \mathrm{Hom}(c,c'), \varphi \in \mathrm{Hom}_{X(c')}(X(f)x, x')\}$.

Composition is defined by $(f',\varphi') \circ (f,\varphi) = (f' \circ f, \varphi' \circ X(f')\varphi)$

$$(c,x) \xrightarrow{(f,\varphi)} (c',x') \xrightarrow{(f',\varphi')} (c'',x'')$$

This is well-defined. The identity is given by $(\mathrm{Id}_c, \mathrm{Id}_{X(c)})$. And the composition is $c \xrightarrow{f} c' \xrightarrow{f'} c''$ and $X(f'f)(x) = X(f')X(f)(x) \xrightarrow{X(f')\varphi} X(f')(x') \xrightarrow{\varphi'} x''$.

Composition is associative because the composition $c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \xrightarrow{f_3} c_3$ in $\mathscr{C}$ is associative and X is functor,

$$X(f_3f_2f_1)(x_0) \xrightarrow{X(f_3f_2)\varphi_1 = X(f_3)X(f_2)\varphi_1} X(f_3f_2)(x_1) \xrightarrow{X(f_3)\varphi_2} X(f_3)(x_2) \xrightarrow{\varphi_3} x_3.$$

If $\xi : X \Rightarrow X'$ is a morphism of diagrams $\{\xi_c : X(c) \to X'(c)\}_{c\in Ob(\mathscr{C})}$, we define the functor $\mathscr{C}\int\xi : \mathscr{C}\int X \to \mathscr{C}\int X'$ on

Objects: $(c,x) \mapsto (c, \xi_c(x))$

Morphisms: $(f,\varphi) \mapsto (f, \xi_{c'}(\varphi))$

This gives a functor

$$\begin{array}{rccc} \mathscr{C}\int - : & \mathbf{Fun}(\mathscr{C},\mathbf{Cat}) & \longrightarrow & \mathbf{Cat} \\ & X & \longmapsto & \mathscr{C}\int X \\ & \xi & \longmapsto & \mathscr{C}\int \xi \end{array}$$

Theorem 2.2. *[Thomson's Theorem] We have the composition*

$$\mathscr{C} \xrightarrow{X} \mathbf{Cat} \xrightarrow{\mathcal{N}} \mathbf{sSet} \xrightarrow{|-|} \mathbf{Top}$$

and

$$\left|\underset{\mathscr{C}}{hocolim}(\mathcal{N}_*X)\right| \simeq \left|\mathcal{N}_*\left(\mathscr{C}\int X\right)\right|$$

the nerve of $\mathscr{C}\int X$ is a simplicial model of $\underset{\mathscr{C}}{hocolim}(\mathcal{N}_*X)$.

This can be restated and proved in terms of simplicial sets (combinatorial).

2.4 Quasi-categories

Question: given a simplicial set X_*, does there exists a small category $\mathscr{C}$ such that $X_* \simeq \mathcal{N}_*\mathscr{C}$?

We want to characterize the image of $\mathcal{N}_* : \mathbf{Cat} \to \mathbf{sSet}, \mathscr{C} \mapsto \mathcal{N}_*\mathscr{C}$. And we can generalize the definition of categories in this way.

Terminology

Definition 2.13. If $\mathscr{C}$ is a category, $\mathscr{M} \subseteq Mor(\mathscr{C})$ is a class of morphisms, we say that a morphism $f : A \to B$ has a *right lifting property* with respect to $\mathscr{M}$ ($f \in RLP(\mathscr{M})$) if for any commutative diagram

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "M\ni g"'] & A \arrow[d, "f"] \\ D \arrow[r, "k"] \arrow[ur, dashed, "\tilde{k}"] & B \end{tikzcd}$$

there exists $\tilde{k} : D \to A$ such that $f\tilde{k} = k$ and $\tilde{k}g = h$.

Dually, a morphism $f : A \to B$ has a *left lifting property* with respect to $\mathscr{M}$ ($f \in LLP(\mathscr{M})$) if for any commutative diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "g \in M"] \\ B \arrow[r, "k"] \arrow[ur, dashed, "\tilde{k}"] & D \end{tikzcd}$$

there exists $\tilde{k} : B \to C$ such that $g\tilde{k} = k$ and $\tilde{k}f = h$.

Consider the inner horn inclusions

$$\left\{ i_n^k : \Lambda_k [n]_* \hookrightarrow \Delta [n]_* \right\}_{0<k<n, n\geq 2} \subseteq Mor\left(\mathbf{sSet}\right)$$

Definition 2.14. (Joyal, J. Lurie) A small *quasi-category* is a simplicial set X such that the canonical projection $X \to \bullet$ (where $\bullet$ is the discrete simplicial one-element set) has the RLP with respect to the inner horn inclusion.

$$\begin{tikzcd} \Lambda_k [n]_* \arrow[r] \arrow[d, hook, "i_n^k"'] & X_* \arrow[d] \\ \Delta [n]_* \arrow[r, "p_n"] \arrow[ur, dashed, "\tilde{p}_n"] & \bullet \end{tikzcd}$$

Note that $i_n^k : \Lambda_k [n]_* \to \Delta [n]_*$ gives

$$\left(i_n^k\right)^* : \mathrm{Hom}_{\mathbf{sSet}}\left(\Delta [n]_*, X_*\right) \to \mathrm{Hom}_{\mathbf{sSet}}\left(\Lambda_k [n]_*, X_*\right)$$

where $X_n \coloneqq X_*([n]) \cong \mathrm{Hom}_{\mathbf{sSet}}\left(\Delta [n]_*, X_*\right)$ by Yoneda lemma, and we denote $\Lambda_k^n(X) \coloneqq \mathrm{Hom}_{\mathbf{sSet}}\left(\Lambda_k [n]_*, X_*\right)$.

Hence the definition can be restated as $\left(i_n^k\right)^* : X_n \to \Lambda_k(X)$ is surjective for any $0 < k < n, n \geq 2$. We say that each k-horn in X can be filled to an n-simplex.

Theorem 2.3. *(Characterization of $\mathcal{N}_*\mathscr{C}$) A simplicial set X is isomorphic to the nerve of a small category $\mathscr{C}$. (necessarily unique up to isomorphism) if and only if $\left(i_n^k\right)^* : X_n \to \Lambda_k^n(X)$ is a bijection for any $0 < k < n, n \geq 2$.*

Thus quasi-categories are generalization of categories.

Remark 2.11. The condition $\left(i_n^k\right)^* : X_n \to \Lambda_k^n(X)$ is surjective for any $0 < k < n, n \geq 2$ is called weak Kan condition, so quasi-categories are weak Kan complexes.

The usual (strong) Kan condition is that $\left(i_n^k\right)^* : X_n \to \Lambda_k^n(X)$ is surjective for any $0 \leq k < n, n \geq 1$.

Examples: simplicial groups (Moore Theorem); simplicial complex of a topological space $\mathrm{Sing}_*(X) \coloneqq \left\{\mathrm{Hom}_{\mathbf{Top}}\left(\Delta^n, X\right)\right\}_{n\geq 0}$.

How does a quasi-category look like?

- X_0 = the set of objects
- X_1 = the set of morphisms
- d_0, d_1 are the source and target maps respectively,

$$X_0 \underset{d_1}{\overset{d_0}{\leftleftarrows}} X_1$$

- composable $f, g \in X_1$, i.e. $d_0 g = d_1 f$,

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, dashed, "h"] & & z \end{tikzcd}$$

gives an element $\sigma_0 \in \Lambda_1^2(X)$. The weak Kan condition implies that there exists $\sigma \in X_2$ "filling" σ_0, so $h =$ "$g \circ f$" is defined by taking face of σ. h is well-defined "up to homotopy" specified by weak Kan condition. Associativity holds only up to homotopy.

It turns out that basic constructions of category theory can be extended to quasi-categories (thanks to Lurie).

Theorem 2.4. *[J01] There is a model structure on the full subcategory of quasi-categories in* **sSet** *(and thus is one of the models for* $(\infty, 1)$*-categories.*

Definition 2.15. A $(\infty, 1)$-*category* is a higher category with all k-morphisms ($k \geq 2$) being invertible.

Motivation. To give $\mathrm{Sing}_*(X)$ a category-like structure.

Quasi-categories = ∞-categories = topological spaces (from the point of view of homotopy theory).

Part II

Review of Some Categorical Constructions

3 Categorical Constructions

3.1 Adjoint Functors

Fact 3.1. *(Categorical) If* $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ *are adjoint functors, then we have natural isomorphisms*

$$\Psi_{A,B} : Hom_{\mathcal{D}}(FA, B) \xrightarrow{\cong} Hom_{\mathcal{C}}(A, GB) \tag{3}$$

together with adjunction morphisms

$$\begin{cases} \eta : Id_{\mathcal{C}} \Rightarrow GF & \textit{unit} \\ \varepsilon : FG \Rightarrow Id_{\mathcal{D}} & \textit{counit} \end{cases} \tag{4}$$

such that the compositions $\varepsilon F \circ F\eta$ *and* $G\varepsilon \circ \eta G$ *are identities.*

$$\begin{aligned} \left(F \overset{F\eta}{\Rightarrow} FGF \overset{\varepsilon F}{\Rightarrow} F\right) &= Id_F \\ \left(G \overset{\eta G}{\Rightarrow} GFG \overset{G\varepsilon}{\Rightarrow} G\right) &= Id_G \end{aligned} \tag{5}$$

Lemma 3.1. *If* $F : \mathcal{C} \to \mathcal{D}$ *and* $G : \mathcal{D} \to \mathcal{C}$ *is a pair of functors given with morphisms 4 satisfying 5, then* $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ *are adjoint functors, with 3 given by*

$$\Psi_{A,B}(f : FA \to B) = \left(A \xrightarrow{\eta_A} GFA \xrightarrow{Gf} GB\right).$$

Proof. The inverse $\Psi^{-1}_{A,B}$ is given by

$$\Psi^{-1}_{A,B}(g : A \to GB) = \left(FA \xrightarrow{Fg} FGB \xrightarrow{\varepsilon_B} B\right).$$

□

Proposition 3.1. *If* $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ *are adjoint functors. Take any category* $\mathcal{A}$ *and consider*

$$\begin{array}{rccc} G^* : & \mathbf{Fun}(\mathcal{C},\mathcal{A}) & \rightleftharpoons & \mathbf{Fun}(\mathcal{D},\mathcal{A}) : F^* \\ & \left(\mathcal{C} \xrightarrow{H} \mathcal{A}\right) & \longmapsto & \left(\mathcal{D} \xrightarrow{G} \mathcal{C} \xrightarrow{H} \mathcal{A}\right) \\ & \left(\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{K} \mathcal{A}\right) & \longleftarrow\!\shortmid & \left(\mathcal{D} \xrightarrow{K} \mathcal{A}\right) \end{array}$$

$$\begin{array}{rccc} F_* : & \mathbf{Fun}(\mathcal{A},\mathcal{C}) & \rightleftharpoons & \mathbf{Fun}(\mathcal{A},\mathcal{D}) : G_* \\ & \left(\mathcal{A} \xrightarrow{H} \mathcal{C}\right) & \longmapsto & \left(\mathcal{A} \xrightarrow{H} \mathcal{C} \xrightarrow{F} \mathcal{D}\right) \\ & \left(\mathcal{A} \xrightarrow{K} \mathcal{D} \xrightarrow{G} \mathcal{C}\right) & \longleftarrow\!\shortmid & \left(\mathcal{A} \xrightarrow{K} \mathcal{D}\right) \end{array}$$

then (G^*, F^*) *and* (F_*, G_*) *are adjoint pairs.*

Proof. Given η, ε, for $H : \mathcal{C} \to \mathcal{A}$ define $H\eta : H \Rightarrow HGF$ by

$$H\eta = \{H(\eta_A) : H(A) \to HGF(A)\}_{A \in Ob(\mathcal{C})}$$

Varying $H \in \mathbf{Fun}(\mathcal{C},\mathcal{A})$ we get $\eta^* : \mathrm{Id}_{\mathbf{Fun}(\mathcal{C},\mathcal{A})} \Rightarrow F^*G^*$. Similarly we have $\varepsilon^* : G^*F^* \Rightarrow \mathrm{Id}_{\mathbf{Fun}(\mathcal{D},\mathcal{A})}$ and they satisfy 5, so (G^*, F^*) is an adjoint pair.

In a similar manner we can get $\eta_* : \mathrm{Id}_{\mathbf{Fun}(\mathcal{A},\mathcal{C})} \Rightarrow G_*F_*$ and $\varepsilon_* : F_*G_* \Rightarrow \mathrm{Id}_{\mathcal{D}}$ and they satisfy 5, so (F_*, G_*) is an adjoint pair. □

In the rest of this section, all categories $\mathcal{C}$ are locally small, i.e. $\mathrm{Hom}(x, y)$ is a set for any $x, y \in Ob(\mathcal{C})$, unless stated otherwise.

3.2 (Co)limits

(Co)limits can be generalized by "local" UMP (objects) or "global" UMP (diagrams).

Let $\mathcal{C}, \mathcal{D}$ be two categories and $d \in Ob(\mathcal{D})$. Define a constant at d to be the functor

$$\begin{array}{rccc} \mathrm{const}_d : & \mathcal{C} & \longrightarrow & \mathcal{D} \\ & c & \mapsto & d \\ & f & \mapsto & \mathrm{Id}_d \end{array}$$

Definition 3.1. (Local) Given any functor $F:\mathcal{C}\to\mathcal{D}$, the colimit $\mathrm{colim}_{\mathcal{C}}(F)$ is the morphism

$$\begin{tikzcd} F \arrow[rr, Rightarrow] \arrow[dr, Rightarrow, "\eta"'] & & \mathrm{const}_d \\ & \mathrm{const}_{\mathrm{colim}_{\mathcal{C}}(F)} \arrow[ur, Rightarrow, "\mathrm{const}_f"'] & \end{tikzcd}$$

which is universal among all morphisms from F to constants in $\mathcal{D}$ for some unique $f:\mathrm{colim}_{\mathcal{C}}(F)\to d$.

Definition 3.2. (Global) Define a functor

$$\begin{aligned} \mathrm{const}: \quad \mathcal{D} &\to \mathcal{D}^{\mathcal{C}} \\ d &\mapsto \mathrm{const}_d \\ f &\mapsto \mathrm{const}_f \end{aligned}$$

then colimit and limit can be defined as left and right adjoint to const.

$$\begin{tikzcd} \mathcal{D} \arrow[d, "\mathrm{const}"'] \\ \mathcal{D}^{\mathcal{C}} \arrow[u, bend left=60, "\mathrm{colim}"] \arrow[u, bend right=60, "\mathrm{lim}"'] \end{tikzcd} .$$

Thus we have natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}}\left(\mathrm{colim}_{\mathcal{C}}(F), d\right) &\cong \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}\left(F, \mathrm{const}_d\right) \\ \mathrm{Hom}_{\mathcal{D}}\left(d, \lim_{\mathcal{C}}(F)\right) &\cong \mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}\left(\mathrm{const}_d, F\right) \end{aligned}$$

Remark 3.1. If the category $\mathcal{D}$ is (co)complete, the two definitions agree. For ordinary (co)limits, these two definitions are essentially equivalent, but once we pass to homotopy (co)limits, there is an important (and very confusing) difference between these two approaches.

(Co)product

Let $\mathcal{C}$ be a discrete category.

$Ob(\mathcal{C}) = I = \{i\}_{i\in I}$.

$Mor(\mathcal{C}) = \{\mathrm{Id}_i\}_{i\in I}$.

A functor $F:\mathcal{C}\to\mathcal{D}$ is given by $F=\{X_i\}_{i\in I}$ then $\mathrm{colim}_{\mathcal{C}}(F)=\coprod_{i\in I}X_i$ and $\lim_{\mathcal{C}}(F)=\prod_{i\in I}X_i$.

(Co)equalizer

Let $\mathcal{C} = \{ \bullet \rightrightarrows \bullet \}$. A functor $F : \mathcal{C} \to \mathcal{D}$ is given by $\left\{ X_0 \underset{g}{\overset{f}{\rightrightarrows}} X_1 \right\}$.

$\operatorname{colim}_{\mathcal{C}}(F) = \operatorname{coeq}\left\{ X_0 \underset{g}{\overset{f}{\rightrightarrows}} X_1 \right\}$ is universal in the following diagram

$$X_0 \underset{g}{\overset{f}{\rightrightarrows}} X_1 \xrightarrow{\pi} X$$

$\lim_{\mathcal{C}}(F) = \operatorname{eq}\left\{ X_0 \underset{g}{\overset{f}{\rightrightarrows}} X_1 \right\}$ is universal in the following diagram

$$X \xrightarrow{i} X_0 \underset{g}{\overset{f}{\rightrightarrows}} X_1$$

Exercise 3.1. If $\mathcal{D}$ is additive, then $\operatorname{colim}_{\mathcal{C}}(F) = \operatorname{Coker}(f - g)$.

Fact 3.2. *Any colimit (respectively. limit) can be constructed using coproducts and coequalizers. But this is not true for homotopy colimits.*

3.3 (Co)ends

(Co)ends are special (co)limits.

Let $\mathcal{C}$ be a small category, $\mathcal{D}$ a locally small, cocomplete (all small colimits exist) category.

Definition 3.3. For a bifunctor $S : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$, the coend of S is defined as

$$\int^{c \in Ob(\mathcal{C})} S(c, c) := \operatorname{coeq}\left\{ \coprod_{Mor(\mathcal{C}) \ni f : c_0 \to c_1} S(c_1, c_0) \underset{f_*}{\overset{f^*}{\rightrightarrows}} \coprod_{c \in Ob(\mathcal{C})} S(c, c) \right\}$$

where

$$f^* = S(f, 1) : S(c_1, c_0) \to S(c_0, c_0)$$
$$f_* = S(1, f) : S(c_1, c_0) \to S(c_1, c_1)$$

The Universal Mapping Property (UMP) of Coend

The coend $d \coloneqq \int_{c\in Ob(\mathcal{C})} S(c,c) \in Ob(\mathcal{D})$ with $\{\varphi_c : S(c,c,) \to d\}_{c\in Ob(\mathcal{C})}$ such that for any morphism $f : c_0 \to c_1$ in $\mathcal{C}$, we have a diagram

$$\begin{tikzcd}
S(c_1,c_0) \arrow[r, "f^*"] \arrow[d, "f_*"'] & S(c_0,c_0) \arrow[d, "\varphi_{c_0}"] \arrow[ddr, dashed, bend left] & \\
S(c_1,c_1) \arrow[r, "\varphi_{c_1}"] \arrow[drr, dashed, bend right] & d \arrow[dr, "\exists!"] & \\
& & d'
\end{tikzcd}$$

which is universal with respect to this property.

Examples

Colimit.

Given $F : \mathcal{C} \to \mathcal{D}$, define

$$\begin{array}{rccc} S: & \mathcal{C}^{op}\times\mathcal{C} & \to & \mathcal{D} \\ & (c',c) & \mapsto & F(c) \end{array}$$

then

$$\int_{c\in Ob(\mathcal{C})} S(c,c) = \mathrm{coeq}\left\{ \coprod_{Mor(\mathcal{C})\ni f:c_0\to c_1} S(c_1,c_0) \underset{f_*}{\overset{f^*=\mathrm{Id}}{\rightrightarrows}} \coprod_{c\in Ob(\mathcal{C})} S(c,c) \right\} = \mathrm{colim}_{\mathcal{C}}(F).$$

Additive functor tensor product.

Let R be a (not necessary commutative) ring and $\mathcal{C}$ be a small category.

Let $F : \mathcal{C}^{op} \to \mathbf{Mod}\text{-}R$ and $G : \mathcal{C} \to R\text{-}\mathbf{Mod}$ be two additive functors. From this we can define a additive bifunctor

$$\begin{array}{rccc} S \coloneqq F \underset{R}{\boxtimes} G: & \mathcal{C}^{op}\times\mathcal{C} & \longrightarrow & \mathbf{Ab} \\ & (c',c) & \longmapsto & F(c')\otimes_R G(c) \\ & (f',f) & \longmapsto & \left(\begin{array}{rccc} F(f')\otimes_R G(f): & F(d')\otimes_R G(c) & \to & F(c')\otimes_R G(d) \\ & x'\otimes y & \mapsto & F(f')\,x'\otimes G(f)\,y \end{array} \right) \end{array}$$

Definition 3.4. Define $F \underset{R}{\bigotimes} G \coloneqq \int^{c\in Ob(\mathcal{C})} F(c)\otimes_R G(c)$. Explicitly, we have

$$F \underset{R}{\bigotimes} G \cong \bigoplus_{c\in Ob(\mathcal{C})} F(c)\otimes_R G(c) \,/\, \langle F(f)\,x'\otimes y - x'\otimes G(f)\,y \,|\, (f : c\to c')\in Mor(\mathcal{C}),\, x'\in F(c'),\, y\in G(c)\rangle.$$

Exercise 3.2. Fix $c \in Ob(\mathcal{C})$, consider

$$R^{op}(h_c) = R^{op}[\mathrm{Hom}_{\mathcal{C}}(-,c)] : \mathcal{C}^{op} \to \mathbf{Mod}\text{-}R$$

which sends $d \in Ob(\mathcal{C})$ to the free right R-module generated by $\mathrm{Hom}_{\mathcal{C}}(d,c)$. Similarly we have

$$R(h^c) = R[\mathrm{Hom}_{\mathcal{C}}(c,-)] : \mathcal{C} \to R\text{-}\mathbf{Mod}.$$

Check that

$$R^{op}(h_c) \bigotimes\nolimits_{\mathcal{C},R} G \cong G(c)$$
$$F \bigotimes\nolimits_{\mathcal{C},R} R(h^c) \cong F(c)$$

Proof. By construction

$$\begin{aligned} R^{op}(h_c) \bigotimes\nolimits_{\mathcal{C},R} G &\cong \bigoplus\nolimits_{d \in Ob(\mathcal{C})} R^{op}[\mathrm{Hom}_{\mathcal{C}}(d,c)] \otimes_R G(d) / \langle (g' \circ f) \otimes y - g' \otimes G(f) y | \rangle \\ &\cong \bigoplus\nolimits_{d \in Ob(\mathcal{C})} \bigoplus\nolimits_{f:d \to c} R^{op}[f] \otimes G(d)_f / \langle f \otimes y - 1 \otimes G(f) y | y \in G(c) \rangle \\ &\cong G(c) \end{aligned}$$

And similarly we have $F \bigotimes_{\mathcal{C},R} R(h^c) \cong F(c)$. ☐

Exercise 3.3. Let $\underline{R} \coloneqq \mathrm{const}_{R_R} : \mathcal{C}^{op} \to \mathbf{Mod}\text{-}R, c \mapsto R_R$, then $\underline{R} \bigotimes_{\mathcal{C},R} G \cong \mathrm{colim}_{\mathcal{C}}(G)$.

Proof. By construction $\underline{R} \bigotimes_{\mathcal{C},R} G \cong \bigoplus_{c \in Ob(\mathcal{C})} R_R \otimes_R G(c) / \langle r \otimes y - r \otimes G(f) y | f : c \to c', y \in G(c) \rangle \cong \mathrm{colim}_{\mathcal{C}}(G)$.

□

4 Kan Extensions

4.1 Definitions

Problem. Given two functors $F:\mathcal{C}\to\mathcal{D}$ and $G:\mathcal{C}\to\mathcal{E}$, we want to extend F along G. Namely, we want to find a functor $H:\mathcal{E}\to\mathcal{D}$ so that the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{D} \\
\mathcal{E} \arrow[ur, dashed, "H"'] &
\end{tikzcd}$$

"commutes" up to isomorphism, that is $F\cong H\circ G$.

In general, such H does not exist for several reasons. For instance, let $\mathcal{C},\mathcal{D},\mathcal{E}$ be discrete categories. It may happen that there exists some $c'\neq c\in Ob(\mathcal{C})$ such that $F(c)\neq F(c')$ but $G(c)=G(c')$. In this case it's impossible to find such H.

Idea. To find the "closest" universal approximation to the equivalence $F\cong H\circ G$, i.e. construct the universal morphisms $\eta:F\Rightarrow H\circ G$ (left Kan extension) and $\varepsilon:H\circ G\Rightarrow F$ (right Kan extension).

Definition 4.1. (Local) A *left Kan extension* of $F:\mathcal{C}\to\mathcal{D}$ along $G:\mathcal{C}\to\mathcal{E}$ is a functor $L_GF:\mathcal{E}\to\mathcal{D}$ together with a morphism $\eta_{\text{un}}:F\Rightarrow L_GF\circ G$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\eta_{\text{un}}"] & \mathcal{D} \\
\mathcal{E} \arrow[ur, bend right, "L_GF"'] &
\end{tikzcd}$$

which is universal (initial) among all pairs $(H:\mathcal{E}\to\mathcal{D},\eta:F\Rightarrow H\circ G)$ in the sense that there exists a unique $\varphi:L_GF\Rightarrow H$ such that $\eta=(\varphi G)\circ\eta_{\text{un}}$,

$$\begin{tikzcd}
F \arrow[rr, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "\eta_{\text{un}}"'] & & H\circ G \\
& L_GF\circ G \arrow[ur, Rightarrow, "\varphi G"'] &
\end{tikzcd}$$

where φG is the horizontal composition

$$\begin{tikzcd} \mathcal{C} \arrow[r, "G"] & \mathcal{E} \arrow[r, bend left, "L_G F"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\varphi"] & \mathcal{D} \end{tikzcd}$$

given by $\varphi G = \{\varphi_{GX} : L_G F\,(GX) \to H\,(GX)\}_{X \in \mathrm{Ob}(\mathcal{C})}$.

Definition 4.2. (Local) A *right Kan extension* of $F : \mathcal{C} \to \mathcal{D}$ along $G : \mathcal{C} \to \mathcal{E}$ is a functor $R_G F : \mathcal{E} \to \mathcal{D}$ together with a morphism $\varepsilon_{\mathrm{un}} : R_G F \circ G \Rightarrow F$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\Uparrow\varepsilon_{\mathrm{un}}"] & \mathcal{D} \\ \mathcal{E} \arrow[ur, bend right, "R_G F"'] & \end{tikzcd}$$

which is universal (terminal) among all pairs $(K : \mathcal{E} \to \mathcal{D}, \varepsilon : K \circ G \Rightarrow F)$ in the sense that there exists a unique $\varphi : K \Rightarrow R_G F$ such that $\varepsilon_{\mathrm{un}} \circ \varphi G = \varepsilon$,

$$\begin{tikzcd} K \circ G \arrow[rr, Rightarrow, "\varepsilon"] \arrow[dr, Rightarrow, "\varphi G"'] & & F \\ & R_G F \circ G \arrow[ur, Rightarrow, "\varepsilon_{\mathrm{un}}"'] & \end{tikzcd}$$

where φG is the horizontal composition

$$\begin{tikzcd} \mathcal{C} \arrow[r, "G"] & \mathcal{E} \arrow[r, bend left, "R_G F"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Uparrow\varphi"] & \mathcal{D} \end{tikzcd}$$

given by $\varphi G = \{\varphi_{GX} : K\,(GX) \to R_G F\,(GX)\}_{X \in \mathrm{Ob}(\mathcal{C})}$.

Remark 4.1. Given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$, define

$$\mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathcal{D})}\,(F, - \circ G) : \mathbf{Fun}\,(\mathcal{E}, \mathcal{D}) \longrightarrow \mathbf{Set}$$

by $H \mapsto \mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathcal{D})}\,(F, H \circ G)$ on objects. The Left Kan extension $L_G F \in \mathrm{Ob}\,(\mathbf{Fun}\,(\mathcal{E}, \mathcal{D}))$ is precisely the object representing this functor since the UMP precisely gives the isomorphism

$$\mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathcal{D})}\,(F, H \circ G) \cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{E},\mathcal{D})}\,(L_G F, H)$$

for all $H \in \mathrm{Ob}\,(\mathbf{Fun}\,(\mathcal{E}, \mathcal{D}))$.

Definition 4.3. (Global) Fix a functor $G : \mathcal{C} \to \mathcal{E}$ and consider the restriction functor

$$\begin{array}{rccc} G_* := (-) \circ G : & \mathbf{Fun}(\mathcal{E}, \mathcal{D}) & \longrightarrow & \mathbf{Fun}(\mathcal{C}, \mathcal{D}) \\ & H & \longmapsto & H \circ G \\ & \varphi & \longmapsto & \varphi G \end{array}$$

We define the left and right Kan extensions as the left and right adjoints of this functor

$$\begin{tikzcd} \mathbf{Fun}(\mathcal{E}, \mathcal{D}) \arrow[d, "G_*"'] \\ \mathbf{Fun}(\mathcal{C}, \mathcal{D}) \arrow[u, bend left=80, "L_G(-)"] \arrow[u, bend right=80, "R_G(-)"'] \end{tikzcd}$$

i.e. we have natural isomorphisms

$$\mathrm{Hom}_{\mathbf{Fun}(\mathcal{E},\mathcal{D})}(L_G F, H) \cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathcal{D})}(F, H \circ G)$$

$$\mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathcal{D})}(K \circ G, F) \cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{E},\mathcal{D})}(K, R_G F)$$

4.2 Basic Examples

Group representations

Let G be a discrete group and k a field. Let $H \leq G$ be a subgroup. We can think of G as a category $\underline{G}$ with one object $\{*\}$ and $\mathrm{Hom}_{\underline{G}}(*, *) = G$. There is an inclusion $i : \underline{H} \hookrightarrow \underline{G}$.

Let $\mathrm{Rep}_k(G)$ be the category of k-linear representations of G and G-equivariant linear maps.Then $\mathrm{Rep}_k(G)$ can be identified with the category $\mathbf{Fun}(\mathbf{G}, \mathbf{Vect}_k)$.

A linear representation of H in $\mathbf{Vect}_k$ given by $\rho : H \to \mathrm{End}_k(V)$ can be interpreted as a functor

$$\begin{array}{rccc} \rho : & \underline{H} & \to & \mathbf{Vect}_k \\ & * & \mapsto & V \\ & g & \mapsto & \rho(g) \end{array}$$

There are classical functors relating $\mathrm{Rep}_k(H)$ and $\mathrm{Rep}_k(G)$: Res_H^G which is precisely post-composition with the inclusion functor$i : \underline{H} \hookrightarrow \underline{G}$. Define

$$\mathrm{Ind}_H^G := k[G] \otimes_{k[H]} -$$

and

$$\mathrm{Coind}_H^G := \mathrm{Hom}_{k[H]}\left(k[G], -\right).$$

For all $V \in \mathrm{Rep}_k(H)$ and all $W \in \mathrm{Rep}_k(G)$, the Hom-Tensor adjunction gives

$$\mathrm{Hom}_G\left(\mathrm{Ind}_H^G(V), W\right) \cong \mathrm{Hom}_G\left(k[G] \otimes_{k[H]} V, W\right) \cong \mathrm{Hom}_H\left(V, \mathrm{Hom}_{k[G]}(k[G], W)\right) \cong \mathrm{Hom}_H(V, W)$$

where the last isomorphism arises from the isomorphism $\mathrm{Hom}_{k[G]}(k[G], W) \cong \mathrm{Res}_H^G(W) \cong W$ of H-representations. In this way, Ind_H^G is realized as a left adjoint to Res_H^G.

Dually, the isomorphism $\mathrm{Res}_H^G(W) \cong k[G] \otimes_{k[H]} W$ together with the Hom-Tensor adjunction gives

$$\mathrm{Hom}_H\left(W \otimes_{k[H]} k[G], V\right) \cong \mathrm{Hom}_G\left(W, \mathrm{Hom}_{k[H]}(k[G], V)\right) \cong \mathrm{Hom}_G\left(W, \mathrm{Coind}_H^G(V)\right)$$

which realizes Coind_H^G as a right adjoint to Res_H^G.

Identifying $\mathrm{Rep}_k(G)$ and $\mathrm{Rep}_k(H)$ with the functor categories $\mathbf{Fun}(\mathbf{G}, \mathbf{Vect}_k)$ and $\mathbf{Fun}(\mathbf{H}, \mathbf{Vect}_k)$ respectively, then $\mathrm{Res}_H^G = - \circ i =: i_*$ and we obtain the adjunction

$$\begin{tikzcd}
\mathbf{Fun}(\mathbf{G}, \mathbf{Vect}_k) \arrow[d, "i_*"] \\
\mathbf{Fun}(\mathbf{H}, \mathbf{Vect}_k) \arrow[u, bend left=60, "L_{\mathbf{i}}(-)"] \arrow[u, bend right=60, "R_{\mathbf{i}}(-)"']
\end{tikzcd}$$

where $L_{\mathbf{i}} = \mathrm{Ind}_H^G$ and $R_{\mathbf{i}} = \mathrm{Coind}_H^G$.

Remark 4.2. If i is not injective, we can consider homotopy Kan extension (derived version).

(Co)limits

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $\mathcal{E} = \bullet$,

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{D} \\
\bullet \arrow[ur, dashed, "H"'] &
\end{tikzcd}$$

then H is determined by the choice of one object $\in Ob(\mathcal{D})$, so $H \circ G$ is a constant functor and any natural transformation $\eta : F \Rightarrow H \circ G$ is equivalent to a natural transformation $\eta : F \Rightarrow \mathrm{const}_d$. Thus

$$L_G F \cong \mathrm{colim}_{\mathcal{C}} F$$
$$R_G F \cong \lim{}_{\mathcal{C}} F$$

Adjunctions

Given an adjoint pair $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$, then we have the natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(F(c), d) \cong \mathrm{Hom}_{\mathcal{C}}(c, G(d)), \forall c \in Ob(\mathcal{C}), d \in Ob(\mathcal{D}).$$

Take $d = F(c)$, we get the unit $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "F"'] & \mathcal{C} \\ \mathcal{D} \arrow[ur, dashed] & \end{tikzcd}$$

and it gives $(L_F(\mathrm{Id}_{\mathcal{C}}), \eta_{\mathrm{un}}) \cong (G, \eta)$.

take $c = G(d)$ we get the counit $\varepsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\mathrm{Id}"] \arrow[d, "G"'] & \mathcal{D} \\ \mathcal{C} \arrow[ur, dashed] & \end{tikzcd}$$

and it gives $(R_G(\mathrm{Id}_{\mathcal{C}}), \varepsilon_{\mathrm{un}}) \cong (F, \varepsilon)$.

4.3 Properties

Consider the following situation.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{D} \arrow[r, "H"] & \mathcal{F} \\ \mathcal{D} \arrow[ur, "L_G(F)"'] \arrow[urr, bend right, "L_G(H\circ F)"'] & & \end{tikzcd}$$

By UMP of left Kan extension, there exists a natural morphism $\xi : L_G(H \circ F) \Rightarrow H \circ L_G(F)$.

Definition 4.4. The functor H preserves left Kan extension of F along G if both $L_G(F)$ and $L_G(H \circ F)$ exists and ξ is an isomorphism.

Definition 4.5. A left Kan extension $L_G(F)$ is called *absolute* if it is preserved by all functors H.

Nonexample. Consider $i : \underline{H} \hookrightarrow \underline{G}$ and take $U : \mathbf{Vect}_k \to \mathbf{Set}$ the forgetful functor.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\rho"] \arrow[d, "i"'] & \mathcal{D} \arrow[r, "H"] & \mathcal{F} \\ \mathcal{D} \arrow[ur] \arrow[urr, dashed] & & \end{tikzcd}$$

Then $UL_i(\rho) = U\left(k[G] \otimes_{k[H]} V\right)$ and $L_i(U\rho) = G \times_H U(V)$ are very different.

Note that U has no right adjoint, so U is not a left adjoint.

Exercise 4.1. Left adjoints always preserves left Kan extensions.

Proof. Let $F : \mathcal{D} \rightleftarrows \mathcal{F} : G$ be an adjoint pair. We have

$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"] \arrow[d, "K"'] & \mathcal{D} \arrow[r, "F"] & \mathcal{F} \\ \mathcal{E} \arrow[ur] \arrow[urr, dashed] & & \end{tikzcd}$$

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Fun}(\mathcal{E},\mathcal{F})}\left(L_K(F \circ H), N\right) &\cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathcal{F})}\left(F \circ H, N \circ K\right) \\ &\cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathcal{D})}\left(H, G \circ N \circ K\right) \\ &\cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{E},\mathcal{D})}\left(L_K(H), G \circ N\right) \\ &\cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{E},\mathcal{F})}\left(F \circ L_K(H), N\right) \end{aligned}$$

and by Yoneda lemma, $L_K(F \circ H) \cong F \circ L_K(H)$. □

Example 4.1. (Total derived functor) Let $\mathcal{C}$ be a model category and $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a Quillen pair.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\gamma"] & \mathcal{D} \\ \mathrm{Ho}(\mathcal{C}) = \mathcal{C}\left[\mathrm{We}^{-1}\right] \arrow[ur, dashed] & \end{tikzcd}$$

The total left functor is $\mathbb{L}F = R_\gamma(F)$ and total right functor $\mathbb{R}F = L_\gamma(F)$.

Theorem 4.1. *[[Mal07], 2007] Derived functors of Quillen functors are absolute Kan-extensions.*

4.4 Pointwise Kan Extensions

Definition 4.6. A right Kan extension is called *pointwise* if it is preserved by all (covariant) representable functors $h^d = \mathrm{Hom}\,(d, -) : \mathcal{D} \to \mathbf{Set}, d \in Ob\,(\mathcal{D})$.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\Uparrow", pos=0.3] & \mathcal{D} \arrow[r, "h^d"] & \mathbf{Set} \\
\mathcal{D} \arrow[ur, "L_G(F)"'] \arrow[urr, bend right, "L_G\left(h^d \circ F\right)"'] & &
\end{tikzcd}$$

Definition 4.7. A left Kan extension is called *pointwise* if it is mapped to a right Kan extension by all (contravariant) representable functors $h_d = \mathrm{Hom}\,(-, d) : \mathcal{D}^{op} \to \mathbf{Set}, d \in Ob\,(\mathcal{D})$.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow", pos=0.3] & \mathcal{D} \arrow[r, "h_d"] & \mathbf{Set}^{op} \\
\mathcal{D} \arrow[ur, "R_G(F)"'] \arrow[urr, bend right, "L_G(h_d \circ F)"'] & &
\end{tikzcd}$$

Remark 4.3. This is very similar to the property of limits and colimits.

We have the following:

$$\text{Absolute Kan extensions} \underset{\neq}{\subset} \text{pointwise Kan extensions} \underset{\neq}{\subset} \text{Kan extensions.}$$

We will give a characterization (formula) for pointwise Kan extensions.

Comma Category

Definition 4.8. Given a functor $F : \mathcal{C} \to \mathcal{D}$ and an object $d \in Ob\,(\mathcal{D})$, we can define the *comma category* F/d (or $F \downarrow d$) as follows.

Objects: $Ob\,(F/d) = \{(c, f)\,|c \in Ob\,(\mathcal{C})\,, f \in \mathrm{Hom}_{\mathcal{D}}\,(Fc, d)\}$.

Morphisms: $\mathrm{Hom}_{F/d}\,((c, f)\,, (c', f')) = \{\varphi \in \mathrm{Hom}_{\mathcal{C}}\,(c, c')\,|f' \circ F\varphi = f\}$, i.e. morphisms are those $\varphi : c \to c'$ such that the following diagram

$$\begin{tikzcd}
Fc \arrow[rr, "F\varphi"] \arrow[dr, "f"'] & & Fc' \arrow[dl, "f'"] \\
& d &
\end{tikzcd}$$

commutes.

Dually we can define cocomma category $d \backslash F$ (or $d \downarrow F$).

Note that there is a forgetful functor $U : F/d \to \mathcal{C}$ which can be thought as a fibre functor.

Example 4.2. Let $F = \mathrm{Id}_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$, then F/d is the category over d and $d\backslash F$ is the category under d.

Example 4.3. (Category of simplicial sets) Take

$$\begin{array}{rccc} F = h_* : & \Delta & \to & \mathbf{sSet} \\ & [n] & \mapsto & \Delta[n]_* \end{array}$$

For any $X \in Ob(\mathbf{sSet})$ we call the category $\Delta X := h_*/X$ the category of simplices of X, which is given by

Objects: $Ob(\Delta X) = \{([n], x) : [n] \in \Delta, x \in \mathrm{Hom}_{\mathbf{sSet}}(\Delta[n]_*, X) = X_n\} = \coprod_{n \geq 0} X_n$.

Morphisms: $\mathrm{Hom}(([n], x), ([m], y)) = \{f : [n] \to [m] \,|\, X(f)\, y = y \circ h_*(f) = x\}$.

Another way to define ΔX is to consider $X : \Delta^{op} \to \mathbf{Set}$ and take the Grothendieck construction $\Delta_X^{op} = \Delta^{op} \int X$ where

Objects: $Ob(\Delta_X^{op}) = \{([m], y) \,|\, [m] \in Ob(\Delta^{op}) = Ob(\Delta), y \in X([m]) = X_m\}$.

Morphisms: $\mathrm{Hom}(([n], x)\, ([m], y)) = \{f \in \mathrm{Hom}_{\Delta^{op}}([n], [m]) \,|\, X(f)\, x = y\}$.

Hence $\Delta_X^{op} \cong (\Delta X)^{op}$.

Example 4.4. Let $\mathcal{C}$ be a small category and take

$$\begin{array}{rccc} \Delta^* : & \Delta & \hookrightarrow & \mathbf{Cat} \\ & [n] & \mapsto & \overrightarrow{n} = \{0 \to 1 \to \cdots \to n\} \end{array}$$

This is a fully faithful functor. Then $\Delta^*/\mathcal{C} :=$ the simplicial complex over $\mathcal{C}$ defined by

Objects: $Ob(\Delta^*/\mathcal{C}) = \{([n], f) \,|\, [n] \in \Delta, f : \overrightarrow{n} \to \mathcal{C}\} \cong \coprod_{n \geq 0} \mathcal{N}_n \mathcal{C}$.

Morphisms: $\mathrm{Hom}(([n], f), ([m], g)) = \{\varphi : \overrightarrow{n} \to \overrightarrow{m} | g \circ \Delta^*(\varphi) = f\}$.

Therefore $\Delta^*/\mathcal{C} \cong \Delta \mathcal{N} \mathcal{C}$.

Remark 4.4. ΔX and $(\Delta X)^{op}$ are examples of Reedy categories (with fibrant or cofibrant, respectively).

Computing Kan extension via (co)limits

Theorem 4.2. *A left Kan extension is pointwise if and only if it can be computed by the formula*

$$L_G(F)(e) = colim_{G/e}\left(G/e \xrightarrow{U} \mathcal{C} \xrightarrow{F} \mathcal{D}\right).$$

Theorem 4.3. *(Dual version) A right Kan extension is pointwise if and only if it can be computed by the formula*

$$R_G(F)(e) = lim_{G/e}\left(e\backslash G \xrightarrow{U} \mathcal{C} \xrightarrow{F} \mathcal{D}\right).$$

Proof. It suffices (and more convenient) to prove the dual version. Indeed, $L_G(F)$ is characterized by

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Fun}(\mathcal{E},\mathcal{D})}(L_G(F), H) & \cong & \mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathcal{D})}(F, G_*H) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{Hom}_{\mathbf{Fun}(\mathcal{E},\mathcal{D})^{op}}(H, L_G(F)) & & \mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathcal{D})^{op}}(G_*H, F) \end{array}$$

Note that $\mathbf{Fun}(\mathcal{C},\mathcal{D})^{op} = \mathbf{Fun}(\mathcal{C}^{op},\mathcal{D})$, so $R_{G^0}\left(F^0\right) \cong L_G(F)$.

Since limits commutes with representable functors, i.e. given $F : \mathcal{J} \to \mathcal{D}$,

$$\mathrm{Hom}_{\mathcal{D}}\left(d, \lim_{\mathcal{J}} F\right) \cong \lim_{\mathcal{J}}\left(\mathrm{Hom}_{\mathcal{D}}(d, F(-))\right)$$

so if $R_G(F)$ is given by such a formula, then it automatically commutes with representable functors $h^d = \mathrm{Hom}_{\mathcal{D}}(d,-), \forall d \in \mathcal{D}$.

Assume that $R_G(F)$ is pointwise, then for any $d \in Ob(\mathcal{D})$ and any $e \in Ob(\mathcal{E})$,

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}}(d, R_GF(e)) &= h^d(R_GF(e)) = \left(h^d \circ R_GF\right)(e) \\ &= \mathrm{Hom}_{\mathbf{Fun}(\mathcal{E},\mathcal{D})}\left(h^e, h^d \circ R_GF\right) \\ &\cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{E},\mathcal{D})}\left(h^e, R_G\left(h^d \circ F\right)\right) \\ &\cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathbf{Set})}\left(h^e \circ G, h^d \circ F\right) \\ &\cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathbf{Set})}\left(\mathrm{Hom}_{\mathcal{E}}(e, G(-)), \mathrm{Hom}_{\mathbf{Fun}(\mathcal{D},\mathbf{Set})}(d, F(-))\right) \\ &\cong \text{the set of cones under } d \text{ of the functor } FU \\ &= \mathrm{Hom}_{\mathbf{Fun}(e\backslash G,\mathcal{D})}(\mathrm{const}_d, FU) \\ &\cong \mathrm{Hom}_{\mathcal{D}}\left(d, \lim_{e\backslash G} FU\right) \end{aligned}$$

By Yoneda lemma, $R_GF(e) \cong \lim_{e\backslash G}(FU)$. ☐

Corollary 4.1. *If $\mathcal{D}$ is cocomplete then every left Kan extension of $F : \mathcal{C} \to \mathcal{D}$ exists and is pointwise. If $\mathcal{D}$ is complete then every right Kan extension of $F : \mathcal{C} \to \mathcal{D}$ exists and is pointwise.*

Corollary 4.2. *If $\mathcal{D}$ is cocomplete and G is fully faithful, then η_{un} is an isomorphism of functors.*

Proof. Take $c \in Ob(\mathcal{C})$ and consider $G/G(c)$, then G is fully faithful implies that $G/G(c)$ has terminal object. Indeed, $Ob(G/G(c)) = \{(c', f') \,|\, c' \in Ob(\mathcal{C}), f' : Gc' \to Gc\}$. Then $\left(c, \mathrm{Id}_{G(c)}\right)$ is terminal in $G/G(c)$

because

$$\mathrm{Hom}\left(\left(c',f'\right),\left(c,\mathrm{Id}_{G(c)}\right)\right)=\left\{h:c'\to c|G\left(h\right)\circ\mathrm{Id}_{G(e)}=f'\right\}=G^{-1}\left(f'\right)$$

contains only one element.

Recall, by UMP of colimits, if $\mathcal{J}$ has terminal object $*$, then for any $F:\mathcal{J}\to\mathcal{D}$, $\mathrm{colim}_{\mathcal{J}}\left(F\right)=F\left(*\right)$. Now for any $c\in Ob\left(\mathcal{C}\right)$, take $e=G\left(c\right)$ and apply formula

$$L_G\left(F\right)\left(e\right)=\mathrm{colim}_{G/G(e)}\left(FU\right)\cong FU\left(c,\mathrm{Id}_{G(c)}\right)=F\left(c\right)$$

So $L_GF\circ G\cong F$. □

Example 4.5. (Co-Yoneda lemma) Simplest version. Consider

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\mathrm{Id}_{\mathcal{C}}"'] & \mathcal{D} \\ \mathcal{C} \arrow[ur, "L_{\mathrm{Id}_{\mathcal{C}}}F\cong F"'] & \end{tikzcd}$$

then we have $L_{\mathrm{Id}_{\mathcal{C}}}F\cong F$ and $F\left(c\right)\cong\mathrm{colim}_{\mathcal{C}/c}\left(\mathcal{C}/c\xrightarrow{U}\mathcal{C}\xrightarrow{F}\mathcal{D}\right)$.

Example 4.6. Take $\hat{\mathcal{C}}=\mathbf{Fun}\left(\mathcal{C}^{op},\mathbf{Set}\right)$ and Yoneda functor $h_*:\mathcal{C}\hookrightarrow\hat{\mathcal{C}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook, "h_*"] \arrow[d, hook, "h_*"'] & \hat{\mathcal{C}} \\ \hat{\mathcal{C}} \arrow[ur, "L_h(h)\cong\mathrm{Id}_{\hat{\mathcal{C}}}"'] & \end{tikzcd}$$

Every presheaf on a small category $\mathcal{C}$ is canonically a colimit of representable presheaf. For any $X\in Ob\left(\hat{\mathcal{C}}\right)$,

$$X\cong\mathrm{colim}_{h_*/X}\left(h_*/X\xrightarrow{U}\mathcal{C}\xrightarrow{h_*}\hat{\mathcal{C}}\right).$$

Example 4.7. Take $\mathcal{C}=\Delta$, then $\hat{\mathcal{C}}=\mathbf{sSet}$ and $h/X=\Delta X$ is the category of simplices over X.

$Ob\left(\Delta X\right)=\coprod_{n\geq0}X_n$. Note $X_n\cong\mathrm{Hom}_{\mathbf{sSet}}\left(\Delta\left[n\right]_*,X\right)$.

$\mathrm{Hom}_{\Delta X}\left(\left(\left[n\right],x\right),\left(\left[m\right],y\right)\right)=\left\{f:\left[n\right]\to\left[m\right]|X\left(f\right)y=x\right\}$.

$$\begin{tikzcd} \Delta \arrow[r, hook, "h_*"] \arrow[d, "h_*"'] & \hat{\Delta}=\mathbf{sSet} \\ \mathbf{sSet} \arrow[ur, "L_h(h)\cong\mathrm{Id}_{\mathbf{sSet}}"'] & \end{tikzcd}$$

So we have $X = \operatorname{colim}_{\Delta X}\left(\Delta X \xrightarrow{U} \Delta \xrightarrow{h} \mathbf{sSet}\right) = \operatorname{colim}_{\Delta[n]_* \to X} \Delta\, [n]_*$.

Computing Kan extension via coends

Proposition 4.1. *If $\mathcal{C}$ is small, $\mathcal{D}$ is cocomplete and $\mathcal{E}$ is locally small, then*

$$L_G F\,(e) = \int^{c \in Ob(\mathcal{C})} Hom_{\mathcal{E}}\,(Gc, e) \bullet F\,(c)\,.$$

Decoding. $\bullet$ is the bifunctor

$$\begin{array}{cccc} \bullet: & \mathbf{Set} \times \mathcal{D} & \longrightarrow & \mathcal{D} \\ & (X, d) & \longmapsto & \coprod_{x \in X} d \\ & \left(X \xrightarrow{f} Y, d \xrightarrow{g} d'\right) & \longmapsto & \coprod_{x \in X} g : \coprod_{x \in X} d \to \coprod_{y \in Y} d' \end{array}$$

taking (X, d) to the coproduct of copies of d indexed by $x \in X$. ($\mathcal{D}$ is said to be tensored over **Set**.)

This $\int$ is the coend of the bifunctor

$$\begin{array}{cccc} S_e: & \mathcal{C}^{op} \times \mathcal{C} & \longrightarrow & \mathcal{D} \\ & (c', c) & \longmapsto & \mathrm{Hom}_{\mathcal{E}}\,(Gc', e) \bullet F\,(c) = \displaystyle\coprod_{h: Gc' \to e} F\,(c) \\ & \left(c' \xrightarrow{f'} d', c \xrightarrow{f} d\right) & \longmapsto & S_e\,(f', f) : \displaystyle\coprod_{h: Gc' \to e} F\,(c) \to \coprod_{h: Gd' \to e} F\,(d) \end{array}$$

for fixed $e \in Ob\,(\mathcal{E})$.

Exercise 4.2. Test it on group representations. Deduce $L_i\,(\rho) = \mathrm{Ind}_H^G\,(\rho) \cong k\,[G] \bigotimes_{k[H]} V$ from coend formula.

$$\begin{tikzcd} \underline{H} \arrow[r, "\rho"] \arrow[d, hook, "i"'] \arrow[dr, phantom, "\eta\Downarrow"] & \mathbf{Vect}_k \\ \underline{G} \arrow[ur, "L_i(\rho)"'] & \end{tikzcd}$$

Proof. By the formula,

$$L_i \rho\,(*) = \int^{* \in Ob(\underline{H})} \mathrm{Hom}_{\underline{G}}\,(*, *) \bullet \rho\,(*) = \int^{* \in Ob(\underline{H})} G \bullet V \cong \coprod_{g \in G} V / \langle w - hv \rangle_{h \in H, w \in V_{gh}, v \in V_g} \cong k\,[G] \bigotimes_{k[H]} V.$$

$\square$

Exercise 4.3. Compute $L_i(X)$ for

$$\begin{tikzcd} \Delta^{op} \arrow[r] \arrow[d, hook, "i"] & \mathbf{Set} \\ \Delta\mathbf{C}^{op} \arrow[ur] & \end{tikzcd}$$

or equivalently, the left adjoint of

$$\begin{tikzcd} \mathbf{Set}^{\Delta\mathbf{C}^{op}} \arrow[d, "U"] \\ \mathbf{sSet} \arrow[u, bend left, "L_i"] \end{tikzcd}$$

Proof. We can use the coend formula to get

$$L_iX([n]) = \int^{m\geq 0} \mathrm{Hom}_{\Delta\mathbf{C}^{op}}([k],[l])\bullet X_l = \coprod_{h:[m]\to[n]\in Mor(\Delta\mathbf{C})} X_m/\left\langle X(f)\,y - x\right\rangle_{f:[m]\to[k]\in Mor(\Delta),x\in(X_k)_{hf},y\in(X_m)_h}.$$

□

4.5 More Examples of Kan Extensions

Induction from Grothendieck construction

Let $\mathcal{C}$ be a small category and $X:\mathcal{C}\to\mathbf{Set}$ be a diagram of shape $\mathcal{C}$ in $\mathbf{Set}$. Then we define $\mathcal{C}_X\left(=\mathcal{C}\int X\right)$ be the category with

$Ob(\mathcal{C}_X)=\{(c,x)\,|c\in Ob(\mathcal{C}),x\in X(c)\}$.

$\mathrm{Hom}_{\mathcal{C}_X}((c,x),(c',x'))=\{\varphi\in\mathrm{Hom}(c,c')\,|X\varphi(x)=x'\}$.

There is an obvious functor

$$\begin{array}{rccc} F: & \mathcal{C}_X & \to & \mathcal{C} \\ & (c,x) & \mapsto & c \\ & \varphi & \mapsto & \varphi \end{array}$$

Take any abelian category $\mathcal{A}$ (e.g. $\mathbf{Vect}_k$ or R-$\mathbf{Mod}$) . Assume that $\mathcal{A}$ is complete and cocomplete. Write $\mathbf{Mod}_{\mathcal{A}}^{\mathcal{C}}:=\mathbf{Fun}(\mathcal{C},\mathcal{A})$ and we have $F^*:\mathbf{Mod}_{\mathcal{A}}^{\mathcal{C}}\to\mathbf{Mod}_{\mathcal{A}}^{\mathcal{C}_X}$. Since $\mathcal{A}$ is complete and cocomplete, the functor F^* has both left and right adjoints.

$$\begin{tikzcd} \mathbf{Mod}_{\mathcal{A}}^{\mathcal{C}} \arrow[d, "F^*"] \\ \mathbf{Mod}_{\mathcal{A}}^{\mathcal{C}_X} \arrow[u, bend left=60, "F_*"] \arrow[u, bend right=60, "F_!"'] \end{tikzcd}$$

We want to compute F_* explicitly.

Claim 4.1. F_* is given by, for any $M : \mathcal{C}_X \to \mathcal{A}$,

$$\begin{array}{rccc} F_*(M): & \mathcal{C} & \to & \mathcal{A} \\ & d & \mapsto & \bigoplus_{x\in X(d)} M(d,x) \end{array}$$

Consider the diagram

$$\begin{tikzcd} \mathcal{C}_X \arrow[r, "M"] \arrow[d, "F"'] & \mathcal{A} \\ \mathcal{C} \arrow[ur, "L_F(M)=:F_*M"'] & \end{tikzcd}$$

For $d \in Ob(\mathcal{C})$, the category F/d is defined as

$Ob(F/d) = \{(c,x,f) \,|\, c \in Ob(\mathcal{C}), x \in X(c), f : c \to d\}$.

$\mathrm{Hom}((c,x,f),(c',x',f')) = \{\varphi \in \mathrm{Hom}_{\mathcal{C}}(c,c') \,|\, X\varphi(x) = x', f' \circ \varphi = f\}$.

By definition, we have $L_F M(d) = \mathrm{colim}_{F/d}\left(F/d \xrightarrow{U} \mathcal{C}_X \xrightarrow{M} \mathcal{A}\right) =: A \in Ob(\mathcal{A})$ with universal $\eta^{\mathrm{un}} : M \circ U \Rightarrow \mathrm{const}_A$. We claim that $A = \bigoplus_{y\in X(d)} M(d,y)$ with

$$\eta^{\mathrm{un}} = \left\{\eta^{\mathrm{un}}_{(c,x,f)} : M(c,x) \xrightarrow{\varphi_f} \bigoplus_{y\in X(d)} M(d,y)\right\}_{(c,x,f)\in Ob(F/d)}$$

where $f : c \to d$ induces $X(f) : X(c) \to X(d), x \mapsto X(f)x =: y$ and $\varphi_f := f \in \mathrm{Hom}_{F/d}((c,x,f),(d,y,\mathrm{Id}))$. This makes sense because $X(f)x = y$ and $\mathrm{Id} \circ f = f$. So $M(\varphi_f) = M(f) : M(c,x) \to M(d,y) \hookrightarrow A$.

We need to check that η^{un} is a morphism of functors. Given $\varphi : (c,x,f) \to (c',x',f')$ we have

1. a commutative diagram

$$\begin{tikzcd} MU(c,x,f) \arrow[r, "\eta^{\mathrm{un}}_{(c,x,f)}"] \arrow[d, "M(\varphi)"'] & A \arrow[d, equal] \\ MU(c',x',f') \arrow[r, "\eta^{\mathrm{un}}_{(c',x',f')}"] & A \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} M(c,x) \arrow[r, "M(\varphi_f)"] \arrow[d, "M(\varphi)"'] & A \arrow[d, equal] \\ M(c',x') \arrow[r, "M(\varphi_{f'})"] & A \end{tikzcd}$$

given by

$$\begin{tikzcd} m \arrow[r, mapsto] \arrow[d, mapsto] & M(f)m \arrow[d, equal] \\ M(\varphi)m \arrow[r] & M(f')(M(\varphi)m) = M(f'\circ\varphi)(m) = M(f)m \end{tikzcd}$$

2. η^{un} is universal.

   Given $B \in Ob(\mathcal{A})$ and $\eta : MU \Rightarrow \mathrm{const}_B$, $\eta = \left\{\eta_{(c,x,f)} : M(c,x) \to B\right\}_{(c,x,f)\in Ob(F/d)}$, we can define

$$\xi : A = \bigoplus_{y\in X(d)} M(d,y) \to B$$

   by UMP of direct sum. Then $\eta(d, y, \mathrm{Id}) = \xi|_{M(d,y)} : M(d,y) \to B$ and we have a commutative diagram

$$\begin{tikzcd} MU \arrow[rr, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "\eta^{\mathrm{un}}"'] & & \mathrm{const}_B \\ & \mathrm{const}_A \arrow[ur, Rightarrow, "\mathrm{const}_f"'] & \end{tikzcd}$$

Example 4.8. Let $\mathcal{A} = \mathbf{Vect}_k$ and

$$\begin{array}{rccc} M: & \mathcal{C}_X & \to & \mathcal{A} \\ & (c,x) & \mapsto & k \\ & \varphi & \mapsto & \mathrm{Id}_k \end{array}$$

the trivial module, then $F_*(k) = k[X]$ since

$$\begin{array}{rccc} F_*(k): & \mathcal{C} & \to & \mathbf{Vect}_k \\ & d & \mapsto & k[X(d)] \end{array}$$

Exercise 4.4. We have natural isomorphisms

$$\mathrm{ToR}_i^{\mathcal{C}}(F_*M, N) \cong \mathrm{ToR}_i^{\mathcal{C}_X}(M, F^*(N)), \forall i \geq 0.$$

where k is a commutative ring, $M \in \mathbf{Mod}_k^{\mathcal{C}_X}$, $\mathrm{ToR}^{\mathcal{C}}$ are derived tensor products $\left(-\overset{\mathbb{L}}{\underset{\mathcal{C},k}{\bigotimes}}-\right)$.

Skeleton and coskeleton

For a fixed $n \geq 0$, define $\Delta_{\geq n}$ to be the full subcategory of Δ with objects $\{[0], [1], \cdots, [n]\}$. Then $i_n : \Delta_{\leq n} \hookrightarrow \Delta$ is an inclusion.

Given a category $\mathcal{C}$, define $\mathbf{s}\mathcal{C} := \mathbf{Fun}(\Delta, \mathcal{C})$ and $\mathbf{s}_n\mathcal{C} := \mathbf{Fun}(\Delta_{\leq n}, \mathcal{C})$. Then the restriction i_n^* has left and right adjoints if $\mathcal{C}$ is complete and cocomplete.

$$\begin{tikzcd} \mathbf{s}\mathcal{C} \arrow[d, "i_n^*"] \\ \mathbf{s}_n\mathcal{C} \arrow[u, bend left=60, "L_{i_n}"] \arrow[u, bend right=60, "R_{i_n}"'] \end{tikzcd}$$

Definition 4.9. The *n-th skeleton* of $X \in Ob(\mathbf{s}\mathcal{C})$ is $\mathrm{sk}_n(X) \coloneqq L_{i_n}(i_n^* X) \in Ob(\mathbf{s}\mathcal{C})$.

Definition 4.10. The *n-th coskeleton* of $X \in Ob(\mathbf{s}\mathcal{C})$ is $\mathit{co}\mathrm{sk}_n(X) \coloneqq R_{i_n}(i_n^* X) \in Ob(\mathbf{s}\mathcal{C})$.

By our theorem, $\mathrm{sk}_n(X)_m \cong \underset{\varphi:[m]\to[k]}{\mathrm{colim}} \varphi^*(X_k), k \le n$. Since φ can be factored uniquely in Δ as surjections followed by injections, this is equivalent to $\mathrm{sk}_n(X)_m \cong \underset{\varphi:[m]\twoheadrightarrow[k]}{\mathrm{colim}} \varphi^*(X_k), k \le n$.

This implies that when $\mathcal{C} = \mathbf{Set}$, $\mathrm{sk}_n(X)_m = X_m$ if $m \le n$. $\mathrm{sk}_n(X)$ is the simplicial subset of X generated by nondegenerating simplicies of degree at most n.

Notice that we have a filtration $\mathrm{sk}_n(X) \hookrightarrow \mathrm{sk}_{n+1}(X) \hookrightarrow \cdots \hookrightarrow X$, so $\underset{n\to\infty}{\mathrm{colim}}\mathrm{sk}_n(X) \cong X$.

Remark 4.5. If we have a map $X \to Y$ between simplicial sets, we can define $\mathrm{sk}_n^X Y \coloneqq X \bigcup_{\mathrm{sk}_n X} \mathrm{sk}_n Y$ by the push-out

$$\begin{tikzcd} \mathrm{sk}_n(X) \arrow[r, hook, "i_n^*"] \arrow[d, "\mathrm{sk}_n(f)"'] & X \arrow[d] \\ \mathrm{sk}_n(Y) \arrow[r, hook] & \mathrm{sk}_n^X Y \end{tikzcd}$$

Augmented simplicial object

The category Δ has terminal object $[0]$ but no initial object. Define the augmented simplicial set Δ_+ as

$Ob(\Delta_+) = Ob(\Delta) \cup \{[-1]\}$.

$$\mathrm{Hom}_{\Delta_+}([n],[m]) = \begin{cases} \mathrm{Hom}_\Delta([n],[m]) & n,m \ge 0 \\ [-1] \to [m] & n = -1 \\ \emptyset & m = -1, n \ge 0 \end{cases}$$

Definition 4.11. For any category $\mathcal{C}$ we define the *augmented simplicial object* as a functor $X : \Delta_+ \to \mathcal{C}$. Denote $\mathbf{s}_+\mathcal{C} = \mathbf{Fun}(\Delta_+, \mathcal{C})$.

Explicitly, each $X \in Ob(\mathbf{s}_+\mathcal{C})$ is given by $X \in Ob(\mathbf{s}\mathcal{C})$ together with $X_{-1} \in Ob(\mathcal{C})$ and $\varepsilon : X_0 \to X_{-1}$ in $Mor(\mathcal{C})$ such that

$$\begin{tikzcd} X_{-1} & X_0 \arrow[l, "\varepsilon"'] & X_1 \cdots \arrow[l, shift right, "d_1"'] \arrow[l, shift left, "d_0"] \end{tikzcd}$$

agrees in the sense that $\varepsilon d_1 = \varepsilon d_0$. We can denote $\varepsilon = d_0$ and extend $d_i d_j = d_{j-1} d_i, i < j$ for $n = 0$.

Equivalently we can denote $\varepsilon = d_0$ as a constant simplicial object and then augmentation is a morphism $\varepsilon_* : X_* \to X_{-1}$ in $\mathbf{s}\mathcal{C}$.

Note that $i_+ : \Delta \hookrightarrow \Delta_+$ implies $(i_+)^* : \mathbf{s}_+\mathcal{C} \to \mathbf{s}\mathcal{C}$. If $\mathcal{C}$ is complete and cocomplete, this has both left and right adjoints.

Take $\mathcal{C} = \mathbf{Set}$, we have

$$
\begin{tikzcd}
\Delta \arrow[r, "X"] \arrow[d, hook, "i_+"'] \arrow[dr, phantom, "\Downarrow\eta"] & \mathbf{Set} \\
\Delta_+ \arrow[ur, "L_i X"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbf{s}_+\mathbf{Set} \arrow[d, "(i_+)^*"] \\
\mathbf{sSet} \arrow[u, bend left=60, "L"] \arrow[u, bend right=60, "R(\text{trivial})"']
\end{tikzcd}
$$

where L is given by left Kan extension $L_{i_+}(X)$,

$$L_{i+}(X)_n = L_{i_+}(X)[(n)] = \operatorname{colim}_{\Delta^{op}/[n]} \left(\Delta^{op}/[n] \xrightarrow{U} \Delta^{op} \xrightarrow{X} \mathbf{Set} \right).$$

If $n \geq 0$, $Ob(\Delta^{op}/[n]) = \left\{ ([m], f : [m] \to [n]) \,|\, f \in Mor\left(\Delta_+^{op}\right), [m] \in Ob(\Delta^{op}) \right\}$. This category has a terminal object $\left([n], \mathrm{Id}_{[n]}\right)$ so

$$L_{i_+}(X)_n = XU\left([n], \mathrm{Id}_{[n]}\right) = X_n.$$

If $n = -1$, $Ob(\Delta^{op}/[n]) = \left\{ ([m], f : [m] \to [-1]) \,|\, f \in Mor\left(\Delta_+^{op}\right), [m] \in Ob(\Delta^{op}) \right\}$, and $\Delta^{op}/[-1] \cong \Delta^{op}$, then

$$L_{i_+}(X)_{-1} = \operatorname{colim}_{\Delta^{op}} X = \operatorname{coeq} \left\{ \begin{tikzcd} X_0 \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & X_1 \end{tikzcd} \right\} =: \pi_0(X).$$

The right Kan extension is given by

$$R_{i_+}(X)_n = R_{i_+}(X)([n]) = \lim_{[n]\backslash\Delta^{op}} \left([n] \backslash \Delta^{op} \xrightarrow{U} \Delta^{op} \xrightarrow{X} \mathbf{Set} \right).$$

If $n \geq 0$, $Ob([n] \backslash \Delta^{op}) = \left\{ ([m], f) \,|\, f : [n] \to [m] \in Mor\left(\Delta_+^{op}\right), [m] \in \Delta^{op} \right\}$. This category has an initial object $\left([n], \mathrm{Id}_{[n]}\right)$ so

$$R_{i_+}([n]) = XU\left([n], \mathrm{Id}_{[n]}\right) = X_n.$$

If $n = -1$, $Ob([n] \backslash \Delta^{op}) = \left\{ ([m], f) \,|\, f : [-1] \to [m] \in Mor\left(\Delta_+^{op}\right), [m] \in Ob(\Delta^{op}) \right\} = \emptyset$, so $R_{i_+}(X)_{-1} = \emptyset$. Hence $R_{i_+}(X) = X$.

Simplicial and cyclic sets

Recall the category $\Delta\mathbf{C}$ has the same object as Δ, $Ob(\Delta\mathbf{C}) = Ob(\Delta)$, and morphisms are determined by, for any $f \in \mathrm{Hom}_{\Delta\mathbf{C}}([n],[m])$, there exists a unique $\gamma \in \mathrm{Aut}_{\Delta\mathbf{C}}([n])$ and $\varphi \in \mathrm{Hom}_{\Delta}([n],[m])$ such that $f = \varphi \circ \gamma$.

We have an inclusion $i : \Delta \hookrightarrow \Delta\mathbf{C}$, thus we can consider the left (right) Kan extension

$$\begin{tikzcd} \Delta \arrow[r, "X"] \arrow[d, hook, "i"'] & \mathbf{Set} \\ \Delta\mathbf{C} \arrow[ur, "L_i(X)"'] & \end{tikzcd}$$

which is given by the left (right) adjoint of $U = i^* : \mathbf{Set}^{\Delta\mathbf{C}^{op}} \to \mathbf{Set}^{\Delta^{op}}$,

$$\begin{tikzcd} \mathbf{Set}^{\Delta\mathbf{C}^{op}} \arrow[d, "U"] \\ \mathbf{Set}^{\Delta^{op}} \arrow[u, bend left=60, "F"] \arrow[u, bend right=60, "R"'] \end{tikzcd}$$

Recall the trick that for any $g \in \mathrm{Aut}_{\Delta\mathbf{C}}([m])$, $a \in \mathrm{Hom}_{\Delta\mathbf{C}}([n],[m])$ and define $f = g \circ a$. By unique factorization, there exists a unique $\varphi = g^*(a) \in \mathrm{Hom}_{\Delta}([n],[m])$ and a unique $\gamma = a_*(g) \in \mathrm{Aut}_{\Delta\mathbf{C}}([n])$ such that $g \circ a = \varphi \circ \gamma = g^*(a) \circ a_*(g)$.

This is functorial. For fixed $g \in \mathrm{Aut}_{\Delta\mathbf{C}}([m])$, we have

$$\begin{array}{rccc} g^* : & \mathrm{Hom}_{\Delta\mathbf{C}}([n],[m]) & \to & \mathrm{Hom}_{\Delta}([n],[m]) \\ & a & \mapsto & g^*(a) \end{array}$$

and for fixed $a \in \mathrm{Hom}_{\Delta\mathbf{C}}([n],[m])$, we have

$$\begin{array}{rccc} a_* : & \mathrm{Aut}_{\Delta\mathbf{C}}([m]) & \to & \mathrm{Aut}_{\Delta\mathbf{C}}([n]) \\ & g & \mapsto & a_*(g) \end{array}$$

Dually in $\Delta\mathbf{C}^{op}$ we have

$$\begin{array}{rccc} g^* : & \mathrm{Hom}_{\Delta\mathbf{C}^{op}}([m],[n]) & \to & \mathrm{Hom}_{\Delta^{op}}([m],[n]) \\ & a & \mapsto & g^*(a) \\ a_* : & \mathrm{Aut}_{\Delta\mathbf{C}^{op}}([m]) & \to & \mathrm{Aut}_{\Delta\mathbf{C}^{op}}([n]) \\ & g & \mapsto & a_*(g) \end{array}$$

such that $a \circ g = a_* (g) \circ g^* (a)$.

Recall we have a functor

$$\begin{array}{rccc} C_* : & \Delta\mathbf{C}^{op} & \longrightarrow & \mathbf{Set} \\ & [n] & \longmapsto & C_n \\ & (a : [m] \to [n]) & \longmapsto & (a_* : C_m \to C_n) \end{array}$$

where $C_n = \mathrm{Aut}_{\Delta\mathbf{C}^{op}} ([n])$.

Claim 4.2. The left adjoint of $U : \mathbf{Set}^{\Delta\mathbf{C}^{op}} \to \mathbf{Set}^{\Delta^{op}}$ is $F : \mathbf{Set}^{\Delta^{op}} \to \mathbf{Set}^{\Delta\mathbf{C}^{op}}$ given by

On objects: $Y_* \mapsto F(Y_*) = \{C_n \times Y_n\}_{n \geq 0}$. And for any $a \in \mathrm{Hom}_{\Delta\mathbf{C}^{op}}([m], [n])$

$$\begin{array}{rccc} F(Y)(a) : & C_m \times Y_m & \to & C_n \times Y_n \\ & (g, y) & \mapsto & (a_* (g), Y(g^* (a)) y) \end{array}$$

On Morphisms: for any $\alpha : X_* \to Y_*$,

$$\begin{array}{rccc} (F\alpha)_n : & C_n \times X_n & \to & C_n \times Y_n \\ & (g, x) & \mapsto & (g, \alpha(x)) \end{array}$$

The counit of this adjunction $\varepsilon = \mathrm{ev}_* : FU \Rightarrow \mathrm{Id}_{\mathbf{Set}^{\Delta\mathbf{C}^{op}}}$ is given by

$$\mathrm{ev}_* (X) = \left\{ \begin{array}{rccc} \mathrm{ev}_n (X) : & C_n \times X_n & \to & X_n \\ & (g, x) & \mapsto & X(g) x \end{array} \right\}.$$

The unit $\eta : \mathrm{Id}_{\mathbf{Set}^{\Delta^{op}}} \Rightarrow UF$ is given by

$$\eta(Y) = \left\{ \begin{array}{rccc} \eta(Y)_n : & Y_n & \to & C_n \times Y_n \\ & y & \mapsto & (1, y) \end{array} \right\}.$$

For simplicity, we write $X(g)(x) = g_* (x)$.

Proof. First we need to check that ev_* is a morphism of cyclic sets.

For any $a \in \mathrm{Hom}_{\Delta\mathbf{C}^{op}}([m], [n])$, we have a commutative diagram

$$\begin{tikzcd} C_m \times X_m \arrow[r, "\mathrm{ev}_m"] \arrow[d, "FX(a)"'] & X_m \arrow[d, "a_*"] \\ C_n \times X_n \arrow[r, "\mathrm{ev}_n"] & X_n \end{tikzcd}$$

because

$$\begin{aligned}
(\mathrm{ev}_n \circ FX(a))(g,x) &= \mathrm{ev}_n(a_*(g), g^*(a)_*(x)) \\
&= a_*(g)_* g^*(a)_*(x) \\
&= (a_*(g) \circ g^*(a))_*(x) \\
&= (a \circ g)_*(x) \\
&= a_*(g_*(x)) \\
&= a_*(\mathrm{ev}_m(g,x))
\end{aligned}$$

Next we need to show that

$$\begin{array}{rrcll}
\Phi: & \mathrm{Hom}_{\mathbf{Set}^{\Delta \mathbf{C}^{op}}}(FY, X) & \overset{\cong}{\rightleftharpoons} & \mathrm{Hom}_{\mathbf{Set}^{\Delta^{op}}}(Y, UX) & :\Psi \\
& \left(FY \xrightarrow{\alpha} X\right) & \mapsto & \left(Y \xrightarrow{\eta} UFY \xrightarrow{U\alpha} UX\right) & \\
& \left(FY \xrightarrow{F\beta} FUX \xrightarrow{\mathrm{ev}} X\right) & \leftarrow\!\shortmid & \left(Y \xrightarrow{\eta} UX\right) &
\end{array}$$

are inverse to each other.

$$\begin{array}{ccccccccc}
\Psi \circ \Phi(\alpha)_n : & FY_n = C_n \times Y_n & \xrightarrow{F\eta} & (FUFY)_n = C_n \times C_n \times Y_n & \xrightarrow{FU\alpha} & (FUX)_n = C_n \times X_n & \xrightarrow{\mathrm{ev}} & X_n \\
& (g,y) & \mapsto & (1,g,y) & \mapsto & (1,\alpha(g,y)) & \mapsto & \alpha(g,y)
\end{array}$$

$$\begin{array}{ccccccccc}
\Phi \circ \Psi(\beta)_n : & Y_n & \xrightarrow{\eta} & (UFY)_n = C_n \times Y_n & \xrightarrow{UF\beta} & (UFUX)_n = C_n \times X_n & \xrightarrow{U\mathrm{ev}} & (UX)_n = X_n \\
& y & \mapsto & (1,y) & \mapsto & (1,\alpha(y)) & \mapsto & \alpha(y)
\end{array}$$

□

Exercise 4.5. Compute F using colimit.

5 Fundamental Constructions

Given two locally small categories $\mathcal{C}$ and $\mathcal{D}$, define $\mathbf{Adj}(\mathcal{C},\mathcal{D})$ to be the category of adjunctions as

Objects: $\left\{(L,R,\varphi) \mid L:\mathcal{C} \rightleftarrows \mathcal{D}:R\ ,\varphi:\mathrm{Hom}_{\mathcal{D}}(F(-),-) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(-,G(-))\right\}$ where φ is an isomorphism of bifunctors $\mathcal{C}^{op}\times\mathcal{C}\to\mathbf{Set}$.

Morphisms: $\mathrm{Hom}_{\mathbf{Adj}(\mathcal{C},\mathcal{D})}((L,R,\varphi),(L',R',\varphi')) = \{(\alpha,\beta)\,|\alpha:L\Rightarrow L',\beta:R'\Rightarrow R,\varphi'=\beta_*\circ\varphi\circ\alpha^*\}$.

Explicitly, for each $c\in Ob(\mathcal{C}), d\in Ob(\mathcal{D})$, we have a commutative (factorization) diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(L'(c),d) \arrow[r, "(\alpha_c)^*"] \arrow[d, "\varphi_{c,d}"'] & \mathrm{Hom}_{\mathcal{D}}(L(c),d) \arrow[d, "\varphi_{c,d}"] \\
\mathrm{Hom}_{\mathcal{C}}(c,R'(d)) & \mathrm{Hom}_{\mathcal{C}}(c,R(d)) \arrow[l, "(\beta_d)_*"']
\end{tikzcd}$$

Proposition 5.1. *Let $\mathcal{C}$ be a small category, and $\mathcal{D}$ a locally small, cocomplete, then there exists a natural equivalence of categories*

$$\Phi:\mathcal{D}^{\mathcal{C}} \underset{\longleftarrow}{\overset{\cong}{\longrightarrow}} \mathbf{Adj}\left(\hat{\mathcal{C}},\mathcal{D}\right):\Psi$$

where Ψ is defined by restriction

$$\Psi\left(\ L:\hat{\mathcal{C}} \rightleftarrows \mathcal{D}:R\ ,\varphi\right) = \left(h^*(L) = L\circ h:\mathcal{C}\overset{h}{\hookrightarrow}\hat{\mathcal{C}}\xrightarrow{L}\mathcal{D}\right)$$
$$\Psi(\alpha,\beta) = h^*(\alpha) = \alpha\circ h$$

Proof. Construction of Φ. Given $F\in Ob\left(\mathcal{D}^{\mathcal{C}}\right)$, define

$$\Phi(F):\left(\ L(F):\hat{\mathcal{C}} \rightleftarrows \mathcal{D}:R(F)\ ,\varphi\right)$$

where $L(F) = L_h(F)$ and

$$\begin{array}{rccl}
R(F): & \mathcal{D} & \to & \hat{\mathcal{C}} \\
 & d & \mapsto & \mathrm{Hom}_{\mathcal{D}}(F(-),d) = h_d\circ F:\mathcal{C}^{op}\to\mathbf{Set}
\end{array}$$

Take any $c\in Ob(\mathcal{C}), d\in Ob(\mathcal{D})$ and consider $h_c\in Ob\left(\hat{\mathcal{C}}\right)$,

$$\mathrm{Hom}_{\hat{\mathcal{C}}}(h_c,RF(d)) \overset{\text{Yoneda}}{\cong} RF(d)(c) = \mathrm{Hom}_{\mathcal{D}}(Fc,d) = \mathrm{Hom}_{\mathcal{D}}(L(F))h_c,d)$$

where the last equality follows from lemma on left Kan extension along fully faithful functors.

Hence $R(F)$ is right adjoint to $L(F)$ on the representable functors.

To extend it to all presheaves $X : \mathcal{C}^{op} \to \mathbf{Set}$, we need the following facts

1. Co-Yonada lemma.

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook, "h"] \arrow[d, hook, "h"] & \hat{\mathcal{C}} \\ \hat{\mathcal{C}} \arrow[ur, "\mathrm{Id}_{\mathcal{C}}"'] & \end{tikzcd}$$

   Every X is canonically a colimit to h_c's.

$$X \cong \mathrm{colim}_{h/X}\left(h/X \xrightarrow{U} \mathcal{C} \overset{h}{\hookrightarrow} \hat{\mathcal{C}}\right) \cong \mathrm{colim}_{h/X}(h).$$

2. Left Kan extension is given by

$$L(F)X = L_h F(X) := \mathrm{colim}_{h/X}\left(h/X \xrightarrow{U} \mathcal{C} \xrightarrow{F} \mathcal{D}\right) = \mathrm{colim}_{h/X}(F).$$

Since colimit commutes with hom,

$$\begin{aligned} \mathrm{Hom}_{\hat{\mathcal{C}}}(X, RF(d)) &\cong \mathrm{Hom}_{\hat{\mathcal{C}}}\left(\mathrm{colim}_{h/X}(h_c), R(F)(d)\right) \\ &\cong \lim_{h/X} \mathrm{Hom}_{\hat{\mathcal{C}}}(h_c, R(F)(d)) \\ &\cong \lim_{h/X}\left(\mathrm{Hom}_{\mathcal{D}}(LF(h_c), d)\right) \\ &\cong \mathrm{Hom}_{\mathcal{D}}\left(\mathrm{colim}_{h/X} LF(h_c), d\right) \\ &\cong \mathrm{Hom}_{\mathcal{D}}\left(LF\left(\mathrm{colim}_{h/X} h_c\right), d\right) \\ &\cong \mathrm{Hom}_{\mathcal{D}}(LFX, d) \end{aligned}$$

We have $\Phi \circ \Psi\left(L : \hat{\mathcal{C}} \rightleftarrows \mathcal{D} : R, \varphi\right) = \left(L : \hat{\mathcal{C}} \rightleftarrows \mathcal{D} : R, \varphi\right)$ and a natural transformation $F \Rightarrow \Psi \circ \Phi(F) = h \circ L_h F$ by the universal property of left Kan extension.

Check that Φ and Ψ are inverse to each other. $\square$

5.1 Main Application to Simplicial Sets

Corollary 5.1. *Let $\mathcal{D}$ be as in Proposition, $\mathcal{C} = \Delta$, then we have a natural equivalence $\mathcal{D}^{\Delta} \cong \mathbf{Adj}(\mathbf{sSet}, \mathcal{D})$. In particular, every left adjoint functor on simplicial set with values in $\mathcal{D}$ comes from a (co)simplicial object in $\mathcal{D}$.*

Given $(\Delta^{\bullet} : \Delta \to \mathcal{D}) \in Ob\left(\mathcal{D}^{\Delta}\right)$

$$\begin{tikzcd} \Delta \arrow[r, "\Delta^{\bullet}"] \arrow[d, hook, "h"'] & \mathcal{D} \\ \mathbf{sSet} \arrow[ur, "L_h(\Delta)"'] & \end{tikzcd}$$

we have

$$L : \mathbf{sSet} \rightleftarrows \mathcal{D} : R$$

where

$$L(X) = L_h(\Delta^{\bullet})(X) = \mathrm{colim}_{\Delta X}\left(\Delta X \xrightarrow{U} \Delta \xrightarrow{\Delta^{\bullet}} \mathcal{D}\right),$$
$$R(d) = \{R(d)_n = \mathrm{Hom}_{\mathcal{D}}(\Delta^n, d)\}_{n\geq 0}, \Delta^n = \Delta^{\bullet}[n].$$

Note $h/X = \Delta X$ is the category of simplices over X with $Ob(\Delta X) = \coprod_{n\geq 0} X_n$.

Geometric realization and singular set

Let $\mathcal{D} = \mathbf{Top}$, we have

$$\begin{array}{cccc} \Delta^{\bullet}: & \Delta & \hookrightarrow & \mathbf{Top} \\ & [n] & \mapsto & \Delta^n = \left\{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} | \sum x_i = 1, x_i \geq 0\right\} \end{array}$$

and adjunction

$$|-| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : \mathcal{S}$$

where $|X| = \mathrm{colim}_{\Delta X}(\Delta^{\bullet})$ is the geometric realization of X, and $\mathcal{S}(Y) = \{\mathcal{S}(Y)_n = \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, Y)\}_{n\geq 0}$ is the singular set functor.

Cyclic realization

The standard cocyclic space $\Delta^{\bullet} : \Delta \to \mathbf{Top}$ is defined by letting

$$\begin{array}{cccc} \tau_n : & \Delta^n & \to & \Delta^n \\ & e_0 & \mapsto & e_n \\ & e_i & \mapsto & e_{i-1}, i \geq 1 \end{array}$$

In barycentric coordinates,

$$\tau_n(x_0, \cdots, x_n) = \tau_n\left(\sum_{i=0}^{n} x_i e_i\right) = \sum_{i=0}^{n} x_i \tau_n(e_i) = x_0 e_n + \sum_{i=1}^{n} x_i e_{i-1} = (x_1, \cdots, x_n, x_0).$$

Let's check the cyclic identities. It's obvious that $\tau_n^{n+1} = \mathrm{Id}$. Note

$$\begin{array}{rccc} d^i : & \Delta^{n-1} & \to & \Delta^n \\ & (x_0, \cdots, x_{n-1}) & \mapsto & (x_0, \cdots, x_{i-1}, 0, x_i, \cdots, x_n) \end{array}$$

so we have

$$\begin{aligned} \tau_n d^i (x_0, \cdots, x_{n-1}) &= \begin{cases} (x_1, \cdots, x_{i-1}, 0, x_i, \cdots, x_n, x_0) & i \neq 0, \\ (x_1, \cdots, x_{n-1}, 0) & i = 0 \end{cases} \\ d^{i-1} \tau_{n-1} (x_0, \cdots, x_{n-1}) &= (x_1, \cdots, x_{i-1}, 0, \cdots, x_{n-1}, x_0) \end{aligned}$$

thus $\tau_n d^i = d^{i-1} \tau_{n-1}, 1 \leq i \leq n$ and $\tau_n d^0 = d^n$. Similarly we have

$$\begin{array}{rccc} s^j : & \Delta^{n+1} & \to & \Delta^n \\ & (x_0, \cdots, x_{n+1}) & \mapsto & (x_0, \cdots, x_{j-1}, x_j + x_{j+1}, \cdots, x_{n+1}) \end{array}$$

and

$$\begin{aligned} \tau_n s^j (x_0, \cdots, x_{n+1}) &= \begin{cases} (x_1, \cdots, x_j + x_{j+1}, \cdots, x_{n+1}, x_0) & i \neq 0 \\ (x_2, \cdots, x_{n+1}, x_0 + x_1) & i = 0 \end{cases} \\ s^{j-1} \tau_{n+1} (x_0, \cdots, x_{n+1}) &= (x_1, \cdots, x_j + x_{j+1}, \cdots, x_{n+1}, x_0) \\ s^n \tau_{n+1}^2 (x_0, \cdots, x_{n+1}) &= (x_2, \cdots, x_{n+1}, x_0 + x_1) \end{aligned}$$

so we have $\tau_n s^j = s^{j-1} \tau_{n+1}, 1 \leq j \leq n$ and $\tau_n s^0 = s^n \tau_{n+1}^2$.

This yields a cyclic realization of a cyclic set

$$\begin{array}{rccc} |-|^{\mathrm{cy}} : & \mathbf{Set}^{\Delta \mathbf{C}^{op}} & \longrightarrow & \mathbf{Top} \\ & X & \mapsto & |X|^{\mathrm{cy}} = \mathrm{colim}_{h^{\mathrm{cy}}/X} (\Delta^\bullet) \end{array}$$

Remark 5.1. There are at least two other constructions of cyclic realization.

1. A cyclic set is in one-to-one correspondence to an $\mathbb{S}^1$-space $|X|$ via geometric realization, so we can define $|X|^{\mathrm{cy}} \coloneqq E\mathbb{S}^1 \times_{\mathbb{S}^1} |X|$.

2. The "flat" cyclic realization $\|X\|^{\mathrm{cy}} \coloneqq \mathrm{hocolim}_{\Delta \mathbf{C}^{op}} (X) \cong |\mathcal{N} \Delta \mathbf{C}_X^{op}|$ via the BK construction for $X : \Delta \mathbf{C}^{op} \to \mathbf{Set}$.

Nerve construction

Let $\mathcal{D} = \mathbf{Cat}$, we have

$$\begin{aligned} \Delta^{\bullet} : \quad \Delta & \hookrightarrow \mathbf{Cat} \\ [n] & \mapsto \overrightarrow{n} = \{0 \to 1 \to \cdots \to n\} \end{aligned}$$

and adjunction

$$\text{ho} : \mathbf{sSet} \rightleftarrows \mathbf{Cat} : \mathcal{N}$$

where

$$\begin{aligned} \text{ho}\,(X) &= L_h\,(\Delta^{\bullet})\,(X) = \operatorname{colim}_{\Delta X}\left(\Delta X \xrightarrow{U} \Delta \xrightarrow{\Delta^{\bullet}} \mathbf{Cat}\right) \\ \mathcal{N}\,(\mathcal{C}) &= \left\{\mathcal{N}_n\mathcal{C} = \operatorname{Hom}_{\mathbf{Cat}}\left(\overrightarrow{n}, \mathcal{C}\right) = \left\{c_0 \xrightarrow{f_0} c_1 \to \cdots \xrightarrow{f_{n-1}} c_n\right\}\right\}_{n \geq 0} \end{aligned}$$

$\text{ho}\,(X)$ is the category freely generated by the graph

$$\begin{tikzcd} X_0 & X_1 \arrow[l, shift right, "d_0"'] \arrow[l, shift left, "d_1"] \end{tikzcd}$$

module the relations

$$\begin{tikzcd} X_0 \arrow[r, "s_0"] & X_1 & X_2 \arrow[l, shift right=2, "d_0"'] \arrow[l, "d_1" description] \arrow[l, shift left=2, "d_2"] \end{tikzcd}$$

1. $d_0\,(x) \circ d_2\,(x) = d_1\,(x)\,, \forall x \in X_2$.

2. $s_0\,(x) = \text{Id}_x, \forall x \in X_0$.

Equivalently, if $\mathcal{C} = \text{ho}\,(X)$, we have

$$\begin{tikzcd} Ob\,(\mathcal{C}) \arrow[r, "s_0" description] & Mor\,(\mathcal{C}) \arrow[l, shift right=2, "d_0"'] \arrow[l, shift left=2, "d_1"] & Mor\,(\mathcal{C}) \underset{Ob(\mathcal{C})}{\times} Mor\,(\mathcal{C}) \arrow[l, shift right=2, "d_0"'] \arrow[l, "d_1" description] \arrow[l, shift left=2, "d_2"] \end{tikzcd}$$

satisfying for any $f : c \to d$ and $g : d \to e$ in $Mor\,(\mathcal{C})$,

$$\begin{aligned} & d_0\,(f) = c, d_1\,(f) = d \\ & d_0\,(f, g) = g, d_1\,(f, g) = g \circ f, d_2\,(f, g) = f \\ & s_0\,(c) = \text{Id}_c \end{aligned}$$

Remark 5.2. Note

1. $\text{ho} \circ \mathcal{N} \xrightarrow{\cong} \text{Id}_{\mathbf{Cat}}$ is an isomorphism, so $\mathcal{N}$ is fully faithful and we can embed $\mathbf{Cat} \hookrightarrow \mathbf{sSet}$. Small categories can be viewed as simplicial sets.

2. $\mathrm{ho}(X)$ is uniquely determined by X_0, X_1 and X_2, and the morphisms between them.

Question: Can we extend this construction to a "higher fashion"?

Answer: Via simplicial categories.

Function spaces of simplicial sets

Take $\mathcal{D} = \mathbf{sSet}$. Fix $Y \in Ob(\mathbf{sSet})$, we can define a functor

$$\begin{array}{rccc} \Delta^{\bullet}: & \Delta & \longrightarrow & \mathbf{sSet} \\ & [n] & \mapsto & \Delta[n] \times Y \end{array}$$

where "$\times$" is the Cartesian product in **sSet**, $X \times Y = \{X_n \times Y_n\}_{n \geq 0}$. Then we have adjoint pairs

$$L : \mathbf{sSet} \rightleftarrows \mathbf{sSet} : R$$

where

$$\begin{aligned} R(Z) &= \mathbf{Hom}(Y, Z) := \{\mathrm{Hom}_{\mathbf{sSet}}(\Delta[n] \times Y, Z)\}_{n \geq 0} \\ L(X) &= \mathrm{colim}_{\Delta X}(\Delta^{\bullet}) \cong \mathrm{colim}_{\Delta X}(\Delta[-]) \times Y \cong X \times Y \end{aligned}$$

$\mathbf{Hom}(Y, Z)$ is called the function space of simplicial sets.

Thus we have natural isomorphism

$$\mathrm{Hom}_{\mathbf{sSet}}(X \times Y, Z) \cong \mathrm{Hom}_{\mathbf{sSet}}(X, \mathbf{Hom}(Y, Z)) \tag{6}$$

where **Hom** plays the role of internal hom in the category of simplicial sets.

Remark 5.3. Note

1. Yoneda lemma immediately tells us that, taking $X = \Delta[n]$, we must have

$$\mathbf{Hom}(Y, Z)_n = \mathrm{Hom}_{\mathbf{sSet}}(\Delta[n] \times Y, Z).$$

2. The equation 6 is the degree zero part of the following isomorphism of simplicial sets:

$$\mathbf{Hom}(X \times Y, Z) \cong \mathbf{Hom}(X, \mathbf{Hom}(Y, Z)).$$

Fundamental groupoids of a simplicial set

Let **Grd** be the category of small groupoids. We have a left adjoint of the inclusion functor $i : \mathbf{Grd} \hookrightarrow \mathbf{Cat}$,

$$\begin{aligned} \tau_n : \quad \mathbf{Cat} &\longrightarrow \mathbf{Grd} \\ \mathcal{C} &\longmapsto \mathcal{C}\left[Mor\left(\mathcal{C}\right)^{-1}\right] \end{aligned}$$

Take the composition $\pi = \tau \circ \mathrm{ho}$, we have

$$\begin{tikzcd} \mathbf{Grd} \arrow[d, bend left, "i"] \arrow[dd, bend left=70, "\mathcal{N}"] \\ \mathbf{Cat} \arrow[u, bend left, "\tau"] \arrow[d, bend left, "\mathcal{N}"] \\ \mathbf{sSet} \arrow[u, bend left, "\mathrm{ho}"] \arrow[uu, bend left=70, "\pi"] \end{tikzcd}$$

Representation functor

Let G be an affine algebraic group over a field k of characteristic 0, e.g. GL_n or SL_n. Then we have a functor

$$\begin{aligned} G : \quad \mathbf{CommAlg}_k &\longrightarrow \mathbf{Set} \\ A &\longmapsto G\left(A\right) \end{aligned}$$

where the representative of G is denoted by $\mathcal{O}\left(G\right)$,

$$\mathrm{Hom}_{\mathbf{CommAlg}_k}\left(\mathcal{O}\left(G\right), A\right) \cong G\left(A\right).$$

The multiplication, inverse and unit in $G\left(A\right)$

$$\begin{cases} m_A : G\left(A\right) \times G\left(A\right) \to G\left(A\right) \\ i_A : G\left(A\right) \to G\left(A\right) \\ e_A : * \to G\left(A\right) \end{cases}$$

are natural transformations, so they give morphisms

$$\begin{cases} \Delta : \mathcal{O}\left(G\right) \to \mathcal{O}\left(G\right) \times \mathcal{O}\left(G\right) \\ S : \mathcal{O}\left(G\right) \to \mathcal{O}\left(G\right) \\ \varepsilon : \mathcal{O}\left(G\right) \to k \end{cases}$$

which makes $\mathcal{O}\left(G\right)$ a commutative Hopf algebra.

We have an (anti-)equivalence of categories

$$\begin{array}{ccc} \mathbf{AffShm}_k & \leftrightharpoons & \mathbf{CommHopfAlg}_k \\ G & \mapsto & \mathcal{O}(G) \\ \operatorname{Spec}(A) & \leftarrow\!\shortmid & A \end{array}$$

Example 5.1. Additive group

$$\begin{array}{cccc} G_a : & \mathbf{CommAlg}_k & \to & \mathbf{Gr} \\ & A & \mapsto & (A, +, 0) \end{array}$$

Example 5.2. Multiplicative group

$$\begin{array}{cccc} G_m : & \mathbf{CommAlg}_k & \to & \mathbf{Gr} \\ & A & \mapsto & (A^*, \times, 1) \end{array}$$

Example 5.3. For $n \geq 1$,

$$\begin{array}{cccc} GL_n : & \mathbf{CommAlg}_k & \to & \mathbf{Gr} \\ & A & \mapsto & GL_n(A) = M_n(A)^* \end{array}$$

and $\mathcal{O}(GL_n) = k[x_{ij}]_{1 \leq i,j \leq n}\left[\det(x_{ij})^{-1}\right]$.

$$\begin{array}{cccc} SL_n : & \mathbf{CommAlg}_k & \to & \mathbf{Gr} \\ & A & \mapsto & SL_n(A) = \{M \in M_n(A) \,|\, \det(M) = 1\} \end{array}$$

and $\mathcal{O}(GL_n) = k[x_{ij}]_{1 \leq i,j \leq n} / (\det(x_{ij}) - 1)$.

Fix G and define a functor

$$\begin{array}{cccc} B_*G : & \mathbf{CommAlg}_k & \longrightarrow & \mathbf{sSet} \\ & A & \mapsto & \mathcal{N}_*(GA) \end{array}$$

where we regard the group GA as a category with a single object. Then

$$B_*G \in \mathbf{Fun}(\mathbf{CommAlg}_k, \mathbf{sSet}) \cong \mathbf{Fun}(\Delta^{op}, \mathbf{Fun}(\mathbf{CommAlg}_k, \mathbf{Set}))$$

Explicitly,

$$\begin{array}{rccccc} B_*G: & \Delta^{op} & \longrightarrow & \mathbf{Fun}\left(\mathbf{CommAlg}_k, \mathbf{Set}\right) & \overset{\text{Yoneda}}{\hookleftarrow} & \mathbf{CommAlg}_k^{op} \\ & [n] & \longmapsto & \left(A \mapsto B_n\left(GA\right)\right) & & \end{array}$$

Then it induces

$$\begin{array}{rccc} \mathcal{O}\left(B_*G\right): & \Delta & \longrightarrow & \mathbf{CommAlg}_k \\ & [n] & \longmapsto & \mathcal{O}\left(B_n\left(G\right)\right) \cong \mathcal{O}\left(G\right)^{\otimes n} \cong \mathcal{O}\left(G^n\right) \end{array}$$

and $\mathcal{O}\left(B_*G\right) = \left\{\mathcal{O}\left(G^n\right)\right\}_{n\geq 0}$ is a simplicial commutative algebra. Explicitly, $\mathcal{O}\left(B_*G\right)$ is given by

$$k \leftrightarrows \mathcal{O}\left(G\right) \leftleftarrows \mathcal{O}\left(G\right)^{\otimes 2} \cdots$$

$$\begin{array}{rccl} d^i: & \mathcal{O}\left(G^{n-1}\right) & \to & \mathcal{O}\left(G^n\right) \\ & f\left(g_1, \cdots, g_{n-1}\right) & \mapsto & d^i f\left(g_1, \cdots, g_n\right) = \begin{cases} f\left(g_2, \cdots, g_n\right) & i = 0 \\ f\left(g_1, \cdots, g_i g_{i+1}, \cdots, g_n\right) & 1 \leq i \leq n-1 \\ f\left(g_1, \cdots, g_{n-1}\right) & i = n \end{cases} \\ s^j: & \mathcal{O}\left(G^{n+1}\right) & \to & \mathcal{O}\left(G^n\right) \\ & f\left(g_1, \cdots, g_{n+1}\right) & \mapsto & s^j f\left(g_1, \cdots, g_n\right) = f\left(g_1, \cdots, e, \cdots, g_n\right) \end{array}$$

By general principle we have

$$L : \mathbf{sSet} \rightleftarrows \mathbf{CommAlg}_k : R$$

where $\forall A \in \mathbf{CommAlg}_k$,

$$R\left(A\right) = \left\{R\left(A\right)_n \coloneqq \mathrm{Hom}_{\mathbf{CommAlg}_k}\left(\mathcal{O}\left(B_n\left(G\right), A\right)\right) \cong \mathrm{Hom}_{\mathbf{CommAlg}_k}\left(\mathcal{O}\left(G\right)^{\otimes n}, A\right) \cong G\left(A\right)^n = B_n\left(GA\right)\right\}$$

so

$$\begin{array}{rccc} R: & \mathbf{CommAlg}_k & \to & \mathbf{sSet} \\ & A & \mapsto & B_*\left(GA\right) \end{array}$$

and $\forall X \in \mathbf{sSet}$,

$$L\left(X\right) = \mathrm{colim}_{\Delta X}\left(\mathcal{O}\left(B_*G\right)\right) = \underset{\Delta[n]\to X}{\mathrm{colim}}\left(\mathcal{O}\left(G\right)^{\otimes n}\right).$$

In particular, $L\left(\Delta\left[n\right]\right) = \mathcal{O}\left(G\right)^{\otimes n}, \forall n \geq 0$. The realization of Δ in $\mathbf{CommAlg}_k$ depends on G. This is similar to the usual geometric realization.

Take Γ a discrete group, $X = B_*\Gamma \in Ob\left(\mathbf{sSet}\right)$,

$$\begin{aligned}\mathrm{Hom}_{\mathbf{CommAlg}_k}\left(L\left(B_*\Gamma\right), A\right) &\cong \mathrm{Hom}_{\mathbf{sSet}}\left(B_*\Gamma, B_*\left(GA\right)\right) \cong \mathrm{Hom}_{\mathbf{sSet}}\left(\mathcal{N}_*\Gamma, \mathcal{N}_*\left(GA\right)\right) \\ &\cong \mathrm{Hom}_{\mathbf{Cat}}\left(\Gamma, GA\right) \cong \mathrm{Hom}_{\mathbf{Gp}}\left(\Gamma, GA\right) = \mathrm{Rep}_G\left(\Gamma\right)\left(A\right).\end{aligned}$$

Note that the nerve functor is fully faithful. By Yoneda lemma, $L\left(B_*\Gamma\right) = \mathcal{O}\left(\mathrm{Rep}_G\left(\Gamma\right)\right)$.

More generally, if X is reduced, i.e. $X_0 = \{*\}$, then $|X|$ is a pointed connected space. In this case, $L\left(X\right) \cong \mathcal{O}\left(\mathrm{Rep}_G\left(\pi_1\left(|X|, *\right)\right)\right)$ which does not depend on higher homotopies.

Exercise 5.1. Calculate (explicit formula) for any X.

Proof. □

Part III

Homotopy Theory

6 Homotopy Coherent Nerves

Every adjoint pair $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ comes form a cosimplicial object in $\mathcal{D}$.

Ore basic example. We have the adjoint pair

$$\text{ho} : \mathbf{sSet} \rightleftarrows \mathbf{Cat} : \mathcal{N}$$

which comes from

$$\begin{array}{rccc} \Delta : & \Delta & \to & \mathbf{Cat} \\ & [n] & \mapsto & \overrightarrow{n} \end{array}$$

We want to refine this to

$$\mathfrak{C} : \mathbf{sSet} \rightleftarrows \mathbf{sCat}_0 : \mathfrak{N}$$

where $\mathbf{sCat}_0$ is the category of (small) simplicial categories. We need to specify a cosimplicial object in $\mathbf{sCat}_0$.

To start with we will introduce the definition of simplicial categories.

6.1 Simplicial Categories

Let **Cat** be the category of small categories.

Definition 6.1. A *simplicial object* in **Cat** is a functor $C_* : \Delta^{op} \to \mathbf{Cat}$ which consists of $C_* = \{C_n\}_{n\geq 0}$ and morphisms $d_i : C_n \to C_{n-1}, s_j : C_n \to C_{n+1}$.

Definition 6.2. A *simplicial category* $\underline{\mathcal{C}}$ is a simplicial object $\mathcal{C}_*$ in **Cat** such that d_i and s_j are identity maps on objects. Thus $Ob\,(\mathcal{C}_0) = \cdots = Ob\,(\mathcal{C}_n) = \cdots, \forall n \geq 0$. Define $Ob\,(\underline{\mathcal{C}}) = Ob\,(\mathcal{C}_0)$. On morphisms, for any $c_1, c_2 \in Ob\,(\underline{\mathcal{C}})$, $\mathcal{C}_*\,(c_1, c_2) \coloneqq \mathrm{Hom}_{\underline{\mathcal{C}}}\,(c_1, c_2) = \{\mathrm{Hom}_{\mathcal{C}_n}\,(c_1, c_2)\}_{n\geq 0} \in Ob\,(\mathbf{sSet})$ with composition $\circ : \mathcal{C}_*\,(c_1, c_2) \times \mathcal{C}_*\,(c_0, c_1) \to \mathcal{C}_*\,(c_0, c_2)$

Equivalently, simplicial categories are categories enriched in **sSet**.

If $\underline{\mathcal{C}}$ is a simplicial category then $\mathcal{C}_0$ is called the *underlying category* of $\underline{\mathcal{C}}$.

Intuition. $\mathrm{Hom}_{\mathcal{C}_0}(c_1, c_2)$ are morphisms in $\mathcal{C}$ and $\mathrm{Hom}_{\mathcal{C}_n}(c_1, c_2)$ are higher homotopies between the morphisms.

Remark 6.1. We can define such enrichment for "large" categories. $\underline{\mathbf{sSet}}$ is given by $Ob(\underline{\mathbf{sSet}}) = Ob(\mathbf{sSet})$ and

$$\mathrm{Hom}_{\underline{\mathbf{sSet}}}(X, Y) = \mathbf{Hom}(X, Y) = \{\mathrm{Hom}_{\mathbf{sSet}}(\Delta[n] \times X, Y)\}_{n \geq 0}$$

which is a simplicial set.

Definition 6.3. A *simplicial functor* $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ consists of two morphisms

$$\begin{array}{rcccl} F: & Ob(\underline{\mathcal{C}}) & \longrightarrow & Ob(\underline{\mathcal{D}}) & \text{map of objects} \\ & \mathcal{C}_*(c_1, c_2) & \to & \mathcal{C}_*(Fc_1, Fc_2) & \text{map of simplicial sets} \end{array}$$

or equivalently, $F = \{F_n : \mathcal{C}_n \to \mathcal{D}_n\}_{n \geq 0}$.

Definition 6.4. Given simplicial functors $F, G : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$, a simplicial natural transformation $\xi : F \Rightarrow G$ consists of the data

1. $\xi_c : F_0 \Rightarrow G_0$ natural transformation of functors between $F_0, G_0 : \mathcal{C}_0 \Rightarrow \mathcal{D}_0$.

2. $s_0(\xi_c) : F_1 \Rightarrow G_1, \cdots, s_0^n(\xi_c) : F_n \Rightarrow G_n, \cdots$

6.2 Barr-Beck Construction (canonical simplicial resolutions)

Definition 6.5. A *monad* (*triple*) on a category $\mathcal{C}$ is given by an endofunctor $T : \mathcal{C} \Rightarrow \mathcal{C}$ with two morphisms $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow T$ and $\mu : T \circ T \Rightarrow T$ satisfying

1. associativity

$$\begin{tikzcd} T \circ T \circ T \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"'] & T \circ T \arrow[d, Rightarrow, "\mu"] \\ T \circ T \arrow[r, Rightarrow, "\mu"] & T \end{tikzcd}$$

2. unitality

$$\begin{tikzcd} T \arrow[r, Rightarrow, "\eta T"] \arrow[d, Rightarrow, "T\eta"'] & T \circ T \arrow[d, Rightarrow, "\mu"] \\ T \circ T \arrow[r, Rightarrow, "\mu"] & T \end{tikzcd}$$

Remark 6.2. This can be regarded as a "generalized" associative, unital algebras.

Definition 6.6. A *comonad* (*cotriple*) on a category $\mathcal{C}$ is given by an endofunctor $\perp : \mathcal{C} \Rightarrow \mathcal{C}$ with two morphisms $\varepsilon : \perp \Rightarrow \mathrm{Id}_{\mathcal{C}}$ and $\delta : \perp \Rightarrow \perp \circ \perp$ satisfying coassociative and counital diagrams.

Main application

Given a pair of adjoint functors $F:\mathcal{C}\rightleftarrows\mathcal{D}:U$ with unit $\eta:\mathrm{Id}_{\mathcal{C}}\Rightarrow UF$ and counit $\varepsilon:FU\Rightarrow\mathrm{Id}_{\mathcal{D}}$, we can define

$$\begin{array}{ll} T=UF:\mathcal{C}\to\mathcal{C} & \mu:T\circ T\Rightarrow T, UFUF\overset{U\varepsilon F}{\Longrightarrow}UF \\ \perp=FU:\mathcal{D}\to\mathcal{D} & \delta:\perp\to\perp\circ\perp, FU\overset{F\eta U}{\Longrightarrow}FUFU \end{array}$$

Claim 6.1. $(T=UF,\eta,\mu)$ is a monad on $\mathcal{C}$ and $(\perp=FU,\varepsilon,\delta)$ is a comonad in $\mathcal{D}$.

Proof. Use identities 5 for adjunction morphisms. Then we have

$$\begin{gathered}\left(FU\overset{F\eta U}{\Rightarrow}FUFU\overset{\varepsilon FU}{\Rightarrow}FU\right)=\mathrm{Id}_{FU}\\ \left(UF\overset{\eta UF}{\Rightarrow}UFUF\overset{U\varepsilon F}{\Rightarrow}UF\right)=\mathrm{Id}_{UF}\end{gathered}$$

which gives the unitality diagrams. The associativity diagram follows from naturality of the unit and counit functors, i.e. we have

$$\begin{tikzcd} UFUFUFx \arrow[r, "UFU\varepsilon_{Fx}"] \arrow[d, "U\varepsilon_{FUFx}"'] & UFUFx \arrow[d, "U\varepsilon_{Fx}"] \\ UFUFx \arrow[r, "U\varepsilon_{Fx}"] & UFx \end{tikzcd}$$

and

$$\begin{tikzcd} FUy \arrow[r, "F\eta_{Uy}"] \arrow[d, "F\eta_{Uy}"'] & FUFUy \arrow[d, "FUF\eta_{Uy}"] \\ FUFUy \arrow[r, "F\eta_{UFUy}"] & FUFUFUy \end{tikzcd}$$

□

Example 6.1. We have an adjoint pair $F:\mathbf{Set}\rightleftarrows R\text{-}\mathbf{Mod}:U$ where $F(X)=R[X]$ is the free R-module generated over the set X. Then

$$\begin{array}{rccc}\perp=FU: & R\text{-}\mathbf{Mod} & \to & R\text{-}\mathbf{Mod}\\ & M & \mapsto & R[M]\end{array}$$

where $R[M]$ is the free module on the underlying set of M.

Claim 6.2. Every monad in $\mathcal{C}$ gives a functor $\mathcal{C}\to\mathbf{c}\mathcal{C}$ and every comonad gives a functor $\mathcal{D}\to\mathbf{s}\mathcal{D}$.

Proof. Given $(\perp,\varepsilon,\delta)$ on $\mathcal{D}$, and $A\in Ob(\mathcal{D})$, we define

$$\begin{array}{rccc}\perp_*: & \mathcal{D} & \to & \mathbf{s}\mathcal{D}\\ & A & \mapsto & \perp_*A=\{\perp_nA\}_{n\geq0}\end{array}$$

where $\perp_n A = \perp^{n+1} A$ and

$$\begin{aligned} d_i = \perp^i \cdot \varepsilon \cdot \perp^{n-i} : \quad & \perp^{n+1} A \quad \to \quad \perp^n A \\ s_j = \perp^j \cdot \delta \cdot \perp^{n-j} : \quad & \perp^{n+1} A \quad \to \quad \perp^{n+2} A \end{aligned}$$

which is similar to the bar construction. Explicitly, $\perp_* A$ can be expressed as follows

$$\perp A \leftleftarrows \perp^2 A \overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}} \perp^3 A \cdots$$

The simplicial identities are satisfied because of the functoriality of units and counits and the identity for adjunction morphisms. In particular, $d_i s_j = \mathrm{Id}, i = j, j+1$ follows from

$$\begin{aligned} \left(FU \overset{F\eta U}{\Rightarrow} FUFU \overset{\varepsilon FU}{\Rightarrow} FU\right) &= \mathrm{Id}_{FU} \\ \left(FU \overset{F\eta U}{\Rightarrow} FUFU \overset{FU\varepsilon}{\Rightarrow} FU\right) &= \mathrm{Id}_{FU} \end{aligned}$$

$\square$

Example 6.2. [Dold-Kan] The adjoint pair

$$\mathcal{N} : \mathbf{Ab}^{\Delta^{op}} \rightleftarrows \mathbf{Ch}_{\geq 0}(\mathbf{Ab}) : \Gamma$$

comes from a cosimplicial object

$$\begin{array}{ccccccc} \Delta & \xrightarrow{\Delta^*} & \mathbf{sSet} & \xrightarrow{\mathbb{Z}[-]} & \mathbf{sAb} & \xrightarrow{\mathcal{N}} & \mathbf{Ch}_{\geq 0}(\mathbf{Ab}) \\ [n] & \mapsto & \Delta[n] & \mapsto & \mathbb{Z}[\Delta[n]] & \mapsto & \mathcal{N}_*(\mathbb{Z}[\Delta[n]]) \end{array}$$

We apply this to the adjunction

$$F : \mathbf{Quiver} \rightleftarrows \mathbf{Cat} : U$$

where **Quiver** is the category of small reflexive directed graphs. A graph is called reflexive if for any vertex $v \in V(\Gamma)$ there is a specified loop (differential graded) $v \xrightarrow{\text{ev}} v$ called the identity differential graded on v. U is the functor forgetting the composition law in a category but remembers objects, codomains, domains of morphisms and identities.

For a quiver, $F(Q)$ is the free category generated by Q, with objects the vertices in Q, $Ob(F(Q)) = V(Q)$, and (non-identity) morphisms are the paths of non-identity differential graded in Q, and composition $\circ$ is the concatenation of paths.

Example 6.3. If k is a field, $k\langle Q\rangle = k[F(Q)]$ is the algebra generated by the path category. If Q is finite, then $k\langle Q\rangle$ gives the path algebra of Q with unit $1 = \sum_{v\in V} e_v$.

The adjunction $F : \mathbf{Quiver} \rightleftarrows \mathbf{Cat} : U$ gives a functor $\perp = FU : \mathbf{Cat} \to \mathbf{Cat}$ which consists of a comonad (cotriple) $(\perp, \varepsilon, \delta)$ on $\mathbf{Cat}$, inducing a functor

$$Q : \mathbf{Cat} \to \mathbf{sCat}_0$$

called the simplicial thickening of a category. Hence we can compose this with the functor $\Delta^* : \Delta \to \mathbf{Cat}$ to get a functor

$$\begin{aligned} \mathfrak{C}\Delta^* : \quad \Delta &\to \mathbf{sCat}_0 \\ [n] &\mapsto Q\overrightarrow{n} := \perp_* \overrightarrow{n} \end{aligned}$$

and it follows that we have an adjoint pair

$$\mathfrak{C} : \mathbf{sSet} \rightleftarrows \mathbf{sCat}_0 : \mathfrak{N}$$

where

$$\begin{aligned} \mathfrak{C}X &:= \mathcal{L}_h(\mathfrak{C}\Delta^*)(X) \cong \mathrm{colim}_{\Delta X}(\mathfrak{C}\Delta^*) \\ \mathfrak{N}_n(\underline{\mathcal{C}}) &:= \mathbf{Fun}(\mathfrak{C}\Delta^n, \underline{\mathcal{C}}) \end{aligned}$$

Theorem 6.1. *[DS11] Proposition 3.8. For each $n \geq 0$, $\mathfrak{C}\Delta^n$ is a simplicial category with objects $Ob(\mathfrak{C}\Delta^n) = Ob(\overrightarrow{n}) = \{0, \cdots, n\}$ and $Hom_{\mathfrak{C}\Delta^n}(i,j) \cong \mathcal{N}_*(P_{ij})$ the ordinary nerve where P_{ij} is the poset defined by all nonempty subsets in the interval $\{k : i \leq k \leq j\} \subseteq \{0 < 1 < \cdots < n\}$.*

Remark 6.3. Note

1. If $i > j$ then $P_{ij} = \emptyset$ and $\mathrm{Hom}(i,j) = \emptyset$.

2. If $j > i$, P_{ij} is the product of $j-i-1$ copies of $[1] = \{0 < 1\}$

3. $\mathrm{Hom}_{\mathfrak{C}\Delta^n}(i,j) = \begin{cases} \Delta_*[1]^{j-i+1} & j > i \\ \Delta_*[0] & j = i \cdot \\ \emptyset & j < i \end{cases}$

Vista. The adjoint pair $\mathfrak{C} : \mathbf{sSet} \rightleftarrows \mathbf{sCat}_0 : \mathfrak{N}$ has natural model structures on both $\mathbf{sSet}$ and $\mathbf{sCat}_0$.

1. $\mathbf{sCat}_0$ has Dwyer-Kan model structure. $F : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a weak equivalence if
   (a) $F : \mathbf{Hom}_{\underline{\mathcal{C}}} \xrightarrow{\sim} \mathbf{Hom}_{\underline{\mathcal{D}}}$ is a weak equivalence.

(b) $\pi_0(F) : \pi_0\underline{\mathcal{C}} \to \pi_0\underline{\mathcal{D}}$ is an equivalence of categories.

2. **sSet** has model structure induced from the one on $\mathbf{sCat}_0$, which is very different from the standard ones.

3. Fibrant objects in **sSet** are exactly quasi-categories.

4. $\mathfrak{C} : \mathbf{sSet} \rightleftarrows \mathbf{sCat}_0 : \mathfrak{N}$ is a Quillen pair. Quasi-categories and simplicial categories gives two equivalent models of $(\infty, 1)$ categories.

Question. An object $\mathcal{C} \in Ob(\mathbf{Cat})$ induces two simplicial categories, $\perp_* \mathcal{C}$ and $\mathfrak{C}(\mathcal{N}_* \mathcal{C})$, what is the relation between them?

Theorem 6.2. *[Riehl] $\perp_* \mathcal{C} \cong \mathfrak{C}(\mathcal{N}_* \mathcal{C})$ for any $\mathcal{C} \in Ob(\mathbf{Cat})$.*

7 Homotopy Colimits

The references for this part are [MS09] and [DHKS04].

7.1 Symmetric Monoidal Categories

Definition 7.1. A category $\mathcal{S}$ is called *symmetric monoidal* if

1. there is a bifunctor $\otimes : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ called *tensor product*, and

2. there exists an object $\mathbf{1} \in Ob(\mathcal{S})$ called the *unit object* such that for any $a, b, c \in Ob(\mathcal{S})$, there are isomorphisms

$$a \otimes b \cong b \otimes a$$
$$(a \otimes b) \otimes c \xrightarrow[\cong]{\alpha_{a,b,c}} a \otimes (b \otimes c)$$
$$\mathbf{1} \otimes a \xrightarrow[\cong]{\lambda_a} a \xrightarrow[\cong]{\rho_a} a \otimes \mathbf{1}$$

which are natural in a, b, c and compatible in the sense that 2 axioms *triangle*

$$\begin{tikzcd}
(a \otimes \mathbf{1}) \otimes b \arrow[rr, "\alpha_{a,1,b}"] \arrow[dr, "\rho_a^{-1} \otimes \mathrm{Id}"'] & & a \otimes (\mathbf{1} \otimes b) \arrow[dl, "\mathrm{Id} \otimes \lambda_b"] \\
& a \otimes b &
\end{tikzcd}$$

and *pentagon*

$$\begin{tikzcd}
& (a \otimes (b \otimes c)) \otimes d \arrow[dr, "\alpha_{a,b\otimes c,d}"] & \\
((a \otimes b) \otimes c) \otimes d \arrow[ur, "\alpha_{a,b,c} \otimes \mathrm{Id}"] \arrow[d, "\alpha_{a\otimes b,c,d,}"'] & & a \otimes ((b \otimes c) \otimes d) \arrow[d, "\mathrm{Id} \otimes \alpha_{b,c,d}"] \\
(a \otimes b) \otimes (c \otimes d) \arrow[rr, "\alpha_{a,b,c\otimes d}"] & & a \otimes (b \otimes (c \otimes d))
\end{tikzcd}$$

holds.

Definition 7.2. $\mathcal{S}$ is called *closed* if there exists a bifunctor

$$\mathbf{Hom}_{\mathcal{S}}(-,-) : \mathcal{S}^{op} \times \mathcal{S} \to \mathcal{S}$$

such that

$$\mathbf{Hom}_{\mathcal{S}}(a \otimes b, c) \cong \mathbf{Hom}_{\mathcal{S}}(a, \mathbf{Hom}_{\mathcal{S}}(b, c)), \forall a, b, c \in Ob(\mathcal{S}).$$

Thus there exists an (enriched) adjunction

$$- \otimes b : \mathcal{S} \rightleftarrows \mathcal{S} : \mathbf{Hom}_{\mathcal{S}}(b, -) \tag{7}$$

Note that there is a natural map (composition law)

$$\circ : \quad \mathbf{Hom}_{\mathcal{S}}(b,c) \otimes \mathbf{Hom}_{\mathcal{S}}(a,b) \longrightarrow \mathbf{Hom}_{\mathcal{S}}(a,c)$$

which is adjoint (under 7) to the composite map

$$\begin{tikzcd} \mathbf{Hom}_{\mathcal{S}}(b,c) \otimes \mathbf{Hom}_{\mathcal{S}}(a,b) \otimes a \arrow[r, "\mathrm{Id}\otimes\varepsilon"] \arrow[dr] & \mathbf{Hom}_{\mathcal{S}}(b,c) \otimes b \arrow[d, "\varepsilon"] \\ & c \end{tikzcd}$$

via the adjunction map

$$\mathbf{Hom}_{\mathcal{S}}(\mathbf{Hom}_{\mathcal{S}}(b,c) \otimes \mathbf{Hom}_{\mathcal{S}}(a,b) \otimes a, c) \xrightarrow{\cong} \mathbf{Hom}_{\mathcal{S}}(\mathbf{Hom}_{\mathcal{S}}(b,c) \otimes \mathbf{Hom}_{\mathcal{S}}(a,b), \mathbf{Hom}(a,c)).$$

Definition 7.3. $\mathcal{S}$ is called *Cartesian closed* if $\otimes = \times$ and $\mathbf{1} = *$ the terminal object in $\mathcal{S}$.

We will often abuse notation and write $(\mathcal{S}, \times, *)$ in general.

7.2 Main Example

The category of simplicial set $(\mathbf{sSet}, \times, *)$

In $\mathcal{S} = \mathbf{sSet}$, the tensor product is given by

$$\begin{array}{rccc} \otimes = \times : & \mathbf{sSet} \times \mathbf{sSet} & \to & \mathbf{sSet} \\ & (X,Y) & \mapsto & X \times Y = \{X_n \times Y_n\}_{n \geq 0} \end{array}$$

and internal hom is given by

$$\begin{array}{rccl} \mathbf{Hom} : & \mathbf{sSet}^{op} \times \mathbf{sSet} & \to & \mathbf{sSet} \\ & (Y,Z) & \mapsto & \mathbf{Hom}(Y,Z) = \{\mathrm{Hom}_{\mathbf{sSet}}(Y \times \Delta[n], Z)\} \end{array}$$

we have

$$\mathbf{Hom}(X \times Y, Z) \cong \mathbf{Hom}(X, \mathbf{Hom}(Y,Z)).$$

The composition $\circ : \mathbf{Hom}(Y,Z) \times \mathbf{Hom}(X,Y) \to \mathbf{Hom}(X,Z)$ is given explicitly by

$$\left(\left(Y \times \Delta[n] \xrightarrow{f} Z\right), \left(X \times \Delta[n] \xrightarrow{g} Y\right)\right) \mapsto \left(X \times \Delta[n] \xrightarrow{\mathrm{Id}\times\mathrm{diag}} X \times \Delta[n] \times \Delta[n] \xrightarrow{g\times\mathrm{Id}} Y \times \Delta[n] \xrightarrow{f} Z\right).$$

Construction (smash product)

Let $(\mathcal{S}, \times, \bullet)$ be a complete and cocomplete Cartesian closed symmetric monoidal category. Define $\mathcal{S}_\bullet = \bullet \downarrow \mathcal{S}$ with objects $\left\{\bullet \xrightarrow{f} v\right\}_{v \in Ob(\mathcal{S})}$. There is a canonical way to make this a symmetric monoidal category .

Given $\tau_v : \bullet \to v$ and $\tau_w : \bullet \to w$ in $\mathcal{S}_\bullet$ we define

$$\begin{aligned} f_v :& \quad v \to v \times \bullet \xrightarrow{\mathrm{Id}_v \times \tau_w} v \times w \\ f_w :& \quad w \to \bullet \times w \xrightarrow{\tau_v \times \mathrm{Id}_w} v \times w \end{aligned}$$

The smash product of v and w is given by the pushout

$$v \wedge w \coloneqq \mathrm{cofib}\left(f_v \coprod f_w\right) = \mathrm{colim}\left(\bullet \leftarrow v \coprod w \xrightarrow{f_v \coprod f_w} v \times w\right).$$

$$\begin{tikzcd} v \coprod w \arrow[r] \arrow[d] & v \times w \arrow[d] \\ \bullet \arrow[r] & v \wedge w \end{tikzcd}$$

Dually we define the internal hom by the pullback

$$\begin{tikzcd} \mathbf{Hom}_{\mathcal{S}_\bullet}(v,w) \arrow[r] \arrow[d] & \mathbf{Hom}_{\mathcal{S}}(v,w) \arrow[d, "(\tau_v)^*"] \\ \mathbf{Hom}_{\mathcal{S}}(\bullet,\bullet) \arrow[r, "(\tau_w)_*"] & \mathbf{Hom}_{\mathcal{S}}(\bullet,w) \end{tikzcd}$$

Then $(\mathcal{S}_\bullet, \wedge, \mathbf{1}_\bullet, \mathbf{Hom}_{\mathcal{S}_\bullet})$ is a closed symmetric monoidal category, called the category of based objects in $\mathcal{S}$.

There is an adjoint pair of functors

$$(-)_+ : \mathcal{S} \rightleftarrows \mathcal{S}_\bullet : U$$

where $(-)_+$ is strictly monoidal, $\mathbf{1}_\bullet \cong (\bullet)_+$ and $(v_+ \wedge w_+) \cong (v \times w)_+$.

Example 7.1.

1. $(\mathbf{Top}, \times, \bullet) \Longrightarrow (\mathbf{Top}_\bullet, \wedge, \mathbf{1}_\bullet)$.

2. $(\mathbf{sSet}, \times, \bullet) \implies (\mathbf{sSet}_+, \wedge, \mathbf{1}_\bullet)$.

Let $(\mathcal{S}, \times, \bullet)$ be a closed symmetric monoidal category.

Definition 7.4. An $\mathcal{S}$-*category* is a category $\mathcal{M}$ enriched over $\mathcal{S}$, thus

(S1) For any objects X, Y in $\mathcal{M}$, there exists an object $\mathbf{Hom}_\mathcal{M}(X, Y)$ in $\mathcal{S}$

(S2) For any objects X, Y, Z in $\mathcal{M}$, there exists a morphism

$$c_{X,Y,Z} : \mathbf{Hom}_\mathcal{M}(Y, Z) \times \mathbf{Hom}_\mathcal{M}(X, Y) \to \mathbf{Hom}_\mathcal{M}(X, Z)$$

in $\mathcal{S}$ called *composition law*.

(S3) For any object X in $\mathcal{M}$, there is a morphism $i_X : \bullet \to \mathbf{Hom}_\mathcal{M}(X, X)$ called *unit*.

(S4) There is an isomorphism

$$\mathbf{Hom}_\mathcal{S}(\bullet, \mathbf{Hom}_\mathcal{M}(X, Y)) \cong \mathrm{Hom}_\mathcal{M}(X, Y).$$

Thus the usual hom can be recovered from internal hom.

These data satisfy the compatibility axioms (*triangle* and *pentagon*) similar to the ones in $\mathcal{S}$.

Example 7.2. For $\mathcal{S} = (\mathbf{sSet}, \times, \bullet)$, the simplicial category $\mathcal{M} \in Ob(\mathbf{sCat}_0)$ is $\mathcal{S}$-enriched.

Definition 7.5. An $\mathcal{S}$-enriched category is called

1. *tensored over* $\mathcal{S}$ if there exists a bifunctor

$$\boxtimes : \mathcal{S} \times \mathcal{M} \to \mathcal{M}$$

viewed as an action of $\mathcal{S}$ on $\mathcal{M}$, such that

$$\mathbf{Hom}_\mathcal{M}(v \boxtimes x, y) \cong \mathbf{Hom}_\mathcal{M}(v, \mathbf{Hom}_\mathcal{M}(x, y)).$$

2. *cotensored over* $\mathcal{S}$ if there exists a bifunctor

$$(-)^- : \mathcal{S}^{op} \times \mathcal{M} \to \mathcal{M}$$

such that

$$\mathbf{Hom}_\mathcal{M}(v, \mathbf{Hom}_\mathcal{M}(x, y)) \cong \mathbf{Hom}_\mathcal{M}(x, y^v).$$

Example 7.3. Any cocomplete, complete, locally small category $\mathcal{M}$ is tensored and cotensored over $\mathcal{S} = (\mathbf{Set}, \times, \bullet)$.

$$\begin{array}{rccc} \boxtimes: & \mathbf{Set} \times \mathcal{M} & \to & \mathcal{M} \\ & (K, X) & \mapsto & \coprod_{k \in K} X \end{array} \qquad \begin{array}{rccc} (-)^{-}: & \mathbf{Set}^{op} \times \mathcal{M} & \to & \mathcal{M} \\ & (K, Y) & \mapsto & \prod_{k \in K} Y \end{array}$$

Main application

Let $\mathcal{S} = (\mathbf{sSet}, \times, \bullet)$ and $\mathcal{M} = \mathbf{s}\mathcal{C} = \mathbf{Fun}(\Delta^{op}, \mathcal{C})$, then $\mathcal{M}$ has a canonical structure of simplicial category tensored and cotensored over $\mathcal{S}$.

- Tensor

$$\begin{array}{rccc} \boxtimes: & \mathbf{sSet} \times \mathbf{s}\mathcal{C} & \to & \mathbf{s}\mathcal{C} \\ & (K, X) & \mapsto & K \boxtimes X = \left\{ \coprod_{K_n} X_n \right\}_{n \geq 0} \end{array}$$

- Internal hom

$$\mathbf{Hom}_{\mathbf{s}\mathcal{C}}(X, Y) = \{\mathrm{Hom}_{\mathbf{s}\mathcal{C}}(\Delta[n] \boxtimes X, Y)\}_{n \geq 0}$$

  Note that $(K \boxtimes L) \boxtimes X \cong K \boxtimes (L \boxtimes X)$ and $\Delta[0] \boxtimes X \cong X \cong * \boxtimes X$.

  Remark 7.1. If $\mathcal{M}$ is tensored over **sSet**, then for any set K viewed as a discrete simplicial set and $X \in Ob(\mathbf{s}\mathcal{C})$, we have $K \boxtimes X \cong \coprod_K X$ because

$$K \boxtimes X \cong (K \boxtimes \Delta[0]) \boxtimes X \cong K \boxtimes (\Delta[0] \boxtimes X) = \left(\coprod_K \Delta[0] \right) \boxtimes X \cong \coprod_K (\Delta[0] \boxtimes X) \cong \coprod_K X.$$

- Fix $K \in Ob(\mathbf{sSet})$, consider $K \boxtimes - : \mathbf{s}\mathcal{C} \to \mathbf{s}\mathcal{C}$. Since $\mathcal{C}$ is cocomplete, so is $\mathbf{s}\mathcal{C}$, hence $K \boxtimes -$ has right adjoint defined by left Kan extension

$$\begin{tikzcd} \mathbf{s}\mathcal{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "K\boxtimes -"'] & \mathbf{s}\mathcal{C} \\ \mathbf{s}\mathcal{C} \arrow[ur, "L_{K\boxtimes -}(\mathrm{Id}_{\mathbf{s}\mathcal{C}})"'] & \end{tikzcd}$$

  Denote

$$Y^K := L_{K\boxtimes -}(\mathrm{Id}_{\mathbf{s}\mathcal{C}})(Y), \forall Y \in Ob(\mathbf{s}\mathcal{C}),$$

  then by general properties of Kan extensions, we have

$$\mathrm{Hom}_{\mathbf{s}\mathcal{C}}(K \boxtimes X, Y) \cong \mathrm{Hom}_{\mathbf{s}\mathcal{C}}\left(X, Y^K\right).$$

This implies for any $n \geq 0$,

$$\begin{aligned}\mathbf{Hom}_{\mathbf{s}\mathcal{C}}(K \boxtimes X, Y)_n &= \mathrm{Hom}_{\mathbf{s}\mathcal{C}}(\Delta[n] \boxtimes K \boxtimes X, Y) \cong \mathrm{Hom}_{\mathbf{s}\mathcal{C}}(K \boxtimes \Delta[n] \boxtimes X, Y) \\ &\cong \mathrm{Hom}_{\mathbf{s}\mathcal{C}}\left(\Delta[n] \boxtimes X, Y^K\right) =: \mathbf{Hom}_{\mathbf{s}\mathcal{C}}\left(X, Y^K\right)_n.\end{aligned}$$

Special cases

1. Let $\mathcal{C} = \mathbf{Set}, \mathcal{M} = \mathbf{sSet}$.

$$K \boxtimes X := \left\{\coprod_{K_n} X_n\right\}_{n\geq 0} = \{K_n \times X_n\}_{n\geq 0} = K \times X$$

Y^K is defined by the formula

$$\mathrm{Hom}_{\mathbf{sSet}}(K \boxtimes X, Y) \cong \mathrm{Hom}_{\mathbf{sSet}}\left(X, Y^K\right).$$

Put $X = \Delta[n]$, we have

$$\left(Y^K\right)_n = \mathrm{Hom}_{\mathbf{sSet}}\left(\Delta[n], Y^K\right) \cong \mathrm{Hom}_{\mathbf{sSet}}(K \boxtimes \Delta[n], Y) = \mathbf{Hom}(K, Y)_n$$

so $Y^K = \mathbf{Hom}(K, Y)$.

2. Let $\mathcal{C} = \mathbf{Mod}(R)$ where R is a unital (commutative) associative ring. Then $\coprod = \bigoplus$. Let $\mathcal{M} = \mathbf{sMod}(R) \cong \mathbf{Com}_{\geq 0}(R)$. The tensor product is given by

$$\begin{aligned}\bigoplus : \quad \mathbf{sSet} \times \mathbf{sMod}(R) &\to \mathbf{sMod}(R) \\ (K, X) &\mapsto \left\{\bigoplus_{K_n} X_n\right\}_{n\geq 0} = \{R[K_n] \otimes_R X_n\}_{n\geq 0}\end{aligned}$$

where $R[K_n]$ is the free bimodule based on K_n. We need to check that this agrees with simplicial operations. Internal hom is defined by

$$\mathbf{Hom}_{\mathbf{sMod}(R)}(X, Y) := \left\{\mathrm{Hom}_{\mathbf{sMod}(R)}(R[\Delta[n]] \otimes_R X, Y)\right\}_{n\geq 0}.$$

And

$$Y^K := \mathbf{Hom}_{\mathbf{sSet}}(X, Y)$$

where the R-module structure comes from the target.

3. Let $\mathcal{C} = \mathbf{CommAlg}_k$ with k a commutative ring. $\mathcal{M} = \mathbf{sCommAlg}_k$, then $\coprod = \bigotimes_k$. The tensor product is given by
$$K \boxtimes A \cong \left\{ \bigotimes_{K_n} A \right\}_{n \geq 0}.$$
And internal hom is defined as
$$\mathbf{Hom}_{\mathbf{sCommAlg}_k}(A, B) = \left\{ \mathrm{Hom}_{\mathbf{sCommAlg}_k}\left(\bigotimes_{\Delta[n]} A, B \right) \right\}_{n \geq 0}.$$

4. $\mathcal{C}$ can be any algebraic category, e.g. $\mathbf{Gr}, \mathbf{Alg}_k, \mathbf{Lie}$.

5. Let $\mathcal{M} = \mathbf{Top}$ the category of compactly generated weak Hausdorff spaces, then we have
$$\begin{array}{rccc} \boxtimes : & \mathbf{sSet} \times \mathbf{Top} & \to & \mathbf{Top} \\ & (K, X) & \mapsto & |K| \times X \end{array}$$
and
$$\begin{array}{rccc} (-)^{-} : & \mathbf{sSet}^{op} \times \mathbf{Top} & \to & \mathbf{Top} \\ & (K, Y) & \mapsto & Y^K := \mathbf{Map}(|K|, Y) \end{array}$$
Usually, **Top** is viewed as a topological category, but via the adjunction
$$|-| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : S$$
we can convert topological categories to simplicial ones. One key observation is that geometric realization preserves product
$$|X \times Y| \cong |X| \times |Y|.$$

7.3 Functor Tensor Product

Let $\mathcal{S} = (\mathcal{S}, \bigotimes, \mathbf{1})$ be a closed symmetric monoidal category and $\mathcal{C}$ be a small category. $\mathcal{M}$ is a cocomplete $\mathcal{S}$-category (tensored over $\mathcal{S}$).
$$\boxtimes : \mathcal{S} \times \mathcal{M} \to \mathcal{M}$$

Definition 7.6. Given two functors $G : \mathcal{C}^{op} \to \mathcal{S}$ and $F : \mathcal{C} \to \mathcal{M}$, define
$$G \underset{\mathcal{C}}{\boxtimes} F := \int^{c \in Ob(\mathcal{C})} G(c) \boxtimes F(c) = \mathrm{coeq} \left\{ \coprod_{f : c \to c'} G(c') \boxtimes F(c) \underset{f_*}{\overset{f^*}{\rightrightarrows}} \coprod_{c \in Ob(\mathcal{C})} G(c) \boxtimes F(c) \right\}$$

where

$$\begin{array}{llcll} f^*: & G(c') \boxtimes F(c) & \xrightarrow{G(f)\boxtimes \mathrm{Id}} & G(c) \boxtimes F(c) & \hookrightarrow \coprod_{c\in Ob(\mathcal{C})} G(c) \boxtimes F(c) \\ f_*: & G(c') \boxtimes F(c) & \xrightarrow{\mathrm{Id}\boxtimes F(f)} & G(c') \boxtimes F(c') & \hookrightarrow \coprod_{c\in Ob(\mathcal{C})} G(c) \boxtimes F(c) \end{array}$$

Example 7.4. Let $\mathcal{S} = (\mathbf{Ab}, \bigotimes_{\mathbb{Z}}, \mathbb{Z})$ and $\mathcal{M} = \mathcal{S}$ with $\mathbf{Hom}_{\mathcal{S}} = \mathrm{Hom}_{\mathbf{Ab}} = \mathrm{Hom}_{\mathbb{Z}}$. $\mathcal{S}$ is enriched and tensored over itself with $\boxtimes = \bigotimes_{\mathbb{Z}}$.

Take a unital associative ring R, we can think of $\underline{R}$ as the category with one object $\{*\}$ enriched over $\mathcal{S}$.

A left module over R is an $\mathcal{S}$-functor

$$\begin{array}{rccc} \underline{F}: & \underline{R} & \to & \mathbf{Ab} \\ & * & \mapsto & M \\ & R & \mapsto & \mathrm{End}(M) \end{array}$$

A right module over R is an $\mathcal{S}$-functor

$$\begin{array}{rccc} \underline{G}: & \underline{R}^{op} & \to & \mathbf{Ab} \\ & * & \mapsto & N \\ & R^{op} & \mapsto & \mathrm{End}(N) \end{array}$$

Let's think of R as a monoid and take the underlying (unenriched) functors

$$\begin{array}{rccc} F: & R & \to & \mathbf{Ab} \\ G: & R^{op} & \to & \mathbf{Ab} \end{array}$$

Then

$$G \underset{R}{\boxtimes} F = \int^{R} N \bigotimes_{\mathbb{Z}} M \cong \frac{N \bigotimes_{\mathbb{Z}} M}{\langle nr \otimes m - n \otimes rm \rangle} \cong N \bigotimes_{R} M$$

is the usual tensor product of left and right modules.

Example 7.5. Let $\mathcal{S} = (\mathbf{sSet}, \times, *)$ and $\mathcal{C}$ be a small category. $\mathcal{M}$ is a simplicial category tensored over $\mathcal{S}$. Let $F : \mathcal{C} \to \mathcal{M}$ be any functor, and $G : \mathcal{C}^{op} \to \mathcal{S}, c \mapsto *$ be the constant functor at terminal object in $\mathcal{S}$. Then $* \underset{\mathcal{C}}{\boxtimes} F \cong \mathrm{colim}_{\mathcal{C}} F$. This follows from two facts.

1. If $S : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ is constant at first argument, then $\mathrm{coend}(S) \cong \mathrm{colim}(S)$.

2. $*$ acts as an identity on $\mathcal{M}$.

Intuition. In general for any $G : \mathcal{C}^{op} \to \mathbf{sSet}$ and $F : \mathcal{C} \to \mathcal{M}$, $G \underset{\mathcal{C}}{\boxtimes} F$ can be thought as a colimit of F "fattened up" by G.

Example 7.6. Let $\mathcal{C}$ be a small category and $\mathcal{M} = \mathcal{S} = \mathbf{sSet}$. For a fixed object $c \in Ob(\mathcal{C})$, for any $F : \mathcal{C} \to \mathbf{sSet}$ and $h_c = \mathrm{Hom}_{\mathcal{C}}(-, c) : \mathcal{C}^{op} \to \mathbf{Set} \hookrightarrow \mathbf{sSet}$, we have

$$\begin{aligned} h_c \underset{\mathcal{C}}{\boxtimes} F &= \int^{d \in Ob(\mathcal{C})} h_c(d) \boxtimes F(d) \\ &= \mathrm{coeq}\left\{ \coprod_{f : d \to d'} \mathrm{Hom}_{\mathcal{C}}(d', c) \boxtimes F(d) \underset{f_*}{\overset{f^*}{\rightrightarrows}} \coprod_{d \in Ob(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(d, c) \boxtimes F(d) \right\} \\ &= \mathrm{colim}_{\mathcal{C}/c} F \cong F(c) \end{aligned}$$

This also follows from the next example when take $G = \mathrm{Id}_{\mathcal{C}}$.

Dually, for any $G : \mathcal{C}^{op} \to \mathbf{sSet}$ and $F = h^c = \mathrm{Hom}_{\mathcal{C}}(c, -) : \mathcal{C} \to \mathbf{Set} \hookrightarrow \mathbf{sSet}$, we have $G \underset{\mathcal{C}}{\boxtimes} h^c \cong Gc$.

Example 7.7. (Kan extension) Given

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow\eta"] & \mathcal{D} \\ \mathcal{E} \arrow[ur, "\mathcal{L}_G(F)"'] & \end{tikzcd}$$

the left Kan extension can be interpreted as a tensor product

$$\mathcal{L}_G(F)(e) = (G \circ h_e) \underset{\mathcal{C}}{\boxtimes} F, e \in Ob(G).$$

where $G \circ h_e = \mathrm{Hom}_{\mathcal{D}}(G(-), e) : \mathcal{C}^{op} \to \mathbf{Set}$.

Proof. See later. We can also prove it by universal properties $\mathcal{L}_G(F)(e) \cong \mathrm{colim}_{G/e}(F)$.

$$(G \circ h_e) \underset{\mathcal{C}}{\boxtimes} F = \mathrm{coeq}\left\{ \coprod_{f : c \to c'} \mathrm{Hom}_{\mathcal{C}}(G(c'), e) \boxtimes F(c) \underset{f_*}{\overset{f^*}{\rightrightarrows}} \coprod_{c \in Ob(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(G(c), e) \boxtimes F(c) \right\} \cong \mathrm{colim}_{G/e}(F)$$

$\square$

7.4 Geometric Realization

Special case (classical geometric realization)

Given

$$\begin{tikzcd} \Delta \arrow[r, "\Delta^\bullet"] \arrow[d, "h"'] \arrow[dr, phantom, "\Downarrow\eta"] & \mathbf{Top} \\ \mathbf{sSet} \arrow[ur, "|-| = \mathcal{L}_h(\Delta^\bullet)"'] & \end{tikzcd}$$

For $X \in Ob(\mathbf{sSet})$, $\mathcal{L}_h(\Delta^\bullet)(X) \cong \operatorname{colim}_{\Delta X}(\Delta^\bullet)$. Then

$$\mathcal{L}_h(\Delta^\bullet)(X) \cong \mathrm{Hom}_{\mathbf{sSet}}(\Delta[\cdot], X) \underset{\Delta}{\boxtimes} \Delta^\bullet \cong X \underset{\Delta}{\boxtimes} \Delta^\bullet = \coprod_{n \geq 0} X_n \times \Delta^n / \langle (x, f_*(u)) \sim (f^*(x), u) \rangle_{f \in Mor(\Delta)}$$

where X_n is given the discrete topology.

In particular, if $X : \Delta^{op} \to \mathbf{Space}$ is a simplicial space ($\mathbf{Top}$ or $\mathbf{sSet}$) then $|-| : \mathbf{Space}^{\Delta^{op}} \to \mathbf{Space}$ is given by

$$|X| := X \underset{\Delta}{\boxtimes} \Delta^\bullet = \coprod_{n \geq 0} X_n \times \Delta^n / \sim$$

where $X_n \times \Delta^n$ is given the product topology.

Internal geometric realization

Let $\mathcal{M}$ be a simplicial category enriched over $\mathcal{S} = \mathbf{sSet}$, we can define

$$\begin{array}{rccc} |-| : & \mathbf{s}\mathcal{M} & \to & \mathcal{M} \\ & X & \mapsto & \Delta[-] \underset{\Delta^{op}}{\boxtimes} X \end{array}$$

where $\Delta[-] : \Delta \to \mathbf{sSet}$ is a left Δ-module, or a right Δ^{op}-module.

Example 7.8. Let $\mathcal{M} = \mathbf{Top}$ and $\mathcal{S} = \mathbf{sSet}$, then we have

$$\begin{array}{ccc} \mathbf{sSet} \times \mathbf{Top} & \to & \mathbf{Top} \\ (K, Z) & \mapsto & |K| \times Z \end{array}$$

is the usual geometric realization.

Example 7.9. Let $\mathcal{M} = \mathbf{sSet}$, $X : \Delta^{op} \to \mathbf{sSet}$ is a bisimplicial category. $X = \{X_{nm}\}_{n,m \geq 0}$ with $s_j^h, d_i^h, s_j^v, d_i^v$ in the following diagram

$$\begin{tikzcd} X_{01} & X_{11} \arrow[l, shift left] \arrow[l, shift right] & X_{21} \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] \\ X_{00} \arrow[from=u, shift left] \arrow[from=u, shift right] & X_{10} \arrow[l, shift left] \arrow[l, shift right] \arrow[from=u] \arrow[from=u, shift left=2] \arrow[from=u, shift right=2] & X_{20} \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[from=u, shift left] \arrow[from=u, shift right] \arrow[from=u, shift left=3] \arrow[from=u, shift right=3] \end{tikzcd}$$

We have a functor

$$\begin{array}{rccc} \mathrm{diag} : & \mathbf{sSet}^{\Delta^{op}} & \to & \mathbf{sSet} \\ & X_{**} & \mapsto & \left\{X_{nn}, d_i = d_i^v d_i^h, s_j = s_j^v s_j^h\right\}_{n \geq 0} \end{array}$$

then $|X| \cong \mathrm{diag}(X)$.

7.5 Homotopy Colimits

Motivation

Question: Why do we need them?

These come from topology. Usually colimits are used to build complicated spaces from simpler ones by "gluing".

Example 7.10. colim $\left(\mathbb{D}^n \overset{i}{\hookleftarrow} \mathbb{S}^{n-1} \overset{i}{\hookrightarrow} \mathbb{D}^n\right) \cong \mathbb{S}^n$.

Problem arises if we want to glue homotopy types.

$$\begin{array}{ccccccccc} \operatorname{colim} & (\mathbb{D}^n & \overset{i}{\hookleftarrow} & \mathbb{S}^{n-1} & \overset{i}{\hookrightarrow} & \mathbb{D}^n) & = & \mathbb{S}^n \\ & \cong\downarrow & & \| & & \downarrow\cong & & \nparallel \\ \operatorname{colim} & (* & \overset{i}{\hookleftarrow} & \mathbb{S}^{n-1} & \overset{i}{\hookrightarrow} & *) & = & * \end{array}$$

Moral. The objects (spaces) defined by colimits of diagrams, which are defined only up to homotopy, are not well-defined, even up to homotopy type.

Homotopy colimits are replacements of usual colimits when we glue objects (spaces) together with homotopies between gluing maps.

There is another use of homotopy colimits, they provide a natural way to deformation ("quantization") of objects.

Idea. If we want to deform an object (space), decompose (in a natural way) into a homotopy colimit and then, instead of deforming the object itself, we deform the underlying diagram of homotopy colimits.

Example

Which diagrams do appear in practice?

Pushout diagram $\quad \mathcal{C} = \{ \bullet \longleftarrow \bullet \longrightarrow \bullet \}$

1. **Mapping Tori**.

   If $f : X \to X$ is a map of unpointed spaces, the mapping torus is defined by

$$T(X, f) := \operatorname{hocolim}\left\{ X \xleftarrow{(\mathrm{Id}, f)} X \coprod X \xrightarrow{(\mathrm{Id},\mathrm{Id})} X \right\} \cong X \times I / (x, 0) \sim (f(x), 1)$$

This comes with projection onto $\mathbb{S}^1$:

$$T(\bullet,1) := \operatorname{hocolim}\left\{ \bullet \xleftarrow{(\mathrm{Id},f)} \bullet \coprod \bullet \xrightarrow{(\mathrm{Id},\mathrm{Id})} \bullet \right\} \cong \mathbb{S}^1.$$

The commutative diagram

$$\begin{tikzcd} X \arrow[d] & X \coprod X \arrow[l, "(\mathrm{Id},f)"'] \arrow[r, "(\mathrm{Id},\mathrm{Id})"] \arrow[d] & X \arrow[d] \\ \bullet & \bullet \coprod \bullet \arrow[l, "(\mathrm{Id},f)"'] \arrow[r, "(\mathrm{Id},\mathrm{Id})"] & \bullet \end{tikzcd}$$

induces (by functoriality) the map of homotopy colimits $T(X,f) \to T(\bullet,1) = \mathbb{S}^1$.

If X is a closed n-dimensional manifold, $f : X \xrightarrow{\sim} X$ is a smooth automorphism, then $T(X,f) \twoheadrightarrow \mathbb{S}^1$ is $(n+1)$-dimensional manifold fibred over $\mathbb{S}^1$.

Example 7.11. Let $\Sigma = \Sigma_g$ be a closed oriented surface of genus $g \geq 1$.

$$MCG^+(\Sigma) = \left\{ \varphi : \Sigma \xrightarrow{\sim} \Sigma \text{ orientation preserving homeomorphism} \right\} / \text{isotopy}$$

for instance, $MCG^+(\Sigma_1) = SL_2(\mathbb{Z})$.

Theorem 7.1. *Any orientable Σ-bundle over $\mathbb{S}^1$ has the form*

$$M_\varphi(\Sigma) = T(\Sigma,\varphi) \twoheadrightarrow \mathbb{S}^1 \tag{8}$$

for some $\varphi \in MCG^+(\Sigma)$.

Remark 7.2. Any Σ-bundle over $\mathbb{S}^1$ of the form 8 induces a short exact sequence

$$1 \to \pi_1(\Sigma) \to \pi_1(M_\varphi(\Sigma)) \to \pi_1\left(\mathbb{S}^1\right) = \mathbb{Z} \to 0$$

thus

$$\pi_1(M_\varphi(\Sigma)) \cong \pi_1(\Sigma) \rtimes \mathbb{Z} \hookleftarrow \pi_1(\Sigma).$$

Corresponding to $\pi_1(\Sigma) \subseteq \pi_1(M_\varphi(\Sigma))$ is an infinite cyclic covering

$$\begin{array}{ccc} \Sigma \times \mathbb{R} & \to & M_\varphi(\Sigma) \\ (x,\lambda) & \mapsto & [(x,\lambda)] \end{array}$$

with

$$\begin{array}{rccc} t: & \Sigma\times\mathbb{R} & \to & \Sigma\times\mathbb{R} \\ & (x,\lambda) & \mapsto & (\varphi(x),\lambda+1) \end{array}$$

then

$$M_{\varphi}(\Sigma)\cong\Sigma\times\mathbb{R}/\left(x,\lambda\right)\sim\left(\varphi(x),\lambda+1\right).$$

2. **Suspension** (unreduced)

$$\Sigma X=\operatorname{hocolim}\left\{\bullet\longleftarrow X\longrightarrow\bullet\right\}=\frac{X\times I}{\left(\left(x,0\right)\sim\left(x',0\right),\left(x,1\right)\sim\left(x',1\right)\right)}$$

3. **Join operation**

$$X*Y=\operatorname{hocolim}\left\{X\xleftarrow{p_1}X\times Y\xrightarrow{p_2}Y\right\}=\frac{X\coprod\left(X\times I\times Y\right)\coprod Y}{\left(\left(x,0,y\right)\sim x,\left(x,1,y\right)\sim y\right)}$$

Group Actions

If G is a discrete (or topological) group acting on a space X, then we have a natural diagram $X:\underline{G}\to$ **Space** and $\operatorname{colim}_{\underline{G}}(X)\cong X/G$.

The homotopy quotient (homotopy orbit space) is $X/_{h}G=X_{hG}\coloneqq\operatorname{hocolim}_{\underline{G}}(X)=EG\times_G X$.

Example 7.12. Let G be a discrete group.

1. G acts on $\{*\}$ trivially

$$\begin{array}{rccc} *: & \underline{G} & \longrightarrow & \mathbf{Set}\hookrightarrow\mathbf{sSet} \\ & * & \longmapsto & * \\ & g & \longmapsto & \mathrm{Id}_* \end{array}$$

then $*/_{h}G\cong\operatorname{hocolim}_{\underline{G}}(*)\cong BG$.

2. G acts on G by left translation

$$\begin{array}{rccc} G: & \underline{G} & \longrightarrow & \mathbf{Set}\hookrightarrow\mathbf{sSet} \\ & * & \longmapsto & G \\ & g & \longmapsto & \left(\begin{array}{rccc} L_g: & G & \to & G \\ & h & \mapsto & gh\end{array}\right) \end{array}$$

then $G/_{h}G\cong\operatorname{hocolim}_{\underline{G}}(G)\cong EG$.

3. G acts on G by adjoint action

$$\begin{array}{rccc} \mathrm{Ad}: & \underline{G} & \longrightarrow & \mathbf{Set} \hookrightarrow \mathbf{sSet} \\ & * & \longmapsto & G \\ & g & \longmapsto & \left(\begin{array}{cccc} \mathrm{Ad}_g: & G & \to & G \\ & h & \mapsto & ghg^{-} \end{array}\right) \end{array}$$

then $\mathrm{Ad}/_h G \cong \mathrm{hocolim}_{\underline{G}}(\mathrm{Ad}) \cong \mathcal{L}BG$ the free loop space over BG.

Simplicial, cyclic colimits

Given a simplicial object $X : \Delta^{op} \to \mathbf{Set} \hookrightarrow \mathbf{sSet}$, $\| X \| = \mathrm{hocolim}_{\Delta^{op}}(X) \simeq |X|$ (Bousfield-Kan Theorem).

Given a cyclic object $X : \Delta\mathbf{C}^{op} \to \mathbf{Set} \hookrightarrow \mathbf{sSet}$, $\| X \|^{cy} = \mathrm{hocolim}_{\Delta\mathbf{C}^{op}}(X) \simeq |X|^{cy} = E\mathbb{S}^1 \times_{\mathbb{S}^1} X$ ([FL91]).

Poset diagrams

Let $\mathcal{C}$ be a category associated to a poset. Many interesting spaces decompose into homotopy colimits of poset diagrams.

Example 7.13. Let B_n be the poset of all nonempty faces in n-simplex, ordered by inclusion.

$n = 1$, the 1-simplex $0 \to 1$ gives $B_1 = \{0, 1, 01\}$, $|B_1| = 2^2 - 1 = 3$.

$n = 2$, the 2-simplex

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}$$

gives $B_2 = \{0, 1, 2, 01, 02, 12, 012\}$, $|B_2| = 2^3 - 1 = 7$.

Theorem 7.2. *[Z83] Given $\mathcal{X} = \{X_i\}_{i=0}^n$ $n+1$ spaces, define*

$$\begin{array}{rccc} \mathcal{D}_{\mathcal{X}}: & B_n & \longrightarrow & \mathbf{Space} \\ & A & \longmapsto & \prod_{i \in A} X_i \\ & A \supseteq B & \longmapsto & \left(p_{AB} : \prod_{i \in A} X_i \to \prod_{i \in B} X_i\right) \end{array}$$

where $p_{AB} : \prod_{i \in A} X_i \to \prod_{i \in B} X_i$ is the canonical projection. Then $\mathrm{hocolim}_{B_n}(\mathcal{X}) \cong X_0 * \cdots * X_n$.

Assume $X_i = \mathbb{S}^1, \forall 0 \le i \le n$, we can modify $\overline{\mathcal{D}_{\mathcal{X}}}(A) = \left(\mathbb{S}^1\right)^{|A|} /_{diag}\mathbb{S}^1$, then $\mathrm{hocolim}_{B_n}\left(\overline{\mathcal{D}_{\mathcal{X}}}\right) \cong \mathbb{CP}^n$.

More generally, any toric variety can be decomposed in this way.

Example 7.14. $n = 1$. $\mathcal{X} = \{X_0, X_1\}$ and $\mathcal{D}_{\mathcal{X}} = \left\{ X_0 \xleftarrow{p_0} X_0 \times X_1 \xrightarrow{p_1} X_1 \right\}$.

7.6 Homotopical Categories

Definition 7.7. A homotopical category $\mathcal{M}$ is a category equipped with a class of morphisms $\mathcal{W} \subseteq Mor(\mathcal{M})$ called weak equivalences, such that

(W1) all identities are in $\mathcal{W}$, $\forall x \in Ob(\mathcal{M})$, $\mathrm{Id}_x \in \mathcal{W}$.

(W2) 2-of-6 property holds: given a composable triple (f, g, h) in $Mor(\mathcal{M})$, if $gf, hg \in \mathcal{W}$, then $f, g, h, hgf \in \mathcal{W}$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, dashed, "gf"'] \arrow[drr, dotted] & Y \arrow[d, "g"] \arrow[dr, dashed, "hg"] & \\
& Z \arrow[r, "h"] & W
\end{tikzcd}$$

Remark 7.3.

1. Axioms 7.6 and 7.6 implies that all isomorphisms are in $\mathcal{W}$.

   If $gf = \mathrm{Id}_X$ and $fg = \mathrm{Id}_Y$ then we have

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, dashed, "\mathrm{Id}_X"'] \arrow[drr, dotted] & Y \arrow[d, "g"] \arrow[dr, dashed, "\mathrm{Id}_Y"] & \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

   thus $f, g \in \mathcal{W}$.

2. Axiom 7.6 implies the usual 2-of-3 property.

   If $f, g \in \mathcal{W}$, then the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, dashed, "f"'] \arrow[drr, dotted] & Y \arrow[d, "\mathrm{Id}_Y"] \arrow[dr, dashed, "g"] & \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

   shows that $gf \in \mathcal{W}$.

   If $f, gf \in \mathcal{W}$, then the diagram

$$\begin{tikzcd}
X \arrow[r, "\mathrm{Id}_X"] \arrow[dr, dashed, "f"'] \arrow[drr, dotted] & X \arrow[d, "f"] \arrow[dr, dashed, "gf"] & \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

shows that $g \in \mathcal{W}$.

If $g, gf \in \mathcal{W}$, then the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, dashed, "gf"'] \arrow[drr, dotted] & Y \arrow[d, "g"] \arrow[dr, dashed, "g"] & \\
& Z \arrow[r, "\mathrm{Id}"] & Z
\end{tikzcd}$$

shows that $f \in \mathcal{W}$.

This allows us to view $\mathcal{W}$ as a subcategory of $\mathcal{M}$, which is wide in the sense that it contains all objects in $\mathcal{M}$.

Example 7.15. *Minimal homotopical category.* Any category $\mathcal{M}$ can be viewed as homotopical if we take $\mathcal{W} = \mathrm{Iso}(\mathcal{M})$ the class of all isomorphisms. I-2ndeed, $\mathrm{Iso}(\mathcal{M})$ satisfies 2-of-6 property.

Given $f, g, h \in Mor(\mathcal{M})$ such that $hg, gf \in \mathrm{Iso}(\mathcal{M})$, then $\gamma = f(gf)^{-1}$ is a right inverse of g, $g\gamma = \mathrm{Id}$. Note that g is monic (since if $gf_1 = gf_2$, then $hgf_1 = hgf_2$, so $f_1 = f_2$), hence γ is also a left inverse of g ($g\gamma g = g$ implies $\gamma g = \mathrm{Id}$). Thus g is an isomorphism, and so are the others.

Remark 7.4. The fact that isomorphisms in any category satisfy 2-of-6 property is used to prove that homotopy equivalences of spaces are weak homotopy equivalences. If there exists a pair of morphisms

$$\begin{tikzcd}
f : X \arrow[r, shift left] & Y : g \arrow[l, shift left]
\end{tikzcd}$$

such that $gf \simeq \mathrm{Id}_X$ and $fg \simeq \mathrm{Id}_Y$, then $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism for any $n \geq 0$. Since π_n is homotopy invariant, f_*, g_* are isomorphisms of groups.

$$\begin{tikzcd}
\pi_n(X, x) \arrow[r, "f_*"] \arrow[dr, dashed, "\cong"'] & \pi_n(Y, f(x)) \arrow[d, "g_*"] \arrow[dr, dashed, "\cong"] & \\
& \pi_n(X, gf(x)) \arrow[r, "f_*"] & \pi_n(Y, fgf(x))
\end{tikzcd}$$

Example 7.16. Any model category $\mathcal{M}$ is homotopical (or has the underlying homotopical category) with the same class of weak equivalences. By Quillen axioms, the class of weak equivalences $\mathcal{W}$ in a model category $\mathcal{M}$ satisfies 2-of-3 property. But in fact, $\mathcal{W}$ satisfies 2-of-6.

Example 7.17. *Diagrams in homotopical categories.* If $\mathcal{M}$ is a homotopical category, and $\mathcal{C}$ is a small category, then $\mathcal{M}^{\mathcal{C}} := \mathbf{Fun}(\mathcal{C}, \mathcal{M})$ is a homotopical category with $\mathcal{W}\left(\mathcal{M}^{\mathcal{C}}\right)$ defined objectwise.

$$(\alpha : F \Rightarrow F') \in \mathcal{W}\left(\mathcal{M}^{\mathcal{C}}\right) \Leftrightarrow \forall c \in Ob(\mathcal{C}), (\alpha_c : F(c) \to F'(c)) \in \mathcal{W}.$$

7.7 Homotopy Category

Definition 7.8. Let $(\mathcal{M}, \mathcal{W})$ be a homotopical category, then *homotopy category* of $\mathcal{M}$ is $\mathrm{Ho}(\mathcal{M}) := \mathcal{M}\left[\mathcal{W}^{-1}\right]$ the formal localization of $\mathcal{M}$ at $\mathcal{W}$.

It comes with a localization functor $\gamma : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ which is initial among all functors $\gamma' : \mathcal{M} \to \mathcal{N}$ such that $\gamma'(f) \in \mathrm{Iso}(\mathcal{N}), \forall f \in \mathcal{W}$.

Convention. $(\mathcal{M}, \mathcal{W})$ is called *saturated* if

$$\gamma(f) \in \mathrm{Iso}(\mathrm{Ho}(\mathcal{M})) \implies f \in \mathcal{W}. \tag{9}$$

Theorem 7.3. *[Quillen] Any model category is saturated.*

Lemma 7.1. *If $(\mathcal{M}, \mathcal{W})$ is a category with a class of morphisms such that for any $X \in Ob(\mathcal{M})$, $Id_X \in \mathcal{W}$ and the property 9 holds, then $(\mathcal{M}, \mathcal{W})$ satisfies 2-of-6, and thus is homotopical.*

Proof. By 9, $\mathcal{W} = \{f \in Mor(\mathcal{M}) \,|\, \gamma(f) \in \mathrm{Iso}(\mathrm{Ho}(\mathcal{M}))\}$. Given $f, g, h \in Mor(\mathcal{M})$, if $gf, hg \in \mathcal{W}$, then $\gamma(gf), \gamma(hg) \in \mathrm{Iso}(\mathrm{Ho}(\mathcal{M}))$, so $\gamma(f), \gamma(g), \gamma(hgf) \in \mathrm{Iso}(\mathrm{Ho}(\mathcal{M}))$, hence $f, g, hgf \in \mathcal{W}$. $\square$

Corollary 7.1. *[Q67] Any model category is homotopical.*

Question. Given a saturated homotopical category, does it come from a model category?

Example: Noncommutative Poisson structure (not yet from model structure)

Let k be a field of characteristic zero. If A is a commutative algebra over k, then a Poisson structure on A is a bracket

$$\begin{array}{rccc} \{-,-\}: & A \times A & \to & A \\ & (a,b) & \mapsto & \{a,b\} \end{array}$$

such that

1. $(A, \{-,-\})$ is a Lie algebra.
2. $\{-,-\}$ satisfies Leibniz rule, $\{a, bc\} = \{a,b\}c + b\{a,c\}, \forall a,b,c \in A$.

Question: how to extend this definition to NC algebras?

Naive definition (forgetting A us commutative) as above is very restrictive.

Theorem 7.4. *[FL98] If A is a noncommutative Noetherian domain, then any Poisson structure $\{-,-\}$ on A is a (scalar) multiple of $[-,-] : A \times A \to A, (a,b) \mapsto ab - ba$.*

Let $\mathbf{DGA}_k$ be the category of differential graded associative k-algebras. For $A \in Ob\left(\mathbf{DGA}_k\right)$, we have

$$A = \bigoplus_{i\in\mathbb{Z}} A_i, A_i \cdot A_j \subseteq A_{i+j}$$

equipped with differential $d : A \to A, d^2 = 0, |d| = -1$, satisfying

$$d\left(ab\right) = d\left(a\right) \cdot b + \left(-1\right)^{|a|} a \cdot db, \forall a, b \in A.$$

Goal. define a Poisson structure on A.

Idea. instead of A, we put a differential graded Lie algebra structure on $A_\natural := A/\left[A, A\right]$ where $\left[A, A\right] = \mathrm{Span}_k \left\{ [a, b] = ab - \left(-1\right)^{|a||b|} ba, a, b \in A \right\}$.

Although $A_\natural$ is not an algebra, it is naturally a chain complex with differential induced from the differential on A. Indeed,

$$\begin{aligned} d\left([a, b]\right) &= d\left(ab\right) - \left(-1\right)^{|a||b|} d\left(ba\right) \\ &= d\left(a\right) \cdot b + \left(-1\right)^{|a|} a \cdot d\left(b\right) - \left(-1\right)^{|a||b|} \left(d\left(b\right) \cdot a + \left(-1\right)^{|b|} b \cdot d\left(a\right) \right) \\ &= \left[d\left(a\right) \cdot b + \left(-1\right)^{(|a|-1)|b|} b \cdot d\left(a\right) \right] + \left(-1\right)^{|a|} \left[a \cdot d\left(b\right) + \left(-1\right)^{(|b|-1)|a|} d\left(b\right) \cdot a \right] \\ &= \left[d\left(a\right), b\right] + \left(-1\right)^{|a|} \left[a, d\left(b\right)\right] \in \left[A, A\right] \end{aligned}$$

Hence $d : A_\natural \to A_\natural, \overline{a} \mapsto \overline{d\left(a\right)}$ is well-defined.

If $\left(V, d_V\right)$ is any complex, then the graded endomorphism ring is

$$\mathbf{End}\left(V\right) = \bigoplus_{n\in\mathbb{Z}} \mathrm{End}\left(V\right)_n, \mathrm{End}\left(V\right)_n = \left\{f : V \to V | f\left(V_i\right) \subseteq V_{i+n}, \forall i \in \mathbb{Z}\right\} = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_k\left(V_i, V_{i+n}\right).$$

The degree of an endomorphism $|f| = n$ if $f \in \mathrm{End}\left(V\right)_n$.

Define $d : \mathbf{End}\left(V\right) \to \mathbf{End}\left(V\right)$ by $d\left(f\right) = \left[d_V, f\right] = d_V \circ f - \left(-1\right)^{|f|} f \circ d_V$, then

$$d^2\left(f\right) = \left[d_V, \left[d_V, f\right]\right] = \left[d_V, d_V \circ f - \left(-1\right)^{|f|} f \circ d_V \right] = d_V^2 \circ f - \left(-1\right)^{|f|-1} d_V \circ f \circ d_V - \left(-1\right)^{|f|} d_V \circ f \circ d_V - f \circ d_V^2 = 0$$

since $d_V^2 = 0$, so d is a differential. This makes $\mathbf{End}\left(V\right)$ a differential graded algebra with unit Id_V and hence a differential graded Lie algebra with bracket

$$\left[f, g\right] = f \circ g - \left(-1\right)^{|f||g|} g \circ f, \forall f, g \in \mathbf{End}\left(V\right).$$

If $V = A$ is a differential graded algebra, then $\mathbf{End}(V)$ contains a canonical differential graded Lie subalgebra $\mathbf{Der}(A)$ consisting all graded k-linear derivations of A. Recall that a derivation of degree r on A is a k-linear map $\delta : A \to A$ such that $\delta \in \mathbf{End}(A)_r$ and satisfies $\delta(ab) = \delta(a) \cdot b + (-1)^{|a| \cdot r} a \cdot \delta(b)$. Note that $\mathbf{Der}(A)$ acts on A naturally so that A becomes a differential graded Lie module over $\mathbf{Der}(A)$.

Consider $\mathbf{Der}(A)^\natural = \{\delta \in \mathbf{Der}(A) \,|\, \delta(A) \in [A, A]\}$. This is a differential graded Lie ideal in $\mathbf{Der}(A)$ since for any $\delta \in \mathbf{Der}(A), d \in \mathbf{Der}(A)^\natural$

$$[\delta, d](A) = \delta(d(A)) - (-1)^{|\delta||d|} d(\delta(A)) \subseteq \delta([A, A]) - (-1)^{|\delta||d|} d(A) \subseteq [A, A].$$

Define $\mathbf{Der}(A)_\natural \coloneqq \mathbf{Der}(A) / \mathbf{Der}(A)^\natural$ be the quotient algebra. The action of $\mathbf{Der}(A)$ on A induces a Lie action of $\mathbf{Der}(A)_\natural$ on $A_\natural$ so that

$$\begin{array}{cccc} \rho : & \mathbf{Der}(A)_\natural & \longrightarrow & \mathbf{End}(A_\natural) \\ & \delta\left(\mod \mathbf{Der}(A)^\natural \right) & \longmapsto & \left(\begin{array}{cccc} \rho(\delta) : & A_\natural & \to & A_\natural \\ & \overline{a} & \mapsto & \overline{\delta(a)} \end{array} \right) \end{array}$$

is Lie algebra homomorphism.

Definition 7.9. A *noncommutative Poisson structure* on A is given by a differential graded Lie algebra structure on $A_\natural$

$$\{-, -\} : A_\natural \times A_\natural \to A_\natural$$

such that the corresponding adjoint representation

$$\begin{array}{cccc} \mathrm{ad} : & A_\natural & \longrightarrow & \mathbf{End}(A_\natural) \\ & \overline{a} & \longmapsto & \left(\{\overline{a}, -\} : \overline{b} \mapsto \{\overline{a}, \overline{b}\} \right) \end{array}$$

factor through ρ, i.e.

$$\begin{tikzcd} A_\natural \arrow[rr, "\mathrm{ad}"] \arrow[dr, "\alpha"'] & & \mathbf{End}(A_\natural) \\ & \mathbf{Der}(A)_\natural \arrow[ur, "\rho"'] & \end{tikzcd}$$

where $\alpha : A_\natural \to \mathbf{Der}(A)_\natural$ is a Lie algebra homomorphism.

Exercise 7.1. If A is commutative, this definition agrees with the classical one.

Proof. If A is commutative, $A_\natural = A$, the adjoint representation is in fact a derivation on A, i.e. the bracket $\{-, -\}$ satisfies Leibniz rule. $\square$

Exercise 7.2. Let k be a field of characteristic zero. Repeat this construction for simplicial algebras $\mathbf{sAlg}_k$ so that $\mathcal{N} : \mathbf{sAlg}_k \to \mathbf{DGA}_k$.

Example: Kontsevich's Bracket

Let $A = k\langle x_1, \cdots, x_n\rangle$ be a free algebra of rank n.

$$A = \bigoplus_{l\geq 0} A^{(l)}, A^{(l)} = \mathrm{Span}_k\left(\text{noncommutative monomials of length } i \text{ in } x_1, \cdots, x_n\right)$$

Note for each $i \geq 0$, there exists a cyclic operation

$$\begin{array}{rccc} \tau_l : & A^{(l)} & \to & A^{(l)} \\ & w = v_1 \cdots v_l & \mapsto & v_l v_1 \cdots v_{l-1} \end{array}$$

Example 7.18. $n = 3$, $w = x_1^2 x_3^3 x_2$, $\tau_6(w) = x_2 x_1^2 x_3^3$.

Consider $\mathcal{N}_l = 1 + \tau_l + \cdots + \tau_l^{l-1} : A^{(l)} \to A^{(l)}$.

Definition 7.10. A *cyclic word* in A of length l is $a \in A^{(l)}$ such that $\mathcal{N}_l(a) = a$. Define

$$A^{cyc} := \bigoplus_{l\geq 0} A^{\mathrm{cyc}(l)} \subseteq A$$

the subspace spanned by all cyclic words.

Lemma 7.2. *The natural map $A^{cyc} \hookrightarrow A \twoheadrightarrow A_\natural$ is an isomorphism of graded vector spaces.*

Fix $x = x_i, i = 1, \cdots, n$ and define the *cyclic derivative*

$$\begin{array}{rccc} \frac{\partial}{\partial x} : & A & \to & A \\ & w = v_1 \cdots v_l & \mapsto & \sum_{v_m = x} v_{m+1} \cdots v_l v_1 \cdots v_{m-1} \end{array}$$

Example 7.19. $A = k\langle x_1, x_2, x_3\rangle$ and $x = x_1$, then

$$\frac{\partial}{\partial x}\left(x_1^2 x_2 x_1 x_3\right) = x_1 x_2 x_1 x_3 + x_2 x_1 x_3 x_1 + x_3 x_1^2 x_2.$$

This induces a well-defined map $\frac{\partial}{\partial x_i} : A_\natural \to A_\natural$. Now let $A = k\langle x_1, x_2\rangle$ and define

$$\begin{array}{rccl} \{-,-\}_\natural : & A_\natural \times A_\natural & \to & A_\natural \\ & (\overline{a}, \overline{b}) & \mapsto & \left(\frac{\partial a}{\partial x_1}\frac{\partial b}{\partial x_2} - \frac{\partial a}{\partial x_2}\frac{\partial b}{\partial x_1}\right) \mod [A, A] \end{array}$$

Theorem 7.5. *This defines a Poisson structure on $A = k\langle x_1, x_2\rangle$.*

Proof. We need to check $\left(A_\natural, \{-,-\}_\natural\right)$ is a Lie algebra, and the adjoint representation factors through $\mathbf{Der}(A)_\natural$.

Bilinearity and Alternativity are obvious. Jacobi identity

$$\begin{aligned}\left\{\overline{a},\left\{\overline{b},\overline{c}\right\}_\natural\right\}_\natural+\left\{\overline{c},\left\{\overline{a},\overline{b}\right\}_\natural\right\}_\natural+\left\{\overline{b},\left\{\overline{c},\overline{a}\right\}_\natural\right\}_\natural &= \frac{\partial a}{\partial x_1}\frac{\partial}{\partial x_2}\left(\frac{\partial b}{\partial x_1}\frac{\partial c}{\partial x_2}-\frac{\partial b}{\partial x_2}\frac{\partial c}{\partial x_1}\right)-\frac{\partial a}{\partial x_2}\frac{\partial}{\partial x_1}\left(\frac{\partial b}{\partial x_1}\frac{\partial c}{\partial x_2}-\frac{\partial b}{\partial x_2}\frac{\partial c}{\partial x_1}\right)+\\ &\quad \frac{\partial c}{\partial x_1}\frac{\partial}{\partial x_2}\left(\frac{\partial a}{\partial x_1}\frac{\partial b}{\partial x_2}-\frac{\partial a}{\partial x_2}\frac{\partial b}{\partial x_1}\right)-\frac{\partial c}{\partial x_2}\frac{\partial}{\partial x_1}\left(\frac{\partial a}{\partial x_1}\frac{\partial b}{\partial x_2}-\frac{\partial a}{\partial x_2}\frac{\partial b}{\partial x_1}\right)+\\ &\quad \frac{\partial b}{\partial x_1}\frac{\partial}{\partial x_2}\left(\frac{\partial c}{\partial x_1}\frac{\partial a}{\partial x_2}-\frac{\partial c}{\partial x_2}\frac{\partial a}{\partial x_1}\right)-\frac{\partial b}{\partial x_2}\frac{\partial}{\partial x_1}\left(\frac{\partial c}{\partial x_1}\frac{\partial a}{\partial x_2}-\frac{\partial c}{\partial x_2}\frac{\partial a}{\partial x_1}\right)\\ &= 0\end{aligned}$$

The adjoint action $\delta_a = \{\overline{a}, -\}_\natural$ satisfies that $\delta_a\left(\overline{bc}\right) = \delta_a\left(\overline{b}\right)\cdot\overline{c} + \overline{b}\cdot\delta_a\left(\overline{c}\right)$. □

Question. where does it come from?

For a finite dimensional vector space V,

$$\begin{aligned} |-|_V : \quad \mathbf{Alg}_k &\to \mathbf{CommAlg}_k \\ A &\mapsto \mathcal{O}\left(\mathrm{Rep}_V(A)\right)\end{aligned}$$

we can think of this functor as a realization (like geometric realization of simplicial sets), then a a NC Poisson structure induces a unique classical Poisson structure on $|A|_V$ for any V and in a sense, this is the weakest structure on A that does this.

Derived Poisson Algebras

Definition 7.11. A *differential graded Poisson algebra* is a differential graded algebra A equipped with a NC Poisson structure. A morphism of differential graded Poisson algebra is a morphism $f : A \to B$ of differential graded algebras such that $f_\natural : A_\natural \to B_\natural$ is a morphism of differential graded Lie algebras. Write $\mathbf{DGPA}_k$ for the category of differential graded Poisson algebras.

$\mathbf{DGA}_k$ and $\mathbf{DGLA}_k$ are both (cofibrantly generated) model categories with weak equivalences being quasi-isomorphisms and fibrations being degreewise surjective maps. Note that there are two forgetful functors

$$\begin{tikzcd} & \mathbf{DGPA}_k \arrow[dl, "U"'] \arrow[dr, "(-)_\natural"] & \\ \mathbf{DGA}_k & & \mathbf{DGLA}_k \end{tikzcd}$$

Definition 7.12. f is a weak equivalence in $\mathbf{DGPA}_k$ if both $U(f)$ and $(f)_\natural$ are weak equivalences in $\mathbf{DGA}_k$ and $\mathbf{DGLA}_k$ respectively.

Proposition 7.1. $\mathbf{DGPA}_k$ *is a saturated homotopical category.*

Proof. We need to show $\mathcal{W}$ the class of weak equivalences in $\mathbf{DGPA}_k$ is saturated, i.e. $\gamma(f)$ is an isomorphism in $\mathrm{Ho}(\mathbf{DGPA}_k) := \mathbf{DGPA}_k[\mathcal{W}^{-1}]$ if and only if $f \in \mathcal{W}$.

Take f such that $\gamma(f)$ is an isomorphism in $\mathrm{Ho}(\mathbf{DGPA}_k)$. Since $U, (-)_\natural$ preserves weak equivalences, they induce

$$\begin{tikzcd} & \mathrm{Ho}(\mathbf{DGPA}_k) \arrow[dl, "\gamma U"'] \arrow[dr, "\gamma(-)_\natural"] & \\ \mathrm{Ho}(\mathbf{DGA}_k) & & \mathrm{Ho}(\mathbf{DGLA}_k) \end{tikzcd}$$

so $\gamma(Uf)$ and $\gamma(f_\natural)$ are isomorphisms in $\mathrm{Ho}(\mathbf{DGA}_k)$ and $\mathrm{Ho}(\mathbf{DGLA}_k)$ respectively. By Quillen's theorem, $U(f)$ and $(f)_\natural$ are weak equivalences, so f is a weak equivalence. $\square$

Conjecture 7.1.

1. $\mathbf{DGPA}_k$ *has a natural (cofibrantly generated) model structure with weak equivalences being* $\mathcal{W}$.

2. *This model structure should be "minimal" in the sense that there is a functor*

$$\mathbf{DGPA}_k \to \mathbf{DGA}_k \underset{\mathbf{CDGA}_k}{\times^h} \mathbf{CDGPA}_k$$

where $\mathbf{CDGPA}_k$ *is the category of commutative differential graded Poisson algebras and* $\times^h$ *is the homotopy fibre product (in the sense of Töen[2006] such that the induced map between homotopical categories is an isomorphism*

$$Ho(\mathbf{DGPA}_k) \xrightarrow{\sim} Ho\left(\mathbf{DGA}_k \underset{\mathbf{CDGA}_k}{\times^h} \mathbf{CDGPA}_k\right).$$

Remark 7.5. *Töen's construction.* Given

$$\begin{tikzcd} & \mathcal{M}_2 \arrow[d, "F_2"] \\ \mathcal{M}_1 \arrow[r, "F_1"] & \mathcal{M}_3 \end{tikzcd}$$

where $\mathcal{M}_1, \mathcal{M}_2$ are model categories, F_1, F_2 are (Quillen) functors, define the category $\mathcal{M}_1 \underset{\mathcal{M}_3}{\times^h} \mathcal{M}_2$ to be the category with

Objects: $\left\{(A_1, A_2, A_3, u_1, u_2), A_i \in Ob(\mathcal{M}_i), F_1(A_1) \xrightarrow{u_1} A_3 \xleftarrow{u_2} F_2(A_2)\right\}$.

Morphisms: $f = (f_1, f_2, f_3) \in \mathbf{Mor}(A_1) \times \mathbf{Mor}(A_2) \times \mathbf{Mor}(A)$ such that

$$\begin{tikzcd} F_1(A_1) \arrow[r, "u_1"] \arrow[d, "F(f_1)"'] & A_3 \arrow[d, "f_3"] & F_2(A_2) \arrow[l, "u_2"'] \arrow[d, "F_2(f_2)"] \\ F_1(B_1) \arrow[r, "v_1"] & B_3 & F_2(B_2) \arrow[l, "v_2"'] \end{tikzcd}$$

commutes.

Theorem 7.6. *[*Töen*]*

1. *$\mathcal{M}_1 \underset{\mathcal{M}_3}{\times^h} \mathcal{M}_2$ is a model category with levelwise weak equivalences.*
2. *For any finite dimensional vector space V, $(V = k^n, n \geq 0)$, there is a left Quillen functor*

$$\begin{aligned} (-)_V : \quad \mathbf{DGA}_k &\longrightarrow \mathbf{CDGA}_k \\ A &\longmapsto \mathcal{O}(Rep_V(A)) \end{aligned}$$

This induces a homotopy fibred category

$$\begin{tikzcd} \mathbf{DGA}_k \underset{\mathbf{CDGA}_k}{\times^h} \mathbf{CDGPA}_k \arrow[r] \arrow[d] & \mathbf{CDGPA}_k \arrow[d, "U"] \\ \mathbf{DGA}_k \arrow[r, "(-)_V"] & \mathbf{CDGA}_k \end{tikzcd}$$

Definition 7.13. A derived Poisson algebra is an object $A \in Ob(\mathrm{Ho}(\mathbf{DGPA}_k))$ such that $A_\natural = A/[A, A]$ is equipped with $\{-,-\}_\natural : A_\natural \times A_\natural \to A_\natural$ compatible with differential graded algebra structure.

Proposition 7.2. *The reduces cyclic homology $\overline{HC}_*(A)$ of any derived Poisson algebra carries a natural graded Lie algebra structure.*

Proof. See later. □

Theorem 7.7. *If A is any derived Poisson algebra, then for any V, the representation homology $HR_*(A, V)^{GL(V)}$ carries a unique graded Poisson structure such that the derived character map*

$$Tr_V(A) : \overline{HC}_*(A) \to HR_*(A, V)^{GL(V)}$$

is a Lie algebra homomorphism.

Example: Poisson algebra

Take $A = C^{\infty}\left(\mathbb{R}^2\right)$ or $A = k[x,y]$ ($k \supseteq \mathbb{Q}$). $\mathbb{R}^2$ is a symplectic manifold with $\omega = dx \wedge dy$. A is a commutative Poisson algebra with

$$\begin{array}{rccc} \{-,-\}: & A \times A & \to & A \\ & (f,g) & \mapsto & \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y} \end{array}$$

To define NC Poisson structure, we need to replace $A = k[x,y]$ with cofibrant (or free) resolution.

Consider $R = k\langle x,y,t\rangle$ with $dt = [x,y]$, $|x| = |y| = 0, |t| = 1$, then we have a cofibrant resolution

$$\begin{array}{ccc} R & \twoheadrightarrow & A \\ x,y & \mapsto & x,y \\ t & \mapsto & 0 \end{array}$$

Note $R_0 = k\langle x,y\rangle$, by earlier example, $(R_0)_\natural$ carries a Lie algebra bracket defined in terms of cyclic derivatives (Kontsevich bracket). It turns out that $\{-,-\}_\natural$ can be extended to (homologically) graded setting:

$$\{-,-\}_\natural : R_\natural \times R_\natural \to R_\natural$$

so that $\{-,-\}_\natural|_0$ is the Kontsevich bracket.

This makes R a derived Poisson algebra.

Question: what does this induce on $\overline{HC}_*(A) = \overline{HC}_*(R)$?

By HKR theorem, since A is smooth of dimension 2,

$$\overline{HC}_*(A) = \overline{HC}_0(A) \oplus \overline{HC}_1(A)$$

where $\overline{HC}_0(A) = \overline{A} = A/k \cdot 1$ and $\overline{HC}_1(A) = \Omega^1(A)/d\Omega^0(A) = \Omega^1(A)/dA$.

$\Omega^\bullet(A)$ is the de Rham algebra with $\Omega^1(A) = \{f dx + g dy, f, g \in A\}$, $\Omega^0(A) = A$.

$\overline{HC}_*(A) = \overline{A} \oplus \Omega^1(A)/dA$ is a graded Lie algebra where

$$\{-,-\}_\natural : \overline{HC}_*(A) \times \overline{HC}_*(A) \to \overline{HC}_*(A)$$

has only two components

$$\begin{array}{rccc} \{-,-\}_\natural|_0 : & \overline{HC}_0(A)\times\overline{HC}_0(A) & \to & \overline{HC}_0(A) \\ & (\overline{f},\overline{g}) & \mapsto & \overline{\{f,g\}} \end{array}$$

the classical Poisson bracket, and

$$\begin{array}{rccc} \{-,-\}_\natural|_1 : & \overline{HC}_0(A)\times\overline{HC}_1(A) & \to & \overline{HC}_1(A) \\ & (\overline{f},\overline{\omega}) & \mapsto & \mathcal{L}_{\theta_f}(\omega) \end{array}$$

where $\overline{f}=f\,(\mod k)$ in $\overline{A}$, and $\overline{\omega}=fdx+gdy\,(\mod dA)$ 1-form, $\theta_f=\{f,-\}=\frac{\partial f}{\partial x}\frac{\partial}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial}{\partial x}$.

Example: Quillen's rational homotopy theory. (*String topology*).

Let $\mathbf{Top}^1$ be the category of 1-connected topological spaces of finite rational type (i.e. $\dim_{\mathbb{Q}} H_i(X,\mathbb{Q})<\infty,\forall i$. Define the rational homotopy

$$\mathrm{Ho}\left(\mathbf{Top}^1\right)_{\mathbb{Q}} \coloneqq \mathbf{Top}^1\left[\mathcal{W}_{\mathbb{Q}}^{-1}\right]$$

where

$$f:X\to Y\in\mathcal{W}_{\mathbb{Q}}\Longleftrightarrow f_i:\pi_i(X)\bigotimes_{\mathbb{Z}}\mathbb{Q}\xrightarrow{\sim}\pi_i(Y)\bigotimes_{\mathbb{Z}}\mathbb{Q},\forall i\geq 0$$

as $\mathbb{Q}$-vector spaces.

Theorem 7.8. *$Ho\left(\mathbf{Top}^1\right)_{\mathbb{Q}}\cong Ho\left(\mathbf{DGLA}_{\mathbb{Q}}^{red}\right)$ where $\mathbf{DGLA}_{\mathbb{Q}}^{red}$ is the category of reduced differential graded Lie algebras $\mathfrak{a}=\bigoplus_{i>0}\mathfrak{a}_i,\mathfrak{a}_0=0$.*

For any 1-connected topological space X, there exists a reduced differential graded Lie algebra $\mathfrak{a}_X$ which is a complete homotopy invariant.

Example 7.20. Let $X=\mathbb{S}^n, n\geq 0$, $\mathfrak{a}_{\mathbb{S}^n}=$ the free graded Lie algebra $\mathcal{L}(x)$ with single generator of degree $|x|=n-1$ with trivial differential $d=0$. There is a Whitehead product

$$H_*(\mathfrak{a}_{\mathbb{S}^n})\cong\pi_*(\Omega X)\bigotimes_{\mathbb{Z}}\mathbb{Q}\cong\pi_{*+1}(X)\bigotimes_{\mathbb{Z}}\mathbb{Q}.$$

Theorem 7.9. *[Jones, 1985] The reduced cyclic homology of the universal enveloping algebra of $\mathfrak{a}_X$ is weak equivalent to the $\mathbb{S}^1$-equivariant reduced homology of the free loop space $\mathcal{L}X=\mathbf{Map}\left(\mathbb{S}^1,X\right)$ with coefficients in $\mathbb{Q}$,*

$$\overline{HC}(\mathcal{U}\mathfrak{a}_X)\simeq\overline{H}_*^{\mathbb{S}^1}(\mathcal{L}X,\mathbb{Q})=H_*\left(E\mathbb{S}^1\underset{\mathbb{S}^1}{\times}\mathcal{L}X\right).$$

There is a natural fibration

$$X \simeq E\mathbb{S}^1 \times \mathcal{L}X \to E\mathbb{S}^1 \underset{\mathbb{S}^1}{\times} \mathcal{L}X \xrightarrow{\pi} E\mathbb{S}^1 \underset{\mathbb{S}^1}{\times} \{*\} = B\mathbb{S}^1,$$

so $\overline{H}_*^{\mathbb{S}^1}(\mathcal{L}X) = \operatorname{Ker}(\pi_*)$.

Theorem 7.10. *If X is a closed oriented 1-connected manifold of dimension $d \geq 2$, then $\mathcal{U}\mathfrak{a}_X \in Ob\,(Ho\,(\mathbf{DGA}_k))$ has a natural derived Poisson structure of degree $2-d$.*

Proof. (Sketch) A theorem of Lambrechts–Stanley (2008) says that there is a finite dimensional cochain commutative differential graded algebra A over $\mathbb{Q}$ such that $C^*(X,\mathbb{Q}) \simeq A$, where $C^*(X,\mathbb{Q})$ is the differential graded algebra of singular cochains in X with the usual cup product. It comes with a cyclic pairing from Poincaré duality:

$$\langle -,- \rangle : A \otimes A \to A$$

of cohomological degree $n = -d$. "Cyclic" means $\langle a, bc\rangle = \pm \langle ca, b\rangle$ where $\pm$ is the Koszul sign.

Take $C = A^* = \operatorname{Hom}_{\mathbb{Q}}(A,\mathbb{Q})$. Since A is finite dimensional, this is a cocommutative coassociative differential graded coalgebra with coproduct dual to the product on A, such that $C_*(X,\mathbb{Q}) \simeq C$ equipped with cyclic pairing $C \otimes C \to C$ of homological degree $n = -d$. Then

$$\mathcal{U}\mathfrak{a}_X \cong \Omega(C) = (T_k C[-1], d) =: (R, d)$$

is the algebraic cobar construction on C, with d coming from $\Delta : C \to C \otimes C$ and d_C.

The construction with cyclic derivatives can be generalized to the graded free algebra $R = C[-1]$, depending on $\langle -,- \rangle$ on C, and compatible with d_R. This gives a natural Poisson bracket on $R_\natural$, making $\mathcal{U}\mathfrak{a}_X$ a derived Poisson algebra. □

Theorem 7.11. *Under Jone's isomorphism, $\overline{HC}_*(\mathcal{U}\mathfrak{a}_X) \cong \overline{H}_*^{\mathbb{S}^1}(\mathcal{L}X,\mathbb{Q})$, a graded Lie algebra structure on $\overline{HC}_*(\mathcal{U}\mathfrak{a}_X)$ corresponds exactly to the Chas-Sullivan (string topology) Lie algebra structure on $\overline{H}_*^{\mathbb{S}^1}(\mathcal{L}X,\mathbb{Q})$.*

Remark 7.6. The Chas-Sullivan bracket was originally defined geometrically in terms of transversal intersection product on chains on $\mathcal{L}X$.

The theorem gives a (more) algebraic way to define Chas-Sullivan structure.

Problem 7.1. There is a Hodge decomposition on $\overline{HC}_*(\mathcal{U}\mathfrak{a}_X)$ which gives

$$\overline{H}_*^{\mathbb{S}^1}(\mathcal{L}X,\mathbb{Q}) \cong \bigoplus_{p\geq 1} \overline{H}_*^{\mathbb{S}^1,(p-1)}(\mathcal{L}X,\mathbb{Q})$$

where $\overline{H}_*^{\mathbb{S}^1,(p-1)}(\mathcal{L}X,\mathbb{Q})$ are the eigenspaces of endomorphisms coming from $\mathbb{S}^1 \to \mathbb{S}^1, e^{i\varphi} \mapsto e^{in\varphi}$, $n = p-1$. Probe that the Chas-Sullivan bracket preserves Hodge decomposition.

Proof. See [BRZ]. ☐

7.8 Derived Functors

Let $(\mathcal{M},\mathcal{W}_{\mathcal{M}}),(\mathcal{N},\mathcal{W}_{\mathcal{N}})$ be homotopical categories,.

Definition 7.14. A functor $F:\mathcal{M}\to\mathcal{N}$ is *homotopical* if $F(\mathcal{W}_{\mathcal{M}})\subseteq\mathcal{W}_{\mathcal{N}}$.

By universal property of localization, it induces a commutative diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{N} \arrow[d, "\delta"] \\ \mathrm{Ho}(\mathcal{M}) \arrow[r, "\overline{F}"] & \mathrm{Ho}(\mathcal{N}) \end{tikzcd}$$

Note that F is homotopical if and only if δF is homotopical (if $\mathrm{Ho}(\mathcal{N})$ id viewed as a homotopical category with $\mathcal{W}_{\mathrm{Ho}(\mathcal{N})}=\mathrm{Iso}(\mathrm{Ho}(\mathcal{N}))$.

Example 7.21. Let $\mathcal{A},\mathcal{B}$ be abelian categories (e.g. $\mathcal{A}=\mathcal{B}=\mathbf{Mod}(\mathbb{Z})$) and $\mathcal{M}=\mathbf{Com}^+(\mathcal{A}),\mathcal{N}=\mathbf{Com}^+(\mathcal{B})$. Take an additive functor $F:\mathcal{A}\to\mathcal{B}$, we can extend it to $F_\bullet:\mathcal{M}\to\mathcal{N}$.

If we take weak equivalences to be chain homotopy equivalences ($f: X_\bullet \to Y_\bullet$ such that there exists $g: Y_\bullet \to X_\bullet$, $fg\simeq \mathrm{Id}_Y$ and $gf\simeq\mathrm{Id}_X$), then $F_\bullet$ is homotopical.

If we choose weak equivalences to be quasi-isomorphisms, then $F_\bullet$ is usually not homotopical (unless it is exact).

Example 7.22. Let $\mathcal{M}$ be a homotopical category and $\mathcal{C}$ a small category, then $\mathcal{M}^{\mathcal{C}}$is a homotopical category with objectwise weak equivalences,

$$\mathcal{W}_{\mathcal{M}^{\mathcal{C}}} \coloneqq \{\alpha: F\Rightarrow F' | \forall c\in Ob(\mathcal{C}),(\alpha_c: F(c)\to F'(c))\in\mathcal{W}_{\mathcal{M}}\}.$$

In general, $\mathrm{colim}_{\mathcal{C}}:\mathcal{M}^{\mathcal{C}}\to\mathcal{M}$ is not homotopical.

Take $\mathcal{M}=\mathbf{Top}_1$ and $\mathcal{C}=\{\bullet\leftarrow\bullet\to\bullet\}$,

$$\begin{tikzcd} F: \arrow[d, Rightarrow, "\alpha"'] & D^n \arrow[d, "\simeq"'] & \mathbb{S}^{n-1} \arrow[l, hook'] \arrow[r, hook] \arrow[d, equal] & D^n \arrow[d, "\simeq"] \\ F': & \bullet & \mathbb{S}^{n-1} \arrow[l] \arrow[r] & \bullet \end{tikzcd}$$

then

$$\mathrm{colim}_{\mathcal{C}}(\alpha) : \mathrm{colim}_{\mathcal{C}}(F) \cong \mathbb{S}^n \xrightarrow{\not\simeq} \mathrm{colim}_{\mathcal{C}}(F') \cong \bullet$$

is not a weak equivalence.

It is a philosophy of homological algebra to replace non-homotopical functors with universal homotopical approximation.

Definition 7.15. [Quillen] A *total left derived functor* of $F : \mathcal{M} \to \mathcal{N}$ is defined by the right Kan extension

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{N} \arrow[r, "\delta"] & \mathrm{Ho}(\mathcal{N}) \\
\mathrm{Ho}(\mathcal{M}) \arrow[ur, Rightarrow, "\eta"] \arrow[urr, bend right, "\mathbb{L}F := R_\gamma(\delta F)"'] & &
\end{tikzcd}$$

which comes together with a natural transformation $\eta : \mathbb{L}F \circ \gamma \Rightarrow \delta F$ (left approximation).

Definition 7.16. [DHKS04] A *left derived functor* of $F : \mathcal{M} \to \mathcal{N}$ is a functor $\mathbf{L}F : \mathcal{M} \to \mathrm{Ho}(\mathcal{N})$ together with composition morphism $\eta : \mathbf{L}F \Rightarrow \delta F$ such that

1. $\mathbf{L}F$ is homotopical.

2. η is terminal among all homotopical functors $G : \mathcal{M} \to \mathrm{Ho}(\mathcal{N})$ with $\tilde{\eta} : G \Rightarrow \delta F$.

$$\begin{tikzcd}
G \arrow[rr, Rightarrow, "\tilde{\eta}"] \arrow[dr, Rightarrow, "\exists!"'] & & \delta F \\
& \mathbf{L}F \arrow[ur, Rightarrow, "\eta"'] &
\end{tikzcd} \tag{10}$$

Note by universal property of localization $\gamma : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$, giving $\mathbf{L}F$ is equivalent to giving $\mathbb{L}F$, both are defined uniquely up to unique isomorphism.

It's convenient-although not always possible-to lift $\mathbb{L}F$ to the level of homotopical categories.

Definition 7.17. A *pointwise left derived functor* of $F : \mathcal{M} \to \mathcal{N}$ is a functor $\mathcal{L}F : \mathcal{M} \to \mathcal{N}$ given together with $\mathcal{L}F \Rightarrow F$ such that $\eta = \mathbb{L}F := \delta\mathcal{L}F \Rightarrow \delta F$ is a derived functor in the sense of Definition 7.16.

$$\begin{tikzcd}
\mathcal{M} \arrow[r, bend left, "\mathcal{L}F"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{N} \arrow[r, "\delta"] & \mathrm{Ho}(\mathcal{N})
\end{tikzcd}$$

Note $\mathcal{L}F$ may or may not exist, and if it does, it is defined only up to homotopy.

Derived Functors Via Deformations

Question. How to construct derived functors?

Idea. If $F : \mathcal{M} \to \mathcal{N}$ is not homotopical, we can restrict F to a subcategory $\mathcal{M}_Q$ of $\mathcal{M}$ consisting of "good" objects, "adjusted" for F in the sense that $F : \mathcal{M}_Q \hookrightarrow \mathcal{M} \to \mathcal{N}$ becomes homotopical.

Definition 7.18. Let $\mathcal{M}$ be a homotopical category. A *left deformation* of $\mathcal{M}$ is a pair (Q, q) where $Q : \mathcal{M} \to \mathcal{M}$ is an endofunctor on $\mathcal{M}$ and $q : Q \Rightarrow \mathrm{Id}_{\mathcal{M}}$ which is a natural weak equivalence, i.e. for any $X \in Ob(\mathcal{M})$, $(q_X : QX \to X) \in \mathcal{W}_{\mathcal{M}}$.

Note that Q is automatically homotopical. Indeed, for any $f : X \to Y$ in $\mathcal{M}$, there is a commutative diagram

$$\begin{tikzcd} QX \arrow[r, "Qf"] \arrow[d, "q_X"', "\wr"] & QY \arrow[d, "q_Y", "\wr"'] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

If $f \in \mathcal{W}_{\mathcal{M}}$, then $QF \in \mathcal{W}_{\mathcal{M}}$ by 2-of-3.

Definition 7.19. Given a left deformation (Q, q) of $\mathcal{M}$, call any full subcategory $\mathcal{M}_Q \subseteq \mathcal{M}$ such that $\mathrm{Im}(Q) \subseteq \mathcal{M}_Q$ to be a *left deformation retract* of $\mathcal{M}$ with respect to (Q, q).

Lemma 7.3. *If $\mathcal{M}_Q$ is any left deformation of $\mathcal{M}$, then $i : \mathcal{M}_Q \to \mathcal{M}$ induces an equivalences of categories*

$$i : Ho(\mathcal{M}_Q) \xrightarrow{\simeq} Ho(\mathcal{M}).$$

Proof. We have two opposite functors

$$i : \mathcal{M}_Q \rightleftarrows \mathcal{M} : Q$$

which are both homotopical, and hence induce

$$i : \mathrm{Ho}(\mathcal{M}_Q) \rightleftarrows \mathrm{Ho}(\mathcal{M}) : Q$$

which are inverse equivalences. □

Example 7.23. If $\mathcal{M}$ is a cofibrantly generated model category, then Quillen's small object argument implies that cofibration/trivial fibration factorization (MC5) can be made functorial: there exists $Q : \mathcal{M} \to \mathcal{M}$ with $q : Q \Rightarrow \mathrm{Id}_{\mathcal{M}}$ for any $X \in Ob(\mathcal{M})$. We have the functorial cofibrant replacement

$$\emptyset \hookrightarrow QX \underset{\sim}{\overset{q_X}{\twoheadrightarrow}} X.$$

Hence (Q, q) is a left deformation of $\mathcal{M}$. $\mathcal{M}_Q$ is the full subcategory of cofibrant objects.

Definition 7.20. Given $F : \mathcal{M} \to \mathcal{N}$, a *left F-deformation* is a left deformation (Q, q) such that $F|_{\mathcal{M}_Q} : \mathcal{M}_Q \hookrightarrow \mathcal{M} \to \mathcal{N}$ is homotopical. If such a left deformation exists for F, F is called *left deformable*.

Exercise 7.3. A left deformation (Q, q) is an F-deformation if and only if

1. FQ is homotopical,
2. $FqQ : FQ^2 \Rightarrow FQ$ is a natural equivalence of functors.

Proof. $\square$

Exercise 7.4. Any left deformable F has a maximal subcategory $\mathcal{M}_Q \subseteq \mathcal{M}$ such that $F|_{\mathcal{M}_Q}$ is homotopical.

Proof. $\square$

Theorem 7.12. *[DHKS04] If $F : \mathcal{M} \to \mathcal{N}$ admits a left deformation (Q, q) then $\mathbf{L}F \coloneqq \delta F Q : \mathcal{M} \xrightarrow{Q} \mathcal{M} \xrightarrow{F} \mathcal{N} \xrightarrow{\delta} Ho\,(\mathcal{N})$ together with $\eta \coloneqq \delta F q : \mathbb{L}F \Rightarrow \delta F$ define the left derived functor of F.*

Proof. We need to show 2 things.

1. $\mathbf{L}F$ is homotopical.

   $\mathbf{L}F$ can be factored as

$$\begin{tikzcd} \mathcal{M} \arrow[r, "Q"] \arrow[dr, "Q"'] & \mathcal{M} \arrow[r, "F"] & \mathcal{N} \arrow[r, "\gamma"] & \mathrm{Ho}\,(\mathcal{N}) \\ & \mathcal{M}_Q \arrow[u, hook, "i"] \arrow[ur, "F|_{\mathcal{M}_Q}"'] & & \end{tikzcd}$$

   where each component is homotopical, so $\mathbf{L}F$ is homotopical.

2. For any $\tilde{\eta} : G \Rightarrow \delta F$ with G homotopical, the unique factorization 10 exists.

   Since G is homotopical and q is a natural weak equivalence, $Gq : GQ \Rightarrow G$ is an isomorphism of functors. Indeed, for any $X \in Ob\,(\mathcal{M})$, $\left(q_X : QX \xrightarrow{\simeq} X\right) \in \mathcal{W}_{\mathcal{M}}$, and since G is homotopical, $\left(Gq_X : GQX \xrightarrow{\simeq} GX\right) \in \mathcal{W}_{\mathrm{Ho}(\mathcal{N})} = \mathrm{Iso}\,(\mathrm{Ho}\,(\mathcal{N}))$, so $Gq : GQ \Rightarrow G$ is an isomorphism of functors. Thus there exists $(Gq)^{-1} : G \Rightarrow GQ$, and the commutative diagram

$$\begin{tikzcd} G \arrow[r, Rightarrow, "\tilde{\eta}"] & \delta F \\ GQ \arrow[u, Rightarrow, "Gq", "\cong"'] \arrow[r, Rightarrow, "\tilde{\eta}Q"] & \delta F Q = \mathbf{L}F \arrow[u, Rightarrow, "\eta = \delta F q"'] \end{tikzcd}$$

give us the factorization of

$$\tilde{\eta} : G \xRightarrow{(Gq)^{-1}} GQ \xRightarrow{\tilde{\eta}Q} \delta FQ = \mathbf{L}F \xRightarrow{\eta} \delta F$$

as $\tilde{\eta} = \eta \circ \zeta$ with $\zeta \coloneqq \tilde{\eta}Q \circ (Gq)^{-1}$.

It remains to show the uniqueness of ζ. Take any factorization $\tilde{\zeta}$,

$$\begin{tikzcd} G \arrow[r, Rightarrow, "\tilde{\zeta}"] \arrow[dr, Rightarrow, "\tilde{\eta}"'] & \delta FQ \arrow[d, Rightarrow, "\eta=\delta Fq"] \\ & \delta F \end{tikzcd}$$

Note this yields

$$\begin{tikzcd} GQ \arrow[r, Rightarrow, "\tilde{\zeta}Q"] \arrow[dr, Rightarrow, "\tilde{\eta}Q"'] & \delta FQ^2 \arrow[d, Rightarrow, "\eta Q=\delta FqQ"] \\ & \delta FQ \end{tikzcd}$$

Since $F|_{\mathcal{M}_Q}$ is homotopical, ηQ is a natural weak equivalence and hence isomorphism of functors (because $\mathrm{Ho}(\mathcal{N})$ is minimal). This means that ηQ is invertible, so $\tilde{\zeta}Q$ is uniquely determined by $\tilde{\eta}$. By naturality, we have

$$\begin{tikzcd} GQ \arrow[r, Rightarrow, "\tilde{\zeta}Q"] \arrow[d, Rightarrow, "Gq"', "\cong"] & \delta FQ^2 \arrow[d, Rightarrow, "\eta Q", "\cong"'] \\ G \arrow[r, Rightarrow, "\tilde{\zeta}"] & \delta FQ \end{tikzcd}$$

in which both vertical arrows are isomorphisms, so $\tilde{\zeta}$ is uniquely determined by $\tilde{\zeta}Q$ and hence $\tilde{\eta}$.

$\square$

Remark 7.7. The argument shows that $\mathbf{L}F \coloneqq FQ$ is a pointwise left derived functor.

Consider the 2-category $\mathbf{HomCat}^L$ with

Objects(0-cells): 4-tuples $(\mathcal{M}, \mathcal{M}_Q, Q, q)$

HorizontalMorphisms(1-cells): deformable functors $F : \mathcal{M} \to \mathcal{M}'$.

VerticalMoprhisms(2-cells): any natural transformation

$$\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{M}' \end{tikzcd} .$$

There is a pseudo-functor

$$\begin{array}{cccc} \mathbb{L}: & \mathbf{HomCat}^L & \longrightarrow & \mathbf{Cat} \\ & \mathcal{M} & \longmapsto & \mathrm{Ho}\,(\mathcal{M}) \\ & (F:\mathcal{M}\to\mathcal{M}') & \longmapsto & (\mathbb{L}F:\mathrm{Ho}\,(\mathcal{M})\to\mathrm{Ho}\,(\mathcal{M}')) \\ & (\alpha:F\Rightarrow F') & & (\mathbb{L}\alpha:\mathbb{L}F\Rightarrow\mathbb{L}F') \end{array}$$

Proof is similar to Hovey's Theorem 1.3.7.9 for left Quillen functors in case of model categories.

The important point is, if $F:\mathcal{M}\to\mathcal{M}', G:F:\mathcal{M}'\to\mathcal{M}''$ such that

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] & \mathcal{M}' \\ \mathcal{M}_Q \arrow[u, hook] \arrow[r] & \mathcal{M}'_{Q'} \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{M}' \arrow[r, "G"] & \mathcal{M}'' \\ \mathcal{M}'_{Q'} \arrow[u, hook] \arrow[r] & \mathcal{M}''_{Q''} \arrow[u, hook] \end{tikzcd}$$

i.e. F maps $\mathcal{M}_Q$ to $\mathcal{M}'_{Q'}$ on which G is homotopical. Then $\mathbb{L}F := FQ$ and $\mathbb{L}G := GQ'$ compose to

$$\mathbb{L}G\circ\mathbb{L}F = (GQ')\circ(FQ) \xRightarrow[\simeq]{Gq'FQ} GFQ =: \mathbb{L}\,(GF)\,.$$

So $(\mathbb{L}G)\circ(\mathbb{L}F)\cong\mathbb{L}\,(GF)$, but this is in general, not true.

Proposition 7.3. *If F is left deformable, then $\mathbb{L}F : Ho\,(\mathcal{M}) \to Ho\,(\mathcal{N})$ is pointwise right Kan extension,*

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{N} \arrow[r, "\delta"] & Ho\,(\mathcal{N}) \\ Ho\,(\mathcal{M}) \arrow[urr, "\mathbb{L}F:=R_\gamma F"'] & & \end{tikzcd}$$

i.e. $\mathbb{L}F$ can be computed as a limit in $Ho\,(\mathcal{N})$

$$R_\gamma\,(\delta F) \cong \lim_{\gamma X\backslash\gamma}\left(\gamma X\backslash\gamma \xrightarrow{U} \mathcal{M} \xrightarrow{\delta F} Ho\,(\mathcal{N})\right),$$

even though $Ho\,(\mathcal{N})$ is not complete.

We can consider right deformation $R:\mathcal{M}\to\mathcal{M}$ and $r:\mathrm{Id}_{\mathcal{M}}\Rightarrow R$ such that

Definition 7.21. $F:\mathcal{M} \rightleftarrows \mathcal{N}:G$ is *deformable* if F is left deformable and G is right deformable.

Theorem 7.13. *[DHKS04] If $F:\mathcal{M} \rightleftarrows \mathcal{N}:G$ is* deformable *, then $\mathbb{L}F$ and $\mathbb{R}G$ exist and the adjunction* *descends to homotopical categories*

$$\mathbb{L}F : Ho\,(\mathcal{M}) \rightleftarrows Ho\,(\mathcal{N}) : \mathbb{R}G\ .$$

Example 7.24. Quillen pairs between model categories.

Example 7.25. Representation functor $(-)_G : \mathbf{sGr} \to \mathbf{sCommAlg}_k$ is not left Quillen but left deformable. We will study this example in detail in the next part.

Examples of Derived Functors

Classical derived functors

Let A be an associative unital ring or k-algebra. Let $\mathcal{M} = \mathbf{Com}^+\,(A)$ be the category of chain complexes of left A-modules and $\mathcal{W}_{\mathcal{M}}$ be the class of all quasi-isomorphisms in $\mathcal{M}$. $(\mathcal{M}, \mathcal{W}_{\mathcal{M}})$ is a homotopical category.

We want to construct a left deformation $Q : \mathcal{M} \to \mathcal{M}$ with $q : Q \Rightarrow \mathrm{Id}_{\mathcal{M}}$ which is adjusted to any additive functor on $\mathcal{M}$. This is given by the classical (2-sided) bar construction. We recall (Quillen's approach to) this construction.

Let's start with a map of A-bimodules $\varepsilon : E \to A$ and define a differential graded algebra as follows. Take $T_A E = \bigoplus_{n \geq 0} \underbrace{\left(E \bigotimes_A E \bigotimes_A \cdots \bigotimes_A E \right)}_{n}$ with differential $d : T_A E \to T_A E$ of degree -1 extending ε, for any $(z_1, \cdots, z_n) \in E \bigotimes_A E \bigotimes_A \cdots \bigotimes_A E$,

$$d\,(z_1, \cdots, z_n) = \sum_{i=1}^{n} (-1)^{i-1}\,(z_1, \cdots, z_{i-1}\varepsilon\,(z_i)\,, z_{i+1}, \cdots, z_n)$$

where we use the natural isomorphism

$$\begin{array}{ccc} E \bigotimes_A A \bigotimes_A E & \cong & E \bigotimes_A E \\ (z_1, a, z_2) & \mapsto & (z_1 z, z_2) = (z_1, a z_2) \end{array}$$

Then

$$\begin{aligned}
d^2(z_1,\cdots,z_n) &= \sum_{i=1}^{n}(-1)^{i-1}d(z_1,\cdots,z_{i-1}\varepsilon(z_i),z_{i+1},\cdots,z_n)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{i-2}(-1)^{j-1}(-1)^{i-1}(z_1,\cdots,z_{j-1}\varepsilon(z_j),z_{j+1},\cdots,z_{i-1}\varepsilon(z_i),z_{i+1},\cdots,z_n)+\\
&\quad \sum_{i=2}^{n}(-1)^{i-2}(-1)^{i-1}(z_1,\cdots,z_{i-2}\varepsilon(z_{i-1})\varepsilon(z_i),z_{i+1}\cdots,z_n)+\\
&\quad \sum_{i=1}^{n-1}(-1)^{i-1}(-1)^{i-1}(z_1,\cdots,z_{i-1}\varepsilon(z_i)\varepsilon(z_{i+1}),z_{i+2}\cdots,z_n)+\\
&\quad \sum_{i=1}^{n}\sum_{j=i+1}^{n}(-1)^{j-2}(-1)^{i-1}(z_1,\cdots,z_{i-1}\varepsilon(z_i),z_{i+1},\cdots,z_{j-1}\varepsilon(z_j),z_{j+1},\cdots,z_n)\\
&= 0
\end{aligned}$$

so (T_AE,d) is a unital differential graded algebra.

Lemma 7.4. *(Quillen) If ε is surjective, then (T_AE,d) is acyclic, $H_*(T_aE,d)=0$.*

Proof. Fix $z\in E\subseteq T_AE$ of degree 1 such that $dz=\varepsilon(z)=1$, then for any $a\in T_AE$,

$$d(za)=d(z)\cdot a+(-1)^{|z|}z\cdot da=a-z\cdot da.$$

So $a=d(za)+z\cdot da$, hence the map $a\mapsto za$ is a contracting homotopy on T_AE compatible with its right A-module structure.

$$\begin{tikzcd}
 & a \arrow[r, mapsto] \arrow[dl, mapsto] \arrow[d, mapsto, "\mathrm{Id}"] & da \arrow[dl, mapsto] \\
za \arrow[r, mapsto] & a=d(za)+z\cdot da &
\end{tikzcd}$$

Similarly, $d(az)=d(a)\cdot z+(-1)^{|a|}a$, so $a\mapsto(-1)^{|a|}az$ is a contracting homotopy on T_AE compatible with its left A-module structure.

Thus the homology of (T_AE,d) is trivial. ☐

Corollary 7.2. *Any projective A-bimodule E with surjective A-bimodule map $\varepsilon:E\to A$ gives a projective resolution $(T_AE)^+\xrightarrow{\varepsilon}A$ where $(T_AE)^+$ is the positive degree part of T_AE.*

Example 7.26. $E=A\otimes A$, $\varepsilon=m:A\otimes A\to A$ is the multiplication on A. Then

$$BA_\bullet=(T_A(A\otimes A),d=b)$$

is the classical bar construction, with

$$BA_0 = A$$
$$BA_n = A^{\otimes(n+1)}$$

and differential b given by

$$b(a_0, \cdots, a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0, \cdots, a_{i-1}, a_i a_{i+1}, \cdots, a_n).$$

Remark 7.8. Quillen's main observation was to build cyclic homology theory for algebras, we can start with projective resolution of A of the form $(T_A E, d)$ associated with $\varepsilon : E \twoheadrightarrow A$.

Remark 7.9. Notice that $(T_a E, d) \cong \operatorname{hocolim}\left\{ A \xleftarrow{(1,\varepsilon)} T_A E \xrightarrow{(1,0)} A \right\}$ where $(1, \varepsilon)$ is the homomorphism coming from the universal property of the tensor algebra. We think of ε as a deformation parameter.

To give a map $f : T_A E \to B$ of algebras is equivalent to give

1. $f_0 : A \to B$ a map of algebras.

2. $f_1 : E \to B$ a map of A-bimodules, where B is viewed as an A-bimodule via

$$a_1 \cdot b \cdot a_2 = f(a_1)\, b f(a_2).$$

Then we have

$$\begin{array}{rccc} T_A : & \mathbf{Bimod}(A) & \rightleftharpoons & A \downarrow \mathbf{Alg} : U \\ & {}_f B_f & \leftarrow\!\shortmid & \left(A \xrightarrow{f} B\right) \\ & E & \mapsto & (A \to T_A E) \end{array}$$

Example 7.27. Let $A = \mathcal{O}(X) \cong k[q]$ where $X = \mathbb{A}^1_k$, then $\mathcal{O}(T^* X) \cong k[p, q]$.

$$\mathcal{O}(T^* X) = \operatorname{hocolim}\left\{ A \xleftarrow{(1,0)} T_A E \xrightarrow{(1,0)} A \right\}$$

for $E = \mathbb{R}\mathrm{Hom}_{A^e}(A, A^e) = A^!$. The homomorphism $\varepsilon : E \to A$ is in $\mathcal{D}^b(A^e)$,

$$(\varepsilon : E \to A) \in \mathrm{Hom}_{\mathcal{D}^b(A^e)}\left(A^!, A\right) \cong H^0\left(\mathbb{R}\mathrm{Hom}_{A^e}(A, A^e), A\right) = H_0\left(A^{!!} \overset{\mathbb{L}}{\underset{A^e}{\otimes}} A\right) \cong HH_0(A) \cong A.$$

So the deformation parameters ε are exactly elements of the 0-th Hochschild homology, which in this case is just A itself.

We have deformation

$$\operatorname{hocolim}\left\{ A \xleftarrow{(1,\hbar)} T_A A^! \xrightarrow{(1,0)} A \right\} \cong \mathcal{D}_\hbar(X).$$

Claim 7.1. The functor $Q := BA \otimes_A - : \mathbf{Com}^+(A) \to \mathbf{Com}^+(A)$ with $q = \left\{ q_M : BA \otimes_A M \xrightarrow{\varepsilon_*} A \otimes_A M \cong M \right\}_{M \in Ob\left(\mathbf{Com}^+(A\right.}$ is a left deformation of $\mathcal{M} = \mathbf{Com}^+(A)$, adjusted to any additive functor.

Proof. This is essentially an classical result from homological algebra.

1. q_M is a natural quasi-isomorphism because $T_A (A \otimes A) \otimes_A M$ is acyclic.

2. $QM = BA \otimes_A M$ is a complex of free left A-modules and hence projective. Take $\mathcal{M}_Q$ to be the chain complexes with projective terms, then $\mathcal{M}_Q \supseteq \mathrm{Im}(Q)$.

3. Given another additive functor $F : \mathbf{Mod}(A) \to \mathbf{Mod}(A)$, the additive functor

$$F_\bullet : \mathbf{Com}^+(A) \to \mathbf{Com}^+(A)$$

preserves homotopy equivalences, so since any quasi-isomorphism between (nonnegatively graded) projective complexes is a homotopy equivalence, $F_\bullet|_{\mathcal{M}_Q}$ is a homotopical functor.

□

Simplicial groups and spaces

Twisted Cartesian products and principal bundles

Let $\mathbf{sGr} = \mathbf{Fun}(\Delta^{op}, \mathbf{Gr})$ be the category of simplicial groups.

Let $G_* = \{G_n\}_{n \geq 0} \in Ob(\mathbf{sGr})$ and $X_* \in Ob(\mathbf{sSet})$.

Definition 7.22. A *twisting function* $\tau : X_* \to G_{*-1}$ is a family of maps $\{\tau_n : X_n \to G_{n-1}\}_{n \geq 1}$ such that

$$\begin{array}{rcll} d_0(\tau(x)) & = & \tau(d_0 x)^{-1} \tau(d_1 x) & \\ d_i(\tau(x)) & = & \tau(d_{i+1} x) & i \geq 1 \\ s_j(\tau(x)) & = & \tau(s_{j+1} x) & j \geq 0 \\ \tau(s_0(x)) & = & 1_{G_n} & \forall x \in G_n \end{array}$$

Definition 7.23. A *(principal) twisted Cartesian product* with fibre G_* and base X_*, and twisting function $\tau : X_* \to G_{*-1}$ is a simplicial set $E_* = G_* \times_\tau X_*$ with

$$E_n := G_n \times X_n, n \geq 0$$

and

$$\begin{aligned} d_i(g,x) &= \begin{cases} (\tau(x)\cdot d_0 g, d_0 x) & i=0, \\ (d_i g, d_i x) & i>0. \end{cases} \\ s_j(g,x) &= (s_j g, s_j x) \qquad j \geq 0. \end{aligned}$$

Proposition 7.4. *Any principal G_*-fibration $p_* : E_* \to X_*$ with right G_* action on E_* with local cross section $\sigma_* : X_* \to E_*$ (i.e. $\sigma_n : X_n \to E_n$ such that $p_n\sigma_n = Id_{X_n}$ and $d_i\sigma = \sigma d_i, \forall i > 0$, $s_j\sigma = \sigma s_j, \forall j \geq 0$) can be identified with $G \times_\tau X \to X$ where $\tau : X_* \to G_{*-1}$ is determined by $d_0\sigma(x) = \sigma(d_0 x)\cdot\tau(x)$.*

The classifying space of a simplicial group $\overline{\mathcal{W}}$

Given $G_* \in Ob(\mathbf{sGr})$, define a reduced simplicial set $\overline{\mathcal{W}}(G_*)$ by

$$\overline{\mathcal{W}}_0(G) \coloneqq \{*\}, \overline{\mathcal{W}}_n(G) \coloneqq G_{n-1}\times G_{n-2}\times\cdots\times G_n, n \geq 0$$

with

$$\begin{aligned} s_0 : \quad \overline{\mathcal{W}}_0(G) &\to \overline{\mathcal{W}}_1(G) \\ * &\mapsto 1_{G_0} \\ d_0 = d_1 : \quad \overline{\mathcal{W}}_1(G) &\to \overline{\mathcal{W}}_0(G) \\ g &\mapsto * \end{aligned}$$

and for $n \geq 1$,

$$\begin{aligned} d_0(g_{n-1},\cdots,g_0) &= (g_{n-2},\cdots,g_0) \\ d_{i+1}(g_{n-1},\cdots,g_0) &= (d_i g_{n-1},\cdots,d_1 g_{n-i}, g_{n-i-2}\cdot d_0 g_{n-i-1}, g_{n-i-3},\cdots,g_0) \\ s_0(g_{n-1},\cdots,g_0) &= (1, g_{n-1},\cdots,g_0) \\ s_{j+1}(g_{n-1},\cdots,g_0) &= (s_j g_{n-2},\cdots s_0 g_{n-j-1}, 1, g_{n-i-2},\cdots,g_0) \end{aligned}$$

This is a simplicial set with a twisting function

$$\begin{aligned} \tau_n(G) : \quad \overline{\mathcal{W}}_n(G) &\to G_{n-1} \\ (g_{n-1},\cdots,g_0) &\mapsto g_{n-1} \end{aligned}$$

Lemma 7.5. *$\tau(G)$ is a universal twisting function in the sense that any principal twisted product $G\times_\tau X$ can be induced from $G_* \times_{\tau(G)} X_*$ by a unique classifying map $X_* \mapsto \overline{\mathcal{W}}(G_*)$ given by*

$$x \in X_n \mapsto \left(\tau(x), \tau(d_0 x), \cdots, \tau\left(d_0^{n-1}x\right)\right) \in \overline{\mathcal{W}}_n(G_*).$$

Example 7.28. If $G_* = \{G_n\}_{n\geq 0}$ is a discrete simplicial group, $\overline{\mathcal{W}}(G) = B_*G$ the simplicial nerve of G, then

$$\begin{array}{ccc} G \times_{\tau(G)} \overline{\mathcal{W}}(G) & \cong & E_*G \\ G^{n+1} & \longleftrightarrow & E_n(G) \\ (g_0, g_0g_1, \cdots, g_0\cdots g_n) & \longleftrightarrow & (g_0, \cdots, g_n) \end{array}$$

The Kan loop group of simplicial sets Conversely, given $X_* \in Ob(\mathbf{sSet}_0)$ a reduced simplicial set, define the Kan loop group of X, $\mathbb{G}(X)_* \in Ob(\mathbf{sGr})$ by

$$\mathbb{G}_n(X) \coloneqq \mathbb{F}\langle X_{n+1}\rangle / (s_0(x) = 1, \forall x \in X_n)$$

induced by

$$B_n = X_{n+1} \backslash s_0(X_n) \hookrightarrow X_{n+1}$$

(but $\{B_n\}$ do not form a simplicial set), with

$$\begin{array}{ccll} d_i^{\mathbb{G}}(x) & = & \begin{cases} d_1(x)\, d_0(x)^{-1} & i = 0, \\ d_{i+1}x & i > 0. \end{cases} \\ s_j^{\mathbb{G}}(x) & = & s_{j+1}x & j \geq 0. \end{array}$$

Define

$$\tau(X) : X_* \to \mathbb{G}(X)_{*-1}$$

by

$$\tau_n(X) : X_n \hookrightarrow \mathbb{F}\langle X_n\rangle \twoheadrightarrow \mathbb{G}X_{n-1}.$$

Given $X_* \in Ob(\mathbf{sSet})$ and $G_* \in Ob(\mathbf{sGr})$ define

$$\mathrm{Tw}(X_*, G_*) \coloneqq \{\text{twisting functions } \tau : X_* \to G_*\}.$$

Theorem 7.14. *There are natural bijections*

$$\begin{array}{ccccc} Hom_{\mathbf{sGr}}(\mathbb{G}X_*, G_*) & \xrightarrow{\sim} & Tw(X_*, G_*) & \xleftarrow{\sim} & Hom_{\mathbf{sSet}}\left(X_*, \overline{\mathcal{W}}G_*\right) \\ f & \mapsto & f \circ \tau(X) & & \\ & & \tau(G) \circ g & \leftarrow & g \end{array}$$

Hence we have adjunction

$$\mathbb{G} : \mathbf{sSet}_0 \rightleftarrows \mathbf{sGr} : \overline{\mathcal{W}}\ .$$

Theorem 7.14, proposition 7.4 and lemma 7.5 implies

Corollary 7.3. *For fixed $G_* \in Ob\,(\mathbf{sGr})\,, X \in Ob\,(\mathbf{sSet})$, there is a natural bijection between the set of twisting function $Tw\,(X_*, G_*)$ and the isomorphism classes of pairs (E_*, G_*) where E_* is a principal G_*-bundle over X_* with local section $\sigma : X_* \to E_*$. The bijection is given by*

$$\tau \mapsto (G \times_\tau X, \sigma)$$

where $d_0\sigma_n\,(x) = \sigma_{n-1}\,(d_0 x)\,\tau\,(x)$.

Our main theorem is the following.

Theorem 7.15. *(Kan)*

1. *For $X_* \in Ob\,(\mathbf{sSet}_0)$ a reduced simplicial set and $G_* \in Ob\,(\mathbf{sGr})$, there are weak homotopy equivalences of spaces*

$$\begin{aligned} |\mathbb{G}\,(X_*)| &\simeq \Omega\,|X_*| \\ \left|\overline{\mathcal{W}}\,(G_*)\right| &\simeq B\,|G_*| \end{aligned}$$

   This shows that the homotopy type of $|\mathbb{G}\,(X_*)|$ is the loop space of $|X|$, and the homotopy type of $\left|\overline{\mathcal{W}}\,(G_*)\right|$ is the classifying space of $|G_*|$, which is the reason for the name of the two functors.

2. *The adjoint functors $\left(\mathbb{G}, \overline{\mathcal{W}}\right)$ give Quillen pair of model categories*

$$\mathbb{G} : \mathbf{sSet}_0 \rightleftarrows \mathbf{sGr} : \overline{\mathcal{W}}$$

   and

$$Ho\,(\mathbf{sSet}_0) \cong Ho\,(\mathbf{sGr})\,.$$

Relation to spaces

Recall the singular complex

$$\begin{aligned} \mathcal{S} : \quad \mathbf{Top} &\longrightarrow \mathbf{sSet} \\ X &\longmapsto \mathcal{S}\,(X)_* = \{\mathrm{Hom}_{\mathbf{Top}}\,(\Delta^n, X)\}_{n \geq 0} \end{aligned}$$

and the Eilenberg subcomplex

$$\begin{aligned} \mathcal{ES}: \quad \mathbf{Top}_* &\longrightarrow \mathbf{sSet} \\ (X,*) &\longmapsto \mathcal{ES}(X)_* = \{f \in \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, X) \,|\, f(e_i) = *, \forall 0 \le i \le n\}_{n\ge 0} \end{aligned}$$

Lemma 7.6. *(Eilenberg) If $(X,*)$ is connected, then $\mathcal{ES}(X)_* \hookrightarrow \mathcal{S}(X)_*$ is a weak equivalence of simplicial sets, i.e.*

$$|\mathcal{ES}(X)_*| \simeq |\mathcal{S}(X)_*| \simeq X.$$

Corollary 7.4. *Let $\mathbf{Top}_{0,*}$ be the category of pointed connected spaces, then*

$$|-| : \mathbf{sSet}_0 \rightleftarrows \mathbf{Top}_{0,*} : \mathcal{ES}$$

is an adjoint pair which is Quillen equivalence,

$$Ho(\mathbf{sSet}_0) \cong Ho\left(\mathbf{Top}_{0,*}\right).$$

A consequence is

$$\begin{array}{ccccc} \mathrm{Ho}(\mathbf{sGr}) & \cong & \mathrm{Ho}(\mathbf{sSet}_0) & \cong & \mathrm{Ho}\left(\mathbf{Top}_{0,*}\right) \\ \mathbb{G}(\mathcal{ES}(X)_*) & \leftarrow\!\shortmid & \mathcal{ES}(X)_* & \leftarrow\!\shortmid & X \end{array}$$

For reduced CW complexes, we can construct much smaller simplicial group models.

7.9 Free Diagrams

Let $\mathcal{I}$ be a small (indexing) category, $\mathcal{C}$ be any category with all small colimits (hence products).

$\mathcal{C}^{\mathcal{I}} = \mathbf{Fun}(\mathcal{I}, \mathcal{C})$ is the category of $\mathcal{I}$-diagrams in $\mathcal{C}$, which can be thought of as $\mathcal{I}$-modules.

Question. What are free $\mathcal{I}$-diagrams (i.e. analogue if free modules)?

Let $\mathcal{I}^S$ be the category $\mathcal{I}$ "made discrete", with

Objects: $Ob\left(\mathcal{I}^S\right) = Ob(\mathcal{I})$.

Morphisms: $\mathrm{Hom}_{\mathcal{I}^S}(i,j) = \begin{cases} \mathrm{Id}_i & i = j, \\ \emptyset & i \neq j. \end{cases}$

Definition 7.24. An $\mathcal{I}$-diagram $X : \mathcal{I} \to \mathcal{C}$ is *free* if it is the left Kan extension

$$\begin{tikzcd} \mathcal{I}^S \arrow[r, "Y"] \arrow[d, hook, "\text{in}"'] & \mathcal{C} \\ \mathcal{I} \arrow[ur, "\mathcal{L}_{\text{in}} Y \cong X"'] & \end{tikzcd}$$

i.e. for any $i \in Ob(\mathcal{I})$,

$$X(i) \cong \operatorname{colim}_{(f:j\to i)\in Ob(\mathcal{I}/i)} Y(j) \cong \coprod_{f:j\to i} Y(j)$$

the coproduct of $Y(j)$ in $\mathcal{C}$ indexed by $f : j \to i$ in $\mathcal{I}$.

Example 7.29. Let $\mathcal{C} = \mathbf{Set}$ and $\mathcal{I}$ be any small category. $X : \mathcal{I} \to \mathbf{Set}$ is free if there exists a sequence of objects $\mathscr{S} \subseteq Ob(\mathcal{I})$ such that

$$X \cong \coprod_{s\in\mathscr{S}} h^s$$

where $h^s := \operatorname{Hom}_{\mathcal{I}}(s, -) : \mathcal{I} \to \mathbf{Set}$ consists of the elementary free diagrams.

For example, let $\mathcal{I} = \{1 \leftarrow 0 \to 2\}$. the $\mathcal{I}$-diagrams in $\mathbf{Set}$ are

$$X = \left\{ X_1 \xleftarrow{f_1} X_0 \xrightarrow{f_2} X_2 \right\}.$$

When is it free (in terms of f_1, f_2)?

First, let's take a look at the elementary diagrams.

$$\begin{aligned} h^0 &= \left\{ \bullet \hookleftarrow \emptyset \hookrightarrow \emptyset \right\} \\ h^1 &= \left\{ \bullet \hookleftarrow \bullet \hookrightarrow \bullet \right\} \\ h^2 &= \left\{ \emptyset \hookleftarrow \emptyset \hookrightarrow \bullet \right\} \end{aligned}$$

Hence X is free if and only if both f_1, f_2 are injective.

Example 7.30. Let $\mathcal{C} = \mathbf{sSet}$ and $\mathcal{I}$ be any small category. For

$$\begin{aligned} X_* : \mathcal{I} &\to \mathbf{sSet} \\ i &\mapsto X_*(i) = \{X_n(i)\}_{n\geq 0} \end{aligned}$$

or equivalently,

$$X_* = \{X_n : \mathcal{I} \to \mathbf{Set}\}_{n\geq 0}$$

X_* is free if and only if X_n is free in $\mathbf{Set}$, $\forall n \geq 0$.

Example 7.31. Let $\mathcal{C} = \mathbf{Cat}$ and $\mathcal{I}$ be any small category, for any functor

$$\begin{aligned} X : \ \mathcal{I} & \rightarrow \mathbf{Cat} \\ i & \mapsto X(i) \end{aligned}$$

it consists of the following data,

$$\begin{aligned} Ob(X) : \ \mathcal{I} & \rightarrow \mathbf{Set} \\ i & \mapsto Ob(X(i)) \\ Mor(X) : \ \mathcal{I} & \rightarrow \mathbf{Set} \\ i & \mapsto Mor(X(i)) \end{aligned}$$

Thus X is free if and only if both $Ob(X)$ and $Mor(X)$ are free in **Set**.

Example 7.32. Let $\mathcal{C} = \mathbf{Grp} \subset \mathbf{Cat}$ be the category of small groupoids. $X : \mathcal{I} \rightarrow \mathbf{Grp}$ is free if and only if $Ob(X) : \mathcal{I} \rightarrow \mathbf{Set}$ is free.

A free diagram of groupoids need not to be free as a diagram of categories.

Simplicial Diagrams

Let $\Delta_+ \subseteq \Delta$ be the subset of surjective maps in Δ (i.e. $Mor(\Delta_+)$ are generated by codegeneracy maps $s^j : [n+1] \rightarrow [n], 0 \leq j \leq n, n \geq 0$).

Definition 7.25. A simplicial object $X : \Delta^{op} \rightarrow \mathcal{C}$ is called *semifree* if

$$X|_{\Delta_+^{op}} : \Delta_+^{op} \rightarrow \mathbf{Set}$$

is a free diagram.

Assume that $\mathcal{C}$ has adjunction to sets,

$$F : \mathbf{Set} \rightleftarrows \mathcal{C} : U$$

then we can make this definition explicitly.

$X : \Delta^{op} \to \mathcal{C}$ is semifree if and only if $X|_{\Delta_+^{op}} : \Delta_+^{op} \to \mathbf{Set}$ factors as

$$\begin{tikzcd} X|_{\Delta_+^{op}} : \Delta_+^{op} \arrow[rr] \arrow[dr, "\exists B"'] & & \mathcal{C} \\ & \mathbf{Set} \arrow[ur, "F"'] & \end{tikzcd}$$

Example 7.33. Let $\Gamma_* : \Delta^{op} \to \mathbf{Gr}$ be a simplicial group, then Γ_* is semifree if and only if there exists a subset $B_n \subseteq \Gamma_n, \forall n \geq 0$ such that

1. $\Gamma_n = \mathbb{F}\langle B_n \rangle$.
2. $B = \bigcup_{n \geq 0} B_n$ is closed under $s_j : \Gamma_n \to \Gamma_{n+1}$, i.e. $s_j(B_n) \subseteq B_{n+1}, \forall 0 \leq j \leq n, n \geq 0$.

Example 7.34. The Kan loop group of a reduced simplicial set $X \in Ob(\mathbf{sSet})$ is semifree.

$$\mathbb{G}(X)_n = \mathbb{F}\langle X_{n+1} \rangle / (s_0 x = 1, \forall x \in X_n) \cong \mathbb{F}\langle B_n \rangle$$

where $B_n = X_{n+1} \backslash s_0(X_n)$.

Definition 7.26. If Γ_* is semifree, then we can define

$$\overline{B}_n \coloneqq B_n \backslash \bigcup_{j=0}^{n-1} s_j(B_{n-1})$$

the set of nondegenerate generators of Γ in degree n, and

$$\overline{B} \coloneqq \bigcup_{n \geq 0} \overline{B}_n.$$

Note that this gives a semifree simplicial group Γ_*, we need to specify the set $\overline{B}$ and the values of face maps $d_i : \Gamma_n \to \Gamma_{n-1}$ on $\overline{B}_n$,

$$\left\{ d_i(x) \in \Gamma_{n-1}, x \in \overline{B}_n \right\}_{n \geq 1}.$$

Definition 7.27. (Kan) A set of nondegenerate generators of semifree Γ_* is called *CW basis* if $d_i(x) = 1 \subseteq \Gamma_{n-1}, \forall x \in \overline{B}_n, 0 \leq i \leq n-1$ (not including n).

Recall we have

$$\begin{array}{ccccc} \mathrm{Ho}(\mathbf{sGr}) & \cong & \mathrm{Ho}(\mathbf{sSet}_0) & \cong & \mathrm{Ho}\left(\mathbf{Top}_{0,*}\right) \\ \mathbb{G}\left(\mathcal{ES}(X)_*\right) & \leftarrow\!\shortmid & \mathcal{ES}(X)_* & \leftarrow\!\shortmid & X \end{array}$$

Γ_* is called a simplicial group model of X if its homotopy type corresponds to X under the above functor.

For any X we have a "big" functorial model given by

$$\Gamma_*^{\text{big}}(X) \coloneqq \mathbb{G}\left(\mathcal{ES}(X)_*\right).$$

One of the main observation of Kan is the following theorem.

Theorem 7.16. *Let X be a reduced CW complex, ($sk_0(X) = \{*\}, sk_n(X) = X, \forall n \gg 0$). There exists $\Gamma_*^{small}(X)$ a semifree simplicial group model of X such that*

1. $\left|\Gamma_*^{small}(X)\right| \simeq \Omega X$.
2. $\Gamma_*^{small}(X)$ *has a CW basis $\overline{B} = \bigcup_{n\geq 0} \overline{B}_n$ such that there is a one-to-one correspondence*

$$\overline{B}_{n-1} \longleftrightarrow \{n\text{-dimensional cells in } X\}$$

for any $n \geq 1$.

The corresponding attaching element $d_n(x), x \in \overline{B}_{n-1}$ depends only on the homotopy class of attaching maps $[f] \in \pi_{n-1}(sk_{n-1}X)$ of X.

Example 7.35. Let $X = \mathbb{S}^1$, then $\Gamma_*(X) = \{\mathbb{F}_1\}_{n\geq 0}$.

7.10 Homotopy 2-types

Moore Complexes

A simplicial group Γ_* induces a chain complex of nonabelian groups

$$N_n\Gamma = \bigcap_{i=1}^{n} \operatorname{Ker}\left(d_i : \Gamma_n \to \Gamma_{n-1}\right), i \neq 0$$

in the following way.

If $x \in N_n\Gamma$, then $d_i(x) = 1, \forall i > 0$,and thus $d_k d_0 x = d_0 d_{k+1} x, \forall k > 0$. Hence $d_0 x \in N_{n-1}\Gamma$, and $d_0^2 x = d_1 d_0 x = 1$. Therefore we have a chain complex

$$\mathbf{N}_*\Gamma = \left\{ N_0\Gamma \xleftarrow{d} N_1\Gamma \xleftarrow{d} N_2\Gamma \cdots \right\}$$

where $d = d_0|_{N_n\Gamma}, n \geq 0$.

Theorem 7.17. *(J. C. Moore) There are natural isomorphisms $\pi_*\Gamma \coloneqq \pi_*(|\Gamma|) \cong H_*(\mathbf{N}_*\Gamma, d)$.*

Remark 7.10. $|\Gamma|$ is a topological group.

Corollary 7.5. *If X is pointed connected space, let $\Gamma_* = \Gamma_*(X)$ be a simplicial group model of X, then*

$$\pi_i(X) \cong \pi_{i-1}(\Gamma_*(X)) \cong H_{i-1}(\mathbf{N}_*\Gamma, d), \forall i \geq 1.$$

Proof. By Kan's theorem, $\Omega X \simeq |\Gamma_*(X)|$, then

$$\pi_i(X) \cong \pi_{i-1}(\Omega X) \cong \pi_{i-1}(|\Gamma_*(X)|), \forall i \geq 1.$$

$\square$

Homotopy 2-types

Definition 7.28. A connected space X is called *homotopy n-type* (n-coconnected) if $\pi_i(X) = 0, \forall i \geq n+1$. Write $\mathrm{Ho}\left(\mathbf{Top}_{0,*}^{\leq n}\right)$ the homotopy subcategory of n-types.

Since $\mathrm{Ho}(\mathbf{Top}_{0,*}) \cong \mathrm{Ho}(\mathbf{sGr})$ a natural question is to characterize the image of $\mathrm{Ho}\left(\mathbf{Top}_{0,*}^{\leq n}\right)$ in $\mathrm{Ho}(\mathbf{sGr})$.

For $n = 1$, we have aspherical spaces with

$$\begin{array}{ccc} \mathrm{Ho}\left(\mathbf{Top}_{0,*}^{\leq 1}\right) & \cong & \mathbf{Gr} \\ X & \mapsto & \pi_1(X) \\ B\Gamma & \leftarrow\!\shortmid & \Gamma \end{array}$$

so $\mathrm{Ho}\left(\mathbf{Top}_{0,*}^{\leq 1}\right)$ can be identified with discrete simplicial groups $\Gamma_* = \{\Gamma_n\}_{n\geq 0}$.

If Γ_8 is a discrete simplicial group, then

$$N_n\Gamma = \begin{cases} \Gamma & n = 0, \\ 1 & n \neq 0. \end{cases}$$

and the Moore complex is

$$\mathbf{N}_*\Gamma = \left\{ 1 \longleftarrow \Gamma \longleftarrow 1 \longleftarrow 1 \cdots \right\}.$$

We want a similar characterization for $\mathrm{Ho}\left(\mathbf{Top}_{0,*}^{\leq 2}\right)$. This is given in terms of crossed modules of groups.

7.11 Crossed Modules

Recall if A, G are fixed groups with A being abelian, then it is well-known that the equivalence classes of extensions

$$\xi \coloneqq \left[\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] & E \arrow[r, "\pi"'] & G \arrow[l, bend right, dotted, "s"'] \arrow[r] & 0 \end{tikzcd}\right]$$

are in bijection with cohomology classes in $H^2(G, A)$.

To give a class in $H^2(G, A)$, choose a section $G \xrightarrow{s} E$ such that $\pi s = \mathrm{Id}_G$and $s(1) = 1$, and define

$$\begin{array}{ccc} G & \to & \mathrm{Aut}(A) \\ g & \mapsto & \left(\mathrm{Ad}_g : a \mapsto s(g)\, a s(g)^{-1}\right) \end{array}$$

and

$$\begin{array}{cccc} c: & G \times G & \to & A \\ & (g, h) & \mapsto & s(gh)\, s(g)^{-1}\, s(h)^{-1} \end{array}$$

then c is a 2-cocycle, hence $[c] \in H^2(G, A)$.

We want a similar interpretation of $H^3(G, A)$.

Definition 7.29. [Whitehead] A *crossed module* is a group homomorphism $\mu : M \to N$ given together with a left action

$$\begin{array}{cccc} \rho: & N \times M & \to & M \\ & (n, m) & \mapsto & {}^n m \end{array}$$

or equivalently

$$\begin{array}{cccc} \rho: & N & \to & \mathrm{Aut}(M) \\ & n & \mapsto & (m \mapsto^n m) \end{array}$$

satisfying

1. $\mu({}^n m) = n \cdot \mu(m) \cdot n^{-1}$ in N.

2. (Peiffer relation) ${}^{\mu(m)} m' = m \cdot m' \cdot m^{-1}$ in M.

Remark 7.11. The following lifting property is equivalent to axioms 1 and 2.

Given any group homomorphism $\mu : M \to N$ there is a natural commutative diagram

$$\begin{tikzcd} M \arrow[r, "\mathrm{Ad}"] \arrow[d, "\mu"'] & \mathrm{Aut}(M) \arrow[d] \\ N \arrow[r, "\mathrm{Ad}"] \arrow[ur, dashed, "\rho"] & \mathrm{Aut}(N) \end{tikzcd} \qquad \begin{tikzcd} m \arrow[r, mapsto] \arrow[d, mapsto] & \mathrm{Ad}_m \arrow[d, mapsto] \\ \mu(m) \arrow[r, mapsto] & \mathrm{Ad}_{\mu(m)} \end{tikzcd}$$

then μ is a crossed module if and only if there is a lift $\rho : N \to \mathrm{Aut}(M)$ making the diagram commutes.

Example 7.36. (Algebra) Let $M \trianglelefteq N$ be a normal subgroup, then $i : M \hookrightarrow N$ is a crossed module with ${}^{n}m = n \cdot m \cdot n^{-1}$.

Example 7.37. [Whitehead] (Topology) A Serre fibration

$$F \xrightarrow{i} E \xrightarrow{\pi} B$$

yields the natural crossed module

$$\mu = \pi_1(i) : \pi_1(F) \to \pi_1(E).$$

Example 7.38. (classical algebraic K-theory) If R is a unital ring, recall for $n \geq 3$ we can define the n-th Steinberg group

$$\mathrm{St}_n(R) = \frac{\langle x_{ij}(r), r \in R, 1 \leq i \neq j \leq n \rangle}{\left(\begin{array}{c} x_{ij}(r) \cdot x_{ij}(s) = x_{ij}(r+s) \\ [x_{ij}(r), x_{kl}(s)] = \begin{cases} 1 & j \neq k, i \neq l \\ x_{il}(rs) & j = k, i \neq l \\ x_{kj}(-rs) & j \neq k, i = l \end{cases} \end{array} \right)}$$

Note that the elementary matrices satisfy these relations.

So $\mu_n : St_n(R) \to E_n(R) = [GL_n(R), GL_n(R)] \hookrightarrow GL_n(R)$ is a well-defined group homomorphism. We have a natural commutative diagram

$$\begin{tikzcd} \mathrm{St}_n(R) \arrow[r, "\mu_n"] \arrow[d, hook, "\mathrm{in}"] & GL_n(R) \arrow[d, hook, "\mathrm{in}"] \\ \mathrm{St}_{n+1}(R) \arrow[r, "\mu_{n+1}"] & GL_{n+1}(R) \end{tikzcd}$$

which induces

$$\mu : \mathrm{St}(R) \to GL(R)$$

where $\mathrm{St}(R) = \varinjlim_n \mathrm{St}_n(R)$ and $GL(R) = \varinjlim_n GL_n(R)$.

This is naturally a crossed module with respect to the natural conjugation action of $GL(R)$. This follows from

Lemma 7.7. *(Whitehead) $Im(\mu) = E(R) = [GL(R), GL(R)] \triangleleft GL(R)$.*

Theorem 7.18. *(Kervaire–Steinberg)*

1. *$St(R)$ is universal central extension of $E(R)$.*

2. *There is an exact sequence*

$$0 \longrightarrow K_2(R) \longrightarrow St(R) \longrightarrow E(R) \longrightarrow K_1(R) \longrightarrow 0$$

where $K_i(R), i = 1, 2$ are the first and second algebraic K-theory groups of R.

Example 7.39. Any group object in **Cat** naturally determines a crossed module (i.e. $\mathrm{Hom}_{\mathbf{Cat}}(-,\mathcal{G}) :$ $\mathbf{Cat} \to \mathbf{Set}$ factors through groups

$$\begin{tikzcd} \mathbf{Cat} \arrow[rr, "\mathrm{Hom}_{\mathbf{Cat}}(-{,}\mathcal{G})"] \arrow[dr] & & \mathbf{Set} \\ & \mathbf{Gr} \arrow[ur] & \end{tikzcd}$$

Note that $\mathcal{N} : \mathbf{Cat} \to \mathbf{sSet}$ the usual nerve functor is right adjoint, hence it maps group objects in **Cat** to group objects in **sSet**, i.e. simplicial groups. $\mathcal{N}_*(\mathcal{G}) \in Ob(\mathbf{sGr})$. Take the Moore complex of $\mathcal{N}_*\mathcal{G}$,

$$N(\mathcal{N}_*\mathcal{G}) = \left\{ N_0 \xleftarrow{d} N_1 \xleftarrow{d} N_2 \longleftarrow \cdots \right\}$$

then $d : N_1 \to N_0$ is a crossed module.

The Relation of Crossed Module with $H^3(G, A)$

Lemma 7.8. *Given a crossed module $\left(M \xrightarrow{\mu} N, \rho\right)$, define $A := Ker(\mu)$ and $G = Coker(\mu)$. Then*

1. *G and A are groups with A being an abelian central subgroup of M.*

2. *the action $\rho : N \times M \to M$ induces a well-defined action $\overline{\rho} : G \times A \to A$ making A a G-module.*

Proof. By axiom 1 $n \cdot \mu(m) \cdot n^{-1} \in \mathrm{Im}(\mu)$ for any $m \in M, n \in N$, so $\mathrm{Im}(\mu) \trianglelefteq N$, thus $G = N/\mathrm{Im}(\mu)$ is a group.

By axiom 2, for any $a \in A, m \in M$, we have $m =^{\mu(a)} m = a \cdot m \cdot a^{-1}$, so $[a, m] = 1$, thus A is a central subgroup of M.

The action ρ restricts to A since for any $a \in A, n \in N$, $\mu({}^n a) = n \cdot \mu(a) \cdot n^{-1} = 1 \in N$, so ${}^n a \in A$. Furthermore, for any $m \in M$, ${}^{\mu(m)}a == m \cdot a \cdot m^{-1} = a$, so $\mathrm{Im}(\mu)$ acts trivially on A, thus the action descends to $\overline{\rho} : G \times A \to A$ making A a G-module. □

Example 7.40. If we take the Steinberg crossed module $\mathrm{St}(R) \xrightarrow{\mu} GL(R)$, in this case $G = K_1(R)$ and $A = K_2(R)$.

Fix a group G and an abelian group A with a G-module structure, consider the set $\mathbf{CrMod}(G,A)$ of all crossed modules $\left(M \xrightarrow{\mu} N, \rho\right)$ with $G = \mathrm{Coker}(\mu)$ and $A = \mathrm{Ker}(\mu)$, and the induced action $\overline{\rho}$ being the one given.

We say that two such crossed modules $\left(M \xrightarrow{\mu} N, \rho\right)$ and $\left(M' \xrightarrow{\mu'} N', \rho'\right)$ are related if there exist group homomorphisms $\alpha : M \to M'$ and $\beta : N \to N'$ such that

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "\alpha"'] & N \arrow[r] \arrow[d, "\beta"] & G \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & M' \arrow[r] & N' \arrow[r] & G \arrow[r] & 0
\end{tikzcd}$$

commutes.

Let $\mathbf{CrMod}(G,A)/\sim$ be the set of equivalence of classes of crossed modules in $\mathbf{CrMod}(G,A)$ modulo the equivalence generated by the above relation.

Proposition 7.5. *(Mac Lane) There is a natural bijection*

$$\mathbf{CrMod}(G,A)/\sim \cong H^3(G,A).$$

The cohomology class $k(\mu,\rho) \in H^3(G,A)$ corresponds to $\left(M \xrightarrow{\mu} N, \rho\right)$ is called its Mac Lane invariant.

Topological interpretation. We need the notion of s classifying space for crossed modules.

Given a crossed module $\left(M \xrightarrow{\mu} N, \rho\right)$, define a (small) category $\mathcal{C} = \mathcal{C}(\mu,\rho)$ as follows:

Objects: $Ob(\mathcal{C}) := N$

Morphisms: $Mor(\mathcal{C}) := M \rtimes N$ where $M \rtimes N \cong M \times N$ as a set and composition is given by $(m,n)\cdot(m',n') = (m \cdot^n m', nn')$.

The source and target maps are

$$\begin{aligned}
s: \quad Mor(\mathcal{C}) &\longrightarrow Ob(\mathcal{C}) \\
(m,n) &\longmapsto n \\
t: \quad Mor(\mathcal{C}) &\longrightarrow Ob(\mathcal{C}) \\
(m,n) &\longmapsto \mu(m)\,n
\end{aligned}$$

so $(m,n) \in M \rtimes N$ corresponds to the arrow

$$n \xrightarrow{(m,n)} \mu(m)\,n\ .$$

The composition is given by

$$\left(\mu(m)\,n \xrightarrow{(m',\mu(m)n)} \mu(m')\,\mu(m)\,n\right) \circ \left(n \xrightarrow{(m,n)} \mu(m)\,n\right) = \left(n \xrightarrow{(m'm,n)} \mu(m'm)\,n\right)$$

The nerve $\mathcal{N}_*\mathcal{C}$ of the category ;looks as follows

$\mathcal{N}_0\mathcal{C} = N.$

$\mathcal{N}_1\mathcal{C} = M \rtimes N.$

$\cdots.$

$\mathcal{N}_n\mathcal{C} = M \rtimes (M \rtimes (\cdots \rtimes (M \rtimes N))) \cong M^n \times N$ as a set.

All face and degeneracy maps d_i, s_j are group homomorphisms, so $\mathcal{N}_*\mathcal{C}$ is a simplicial group. We denote this simplicial group by $N//M \in Ob(\mathbf{sGr})$.

Definition 7.30. (Mac Lane) The *classifying space of the crossed module* $\left(M \xrightarrow{\mu} N, \rho\right)$ is

$$X(\mu,\rho) := B\,|N//M|$$

the usual classifying space of a topological space $|N//M|$.

We want to understand the homotopy type of $X(\mu,\rho)$.

We will use the Kan loop group adjunction

$$\mathbb{G} : \mathbf{sSet}_0 \rightleftarrows \mathbf{sGr} : \overline{W}\ .$$

In these terms, we can identify $B\,|N//M| \simeq \left|\overline{W}(N//M)\right|$, then use Kan's theorem. for any $i \geq 1$,

$$\pi_i(B\,|N//M|) \cong \pi_i\left(\left|\overline{W}(N//M)\right|\right) \cong \pi_{i-1}\left(\Omega\left|\overline{W}(N//M)\right|\right) \cong \pi_{i-1}\left(\left|\mathbb{G}\overline{W}(N//M)\right|\right) \cong \pi_{i-1}(|N//M|) \cong H_{i-1}(\mathbf{N}_\bullet(N//M),$$

where the second lats isomorphism follows from the fact that the unit of Kan loop adjunction is a weak equivalence.

We need to see what is the Moore complex of $N//M$. Be definition, $N//M$ looks as follows

$$\begin{tikzcd} N \arrow[r, "s_0" description] & M \rtimes N \arrow[l, shift right=2ex, "d_1=s"'] \arrow[l, shift left=2ex, "d_0=t"] \end{tikzcd}$$

where $s_0 : N \to M \rtimes N$is the canonical inclusion so that $d_0 s_0 = d_1 s_0 = \mathrm{Id}_N$. Thus $d_0|_N = d_1|_N = \mathrm{Id}_N$.

A straightforward calculation shows that $\mathbf{N}_\bullet(N//M, d)$ has the form

$$N \xleftarrow{d=\mu} \mathrm{Ker}(d_1) \cong M \longleftarrow 1 \longleftarrow 1 \longleftarrow \cdots$$

where $d = d_0|_{\mathrm{Ker}(d_1)} = \mu$.

Claim 7.2. This is exactly the crossed module we start with.

Lemma 7.9. *If Γ_* is a simplicial group such that its Moore complex has length 1, then the following relation holds.*

$$\Gamma_* = \begin{bmatrix} \begin{tikzcd} \Gamma_0 \arrow[r, "s_0"] & \Gamma_1 \arrow[l, shift right=2, "d_1"'] \arrow[l, shift left=2, "d_0"] \arrow[r, shift left=3] \arrow[r, shift left=1] & \cdots \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] \end{tikzcd} \end{bmatrix}$$

and $[Ker(d_0), Ker(d_1)] = 1$ in Γ_1.

Proof. (Sketch).

For $x \in \mathrm{Ker}(d_0)$ and $y \in \mathrm{Ker}(d_1)$, we consider $\left[s_0(x)\, s_1(x)^{-1}, s_0(y)\right] \in \Gamma_2$ then

$$d_0\left(\left[s_0(x)\, s_1(x)^{-1}, s_0(y)\right]\right) = \left[d_0 s_0(x)\, d_0\left(s_1(x)^{-1}\right), d_0 s_0(y)\right] = \left[x\left(s_0 d_0 x\right)^{-1}, y\right] = [x, y] = 1$$

and

$$d_1\left(\left[s_0(x)\, s_1(x)^{-1}, s_0(y)\right]\right) = \left[d_1 s_0(x)\, d_1\left(s_1(x)^{-1}\right), d_1 s_0(y)\right] = [1, y] = 1$$

so by assumption

$$\left[s_0(x)\, s_1(x)^{-1}, s_0(y)\right] \in \mathrm{Ker}(d_0) \cap \mathrm{Ker}(d_1) = 1$$

Note

$$1 = d_0\left(\left[s_0(x)\, s_1(x)^{-1}, s_0(y)\right]\right) = [x, y].$$

$\square$

Lemma 7.10. *In the case of simplicial group $N//M$, the commutator relation $[Ker(d_0), Ker(d_1)] = 1$ is equivalent to the axiom 2 of crossed module.*

Corollary 7.6. *We have*

$$\begin{aligned} \pi_1(B|N//M|) &\cong H_0(\mathbf{N}_\bullet |N//M|, d) \cong G \\ \pi_2(B|N//M|) &\cong H_1(\mathbf{N}_\bullet |N//M|, d) \cong A \\ \pi_i(B|N//M|) &= 0, i \geq 3. \end{aligned}$$

Moreover, the action of π_1 on π_2 agrees with the G-module structure on A, and

$$\begin{aligned} H^3(B\pi_1;\pi_2) &\cong H^3(G;A) \\ \textit{Postnikov's } k^3\textit{-invariant} &\leftrightarrow k(\mu,\rho) \end{aligned}$$

Recall for any connected CW complex X, the Postnikov decomposition is given by

$$\operatorname{cosk}_1(X) \twoheadleftarrow \operatorname{cosk}_2(X) \twoheadleftarrow \operatorname{cosk}_3(X) \twoheadleftarrow \cdots \twoheadleftarrow X$$

such that

1. $\pi_i(X) \cong \pi_i(\operatorname{cosk}_n(X)), i \leq n.$

   $\pi_i(\operatorname{cosk}_n(X)) = 0, i > n.$

2. For $n \geq 2$, , there is a fibration

$$\operatorname{cosk}_{n-1}(X) \twoheadleftarrow \operatorname{cosk}_n(X) \longleftarrow K(\pi_n(X), n)$$

   with characteristic classes

$$k^{n+1}(X) \in H^{n+1}(\operatorname{cosk}_{n-1}(X); \pi_n(X))$$

   called the $(n+1)$-th Postnikov invariants.

In our case, if $X = X(\mu,\rho) = B|N//M|$, then we have

$$B\pi_1(X) = \operatorname{cosk}_1(X) \twoheadleftarrow \operatorname{cosk}_2(X) \twoheadleftarrow \operatorname{cosk}_3(X) \twoheadleftarrow \cdots \twoheadleftarrow X$$

the Postnikov invariant $k^3 \in H^3(B\pi_1 X, \pi_2 X) \cong H^3(\pi_1 X, \pi_2 X) \cong H^3(G, A)$ coincide with Mac Lane's invariant.

As a summary, we have

Theorem 7.19. *(Loday) The following are equivalent:*

1. *crossed modules of groups* $\left(M \xrightarrow{\mu} N, \rho\right)$
2. *group objects in* **Cat**
3. *simplicial groups* Γ_* *with Moore complex of length 1*

Proof. We have construct the proof of $1 \implies 2 \implies 3 \implies 1$ as follows.

Given a crossed module of groups $\left(M \xrightarrow{\mu} N, \rho\right)$, we can define a group object $\mathcal{C}(\mu, \rho)$.

For a group object $\mathcal{C}$, the nerve $\mathcal{N}_* \mathcal{C}$ is a simplicial group with Moore complex of length 1.

The Moore complex of Γ_*

$$\left[N = \Gamma_0 \xleftarrow{\mu} M = \mathrm{Ker}(d_1) \longleftarrow 1 \longleftarrow \cdots \right]$$

gives a crossed module $\mu : M \to N$. Note the axiom 2 of crossed module is equivalent to $[\mathrm{Ker}(d_0), \mathrm{Ker}(d_1)] = 1$ in Γ_1. □

7.12 Homotopy Normal Maps

The references for this part are [FS10, FH11].

There is another view on this based on the notion of homotopy normal maps.

Motivation: recall an injective group homomorphism $\mu : M \hookrightarrow N$ is normal if it is the kernel of some group homomorphism $N \to \Gamma$. Up to homotopy, and group homomorphism can be viewed as an inclusion, so it's natural to ask how to extend this notion to an arbitrary homomorphism.

Definition 7.31. A group homomorphism $\mu : M \to N$ is *homotopy normal* if the corresponding map of spaces $B\mu : BM \to BN$ is the map of a homotopy fibre of some fibration

$$BM \xrightarrow{B\mu} BN \xrightarrow{\nu} X$$

where X is a pointed connected space. The homotopy class of ν is called the *normal structure* on μ.

Note if $M \trianglelefteq N$ is a normal subgroup in the normal sense, we have a canonical normal structure on $\mu : M \hookrightarrow N$, i.e. $BN \xrightarrow{p} B(N/M)$ where $N \xrightarrow{p} N/M$.

Theorem 7.20. *[FS10] A group homomorphism $\mu : M \to N$ is homotopy normal if and only if there is an action $\rho : N \to Aut(M)$ making $\mu : M \to N$ a crossed module. The corresponding space for which $BM \to BN$ is the homotopy fiber map is $B|N//M|$, the classifying space of $\left(M \xrightarrow{\mu} N, \rho\right)$.*

Alternative way to state this is in terms of homotopy colimits.

Given $\mu : M \to N$, we can consider the diagram

$$\begin{array}{rcccc} \mu: & \underline{M} & \longrightarrow & \mathbf{Cat} & \hookrightarrow \mathbf{sSet} \\ & * & \longmapsto & N & \\ & m & \longmapsto & \left(\begin{array}{cccc} l_m: & N & \to & N \\ & n & \mapsto & \mu(m)\, n \end{array} \right) & \end{array}$$

and take the homotopy quotient (Borel construction) to get a simplicial set

$$N//M = \mathrm{hocolim}_{\underline{M}}(\mu) = E_*M \times_M N$$

with

$$\begin{aligned} (N//M)_0 &= M \times_M N \cong N \\ (N//M)_n &= \underbrace{M \times_M M \times_M \cdots \times_M M}_{n+1} \times_M N \cong M^n \times N \end{aligned}$$

where $N = (N//M)_0$ acts by right multiplication on $(N//M)_n$, so we get an N-simplicial set.

Similarly, for any simplicial group Γ_*, Γ_0 acts on the right via the degeneracy maps s_0 on all Γ_n, so Γ_* is a Γ_0-simplicial set.

Theorem 7.21. *$\mu : M \to N$ is homotopy normal if and only if there is a simplicial group Γ_* such that $\Gamma_0 \cong N$ which extends to an isomorphism of Γ_0-simplicial sets $\Gamma_* \xrightarrow{\cong} N//M$.*

Remark 7.12. This extends to other monoidal model categories.

Example 7.41. A map $\mu : M_* \to N_*$ between simplicial groups is homotopy normal if there exists homotopy fibration of simplicial sets

$$\overline{W}(M_*) \xrightarrow{\overline{W}\mu} \overline{W}(N_*) \xrightarrow{\mu} X_*.$$

Example of Crossed modules

If G is a group, $z \in \mathcal{Z}(G)$ is a central element, then the twisted nerve of G is a cyclic set

$$\begin{array}{rccc} B_*(G,z): & \Delta\mathbf{C}^{op} & \longrightarrow & \mathbf{Set} \\ & [n] & \mapsto & G^n \end{array}$$

such that $B_*(G,z)|_{\Delta^{op}} = B_*G$. The cyclic action is

$$t_n(g_1, \cdots, g_n) = \left(z(g_1 \cdots g_n)^{-1}, g_2, \cdots, g_n \right).$$

The cyclic realization of $B_*(G,z)$ is

$$X(G,z) = |B_*(G,z)|^{cy} := E\mathbb{S}^1 \times_{\mathbb{S}^1} |B_*(G,z)|\,.$$

Question. what is the homotopy type of $B_*(G,z)$?

Aside. Let $C_* = \{\mathbb{Z}/(n+1)\}_{n\geq 0}$ and X_* be a cyclic set. We have a canonical map

$$C_* \times X_* \to X_*$$

via the the action of $\mathrm{Aut}_{\Delta\mathbf{C}^{op}}([n]) \cong \mathbb{Z}/(n+1)$ on X_*, which is not a map of simplicial sets. So we need to replace it by $FUX_n \to X_n$ where

$$F : \mathbf{Set}^{\Delta^{op}} \rightleftarrows \mathbf{Set}^{\Delta\mathbf{C}^{op}} : U\ .$$

Consider the crossed module

$$\begin{array}{rccc} \gamma: & \mathbb{Z} & \longrightarrow & G \\ & n & \mapsto & z^n \end{array}$$

with the trivial G-action on $\mathbb{Z}$.

$$X(G,z) \cong B(G//\mathbb{Z})\,.$$

Thus

$$\begin{array}{rcll} \pi_i(X(G,z)) & = & 0 & i \geq 3 \\ \pi_1(X(G,z)) & = & \mathrm{Coker}\gamma & \\ \pi_2(X(G,z)) & = & \mathrm{Ker}\gamma & \end{array}$$

If z is of finite order,

$$X(G,z) \cong B(G/\langle z\rangle)\,.$$

Quillen's + Construction

We will work with pointed connected CW complexes. Given such a complex X, denote $\pi := \pi_1(X,*)$.

Theorem 7.22. *Let $N \trianglelefteq \pi$ be a normal perfect (i.e. $[N,N]=N$) subgroup of π, then there exists a CW complex X_N^+ with a map $j : X \to X_N^+$ such that*

1. *$\pi_1(j) : \pi \twoheadrightarrow \pi_1(X_N^+)$ is surjective with $\mathrm{Ker}\pi_1(j) = N$.*
2. *$j_* : H_*(X,\mathbb{Z}) \xrightarrow{\sim} H_*(X_N^+,\mathbb{Z})$ is an isomorphism.*

Remark 7.13. The second condition can be restated as, for any $\pi_1\left(X_N^+\right)$-module A (local system on X_N^+), $j^*: H^*\left(X_N^+, A\right) \xrightarrow{\sim} H^*(X, A)$.

Idea. Add 2-cells to "kill" $N \subseteq \pi$ and add 3-cells to neutralize the effect of 2-cells on homology groups.

Remark 7.14. There is an abstract analogue of this construction for various model categories (e.g. differential graded Lie algebras) where "perfect" makes sense.

Analogy.

1. π_* for spaces $\longleftrightarrow$ H_* of differential graded objects.
2. H_* for spaces $\longleftrightarrow$ Quillen's homology of differential graded objects.

Basic properties of + construction

(P1) $\left(X_N^+, j\right)$ is universal in $\mathrm{Ho}\left(\mathbf{Top}_{0,*}\right)$ among all pairs of $(Y, f: X \to Y)$ such that $\pi_1(f)(N) = 1$ in $\pi_1(Y)$, in the sense

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "j"'] & & Y \\ & X_N^+ \arrow[ur, dotted, "\tilde{f}"'] & \end{tikzcd}$$

where $\tilde{f}$ is unique up to homotopy.

(P2) If G is a group such that $N = [G, G]$ is perfect, i.e. $[[G,G],[G,G]] = [G,G]$, then $i: N \hookrightarrow G$ induces $Bi: BN \to BG$ which gives a universal covering $\alpha = (Bi)^+ : (BN)_N^+ \to (BG)_N^+$

$$\begin{tikzcd} BN \arrow[r, "Bi"] \arrow[d, "j"'] & BG \arrow[d, "j"] \\ (BN)_N^+ \arrow[r, "\alpha"] & (BG)_N^+ \end{tikzcd}$$

Application. If R is a unital associative ring, define $GL(R) := \varinjlim_n GL_n(R)$ and $E(R) := \varinjlim_n E_n(R) \trianglelefteq GL(R)$ the elementary subgroup generated by elementary matrices.

Lemma 7.11. *(Whitehead) The derived group of the stable general linear group is the group generated by elementary matrices.*

$$E(R) = [GL(R), GL(R)].$$

Definition 7.32. (Quillen) For $n \geq 1$, the higher algebraic K-theory

$$K_n(R) := \pi_n\left(BGL(R)^+\right)$$

where the + construction is taken with respect to $E(R)$.

Applying property 7.12 to $E_n(R) \trianglelefteq GL(R)$, we get $BE(R)^+ \to BGL(R)^+$ which gives a universal covering

$$\pi_n\left(BE(R)^+\right) \longrightarrow K_n(R), n \geq 2.$$

The Bousfield-Kan Completion of A Space

The Bousfield-Kan completion is a natural extension of pronilpotent completion of groups to spaces.

Recall, if F is a (discrete group), then the lower central series is given by

$$\Gamma = \Gamma_1 \trianglerighteq \Gamma_2 \trianglerighteq \cdots \trianglerighteq \Gamma_n \trianglerighteq \cdots$$

where $\Gamma_1 = \Gamma, \Gamma_2 = [\Gamma_1, \Gamma], \cdots, \Gamma_n = [\Gamma_{n-1}, \Gamma], n \geq 0$.

Γ is abelian if and only if $\Gamma_2 = \{1\}$.

Γ is nilpotent if and only if there exists $n \geq 2$ such that $\Gamma_n = \{1\}$.

Definition 7.33. The pronilpotent completion of Γ is

$$\widehat{\Gamma} = C(\Gamma) := \varprojlim_n \Gamma/\Gamma_n$$

with completion map

$$\begin{array}{rccc} p: & \Gamma & \to & C\Gamma \\ & \gamma & \mapsto & (\gamma\Gamma_1, \gamma\Gamma_2, \cdots) \end{array}$$

where

$$\cdots \longrightarrow \Gamma/\Gamma_3 \xrightarrow{p_2} \Gamma/\Gamma_2 \xrightarrow{p_1} \Gamma/\Gamma_1 = 1\ .$$

Explicitly,

$$C\Gamma = \left\{ (\overline{\gamma}_1, \overline{\gamma}_2, \cdots,) \in \prod_{n\geq 1} \Gamma/\Gamma_n | p_n\left(\overline{\gamma}_{n+1}\right) = \gamma_n, \forall n \geq 1 \right\}.$$

The completion map p is universal among all maps $\Gamma \to N$.

Remark 7.15. In categorical terms, this can be defined as the right Kan extension of the inclusion of nilpotent groups $\mathbf{NGr} \overset{i}{\hookrightarrow} \mathbf{Gr}$ along itself.

$$\begin{tikzcd} \mathbf{NGr} \arrow[r, hook] \arrow[d, hook, "i"'] & \mathbf{Gr} \\ \mathbf{Gr} \arrow[ur, "R_i(i)=C"'] & \end{tikzcd}$$

and

$$C(\Gamma) = \lim_{\Gamma \downarrow \mathbf{NGr}} (i).$$

Next, recall the Kan loop group construction

$$\mathbb{G} : \mathbf{sSet}_0 \rightleftarrows \mathbf{sGr} : \overline{W}.$$

Note

- $\pi_i(\mathbb{G}X) = \pi_i(\Omega|X|) = \pi_{i+1}(|X|), \forall i \geq 0.$
- $X \xrightarrow{\sim} \overline{W}\mathbb{G}X$ is a weak equivalence for any reduced simplicial set X.
- $\mathbb{G}\overline{W}\Gamma \xrightarrow{\sim} \Gamma$ is a weak equivalence for any simplicial group Γ.

Definition 7.34. The Bousfield-Kan(integral) completion of a reduced simplicial set X is defined by

$$\mathbb{Z}_\infty(X) := \overline{W}C\mathbb{G}(X)$$

where $C : \mathbf{sGr} \to \mathbf{sGr}$ is the degreewise pronilpotent completion. If $X \in Ob\left(\mathbf{Top}_{0,*}\right)$, then

$$\mathbb{Z}_\infty(X) := |\mathbb{Z}_\infty(\mathcal{ES}_*(X))|.$$

Basic properties [BK]

(I1) [Lemma I 5.5, p25] For a map $f : X \to Y$ in $\mathbf{sSet}_0$, $\mathbb{Z}_\infty(f) : \mathbb{Z}_\infty(X) \to \mathbb{Z}_\infty(Y)$ is a homotopy equivalence if and only if $f_* : H_*(X, \mathbb{Z}) \to H_*(Y, \mathbb{Z})$ is an isomorphism.

(I2) [Prop V 3.4, P34] For any reduced simplicial set X, the canonical map $i : X \to \mathbb{Z}_\infty(X)$ is a weak equivalence if and only if X is nilpotent, i.e. $\pi_1(X)$ is nilpotent and $\pi_1(X)$ acts nilpotently on higher homotopy groups $\pi_i(X), i \geq 2$.

Proposition 7.6. *[D. Farjoun] Let R be a unital associative ring, then there exists a natural homotopy group equivalence*

1. $\left|\mathbb{Z}_\infty\overline{W}E(R)\right| \cong BE(R)^+$.
2. $\left|\mathbb{Z}_\infty\overline{W}GL(R)\right| \cong BGL(R)^+$.

Moral. $\mathbb{Z}_\infty$ can be viewed as a simplicial realization of $+$ construction.

Corollary 7.7. $K_n(R) = \pi_n \left|\mathbb{Z}_\infty \overline{W} GL(R)\right|, n \geq 1.$

Corollary 7.8. *The functor* $C : \mathbf{sGr} \to \mathbf{sGr}$ *has left deformation* $Q = \mathbb{G}\overline{W} : \mathbf{sGr} \to \mathbf{sGr}$ *and therefore it has derived functor*

$$\mathbb{L}C : Ho(\mathbf{sGr}) \to Ho(\mathbf{sGr})$$

such that

$$\pi_n \mathbb{L}C(GL(R)) \cong K_{n+1}(R), n \geq 0.$$

Remark 7.16. This applies to many functors on groups.

Part IV

Derived Algebraic Geometry

8 Simplicial Presheaves

Motivation. In classical algebraic geometry, basic objects are schemes.

Fix a commutative ring k (base ring). Consider $\mathbf{Sch}_k$ the category of schemes over k and the Yoneda embedding

$$\begin{array}{ccc} \mathbf{Sch}_k & \longrightarrow & \mathbf{Fun}\left(\mathbf{Sch}_k^{op}, \mathbf{Set}\right) \\ X & \longmapsto & h_X \end{array}$$

where h_X is called the *functor of points* of X. X is determined by h_X up to unique isomorphism.

We can refine Yoneda lemma in the following way.

Recall affine schemes over k are schemes of the form $X = \mathrm{Spec}\,(A)$ where A is a commutative k-algebra, thus we have a functor

$$\mathrm{Spec} : \mathbf{CommAlg}_k \longrightarrow \mathbf{Sch}_k$$

with images being affine schemes.

Note we have

$$\begin{tikzcd} \mathbf{Sch}_k \arrow[r, hook, "h"] \arrow[drr, dashed, bend right, "\bar{h}=\mathrm{Res}\circ h"'] & \mathbf{Fun}\left(\mathbf{Sch}_k^{op}, \mathbf{Set}\right) \arrow[r, "\mathrm{Res}"] & \mathbf{Fun}\left(\mathbf{AffSch}_k^{op}, \mathbf{Set}\right) \arrow[d, "\cong"] \\ & & \mathbf{Fun}\left(\mathbf{CommAlg}_k, \mathbf{Set}\right) \end{tikzcd}$$

where $\bar{h} = \mathrm{Res} \circ h$ is the composition of Yoneda functor and restriction functor.

(Enhanced) Yoneda lemma. the restriction of the functor of points to commutative algebras

$$\bar{h} : \mathbf{Sch}_k \longrightarrow \mathbf{Fun}\left(\mathbf{CommAlg}_k, \mathbf{Set}\right)$$

remains fully faithful. Thus the category of schemes over k can be identified with a certain subcategory of functors $F : \mathbf{CommAlg}_k \longrightarrow \mathbf{Set}$.

Question. How to characterize this subcategory?

Example 8.1. Every (co)representable functor

$$\begin{array}{rccl} h^A: & \mathbf{CommAlg}_k & \longrightarrow & \mathbf{Set} \\ & B & \longmapsto & \mathrm{Hom}_{\mathbf{CommAlg}_k}(A,B) \end{array}$$

is a scheme functor (such functors are exactly affine schemes).

For example, take X to be the elliptic curve defined by $y^2 = x^3 + x^2 + 1$, then

$$\begin{array}{rccl} \bar{h}: & \mathbf{CommAlg}_k & \longrightarrow & \mathbf{Set} \\ & B & \longmapsto & \left\{(x,y) \in B \times B | y^2 = x^3 + x^2 + 1\right\} \end{array}$$

However, not only representable functors may occur.

Example 8.2. Fix $0 < d < n, n \geq 1$ and consider the functor

$$\begin{array}{rccl} \mathrm{Gr}(d,n): & \mathbf{CommAlg}_k & \longrightarrow & \mathbf{Set} \\ & B & \longmapsto & \{\text{rank } d \text{ summands of } B^{\oplus n}\} \\ & \left(B_1 \xrightarrow{f} B_2\right) & \longmapsto & \left(\begin{array}{ccc} \mathrm{Gr}(d,n)(B_1) & \to & \mathrm{Gr}(d,n)(B_2) \\ Q & \mapsto & B_2 \otimes_{B_1} Q \end{array}\right) \end{array}$$

This is the functor of points that represents the classical Grassmannian.

However, this is not a representable functor. .

Theorem 8.1. *[Grothendieck] A functor* $F : \mathbf{CommAlg}_k \longrightarrow \mathbf{Set}$ *has the form* h_X *for some* $X \in Ob(\mathbf{Sch}_k)$ *if and only if*

1. *F is a sheaf in the Zariski (Grothendieck) topology in* $\mathbf{AffSch}_k \cong \mathbf{CommAlg}_k^{op}$.

2. *There are commutative k-algebras* $A_i \in Ob(\mathbf{CommAlg}_k)$ *and natural transformations* $\alpha_i \in F(A_i) \cong Hom_{\mathbf{Fun}}\left(h^{A_i}, F\right)$ *such that for any field* $K \supseteq k$,

$$F(K) = \bigcup_{i \in I} \alpha_i\left(h^{A_i}(k)\right)$$

i.e. $F(K)$ *is covered bu* $h^{A_i}(K)$ *via* α_i.

Derived algebraic geometry. Generalize classical schemes (viewed as functors $\mathbf{CommAlg}_k \longrightarrow \mathbf{Set}$) in the following ways.

$$\begin{tikzcd}
\mathbf{CommAlg}_k \arrow[r, "\text{schemes}"] \arrow[dr, "\text{stacks}"] \arrow[dd, hook, "i"'] \arrow[ddr, bend left, "\text{higher stacks}"] & \mathbf{Set} \arrow[d, hook, "i"] \\
 & \mathbf{Grpd} \arrow[d, hook, "\text{nerve}"] \\
\mathbf{sCommAlg}_k \arrow[r, "\text{derived stacks}"'] \arrow[uur, dotted, "\text{derived schemes}"] & \mathbf{sSet}
\end{tikzcd}$$

Goal. Need a homotopy theory on simplicial presheaves, local homotopy theory [Joyal & Jardine].

8.1 Grothendieck Topology

Motivation. If X is a topological space, we define and then replace X by the category $\mathcal{C} = \mathcal{O}(X)$ of open sets in X.

Objects: open sets in X

Morphisms: $\mathrm{Hom}_{\mathcal{C}}(U, V) = \begin{cases} U \hookrightarrow V & U \subseteq V \\ \emptyset & \text{o.w.} \end{cases}$

Note

1. X is the terminal object in $\mathcal{C}$.

2. For any finite index set I, $|I| < \infty$,
$$\bigcap_{i \in I} U_i = \prod_{i \in I} U_i.$$

3. For any index set I,
$$\bigcup_{i \in I} U_i = \coprod_{i \in I} U_i.$$

4. A *presheaf* on X is a functor $F : \mathcal{C}^{op} \to \mathbf{Set}$.

5. A *sheaf* on X is a presheaf $F : \mathcal{C}^{op} \to \mathbf{Set}$ such that for any $U \subseteq X$ open and open covering $\{U_i\}_{i \in I}$ of U, we have
$$F(U) \cong \mathrm{eq}\left\{ \prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j) \right\}.$$

Grothendieck's generalization of this notion of topology consists of replacing a space X (or the corresponding category $\mathcal{O}(X)$) by an abelian category $\mathcal{C}$ in which we specify a system of covering for each object $U \in Ob(\mathcal{C})$, i.e. the data

$$\mathrm{cov}\,(U) = \left\{\{U_i \to U\}_{i\in I} \subseteq Mor\,(\mathcal{C})\right\}$$

satisfying the following 3 axioms

T1. For any $U \in Ob\,(\mathcal{C})$, $\left\{U \xrightarrow{\mathrm{Id}} U\right\} \in \mathrm{cov}\,(U)$.

T2. For any $f : V \to U$, and $\{U_i \to U\}_{i\in I} \in \mathrm{cov}\,(U)$, all fibre products $\{U_i \times_U V \to V\}_{i\in I} \in \mathrm{cov}\,(V)$, i.e. pulling back of coverings are still coverings.

T3. For any $\{U_i \to U\}_{i\in I} \in \mathrm{cov}\,(U)$ and $\{U_{ij} \to U_i\}_{j\in J} \in \mathrm{cov}\,(U_i)$, we have $\{U_{ij} \to U\}_{(i,j)\in I\times J} \in \mathrm{cov}\,(U)$.

Modification.

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \hat{\mathcal{C}} = \mathbf{Fun}\,(\mathcal{C}^{op}, \mathbf{Set}) \\ U & \longmapsto & h_U \end{array}$$

we can define covering in terms of subfunctors of h_U, called sieves.

Simplicial Presheaves

Let $\mathcal{C}$ be a category. Denote $\mathbf{Pr}\,(\mathcal{C}) := \mathbf{Fun}\,(\mathcal{C}^{op}, \mathbf{Set})$ the category of presheaves on $\mathcal{C}$. For $X \in Ob\,(\mathcal{C})$, $h_X := \mathrm{Hom}_{\mathcal{C}}\,(-, X) : \mathcal{C}^{op} \to \mathbf{Set}$ is a presheaf on X. The assignment $X \mapsto h_X$ gives a functor $h : \mathcal{C} \hookrightarrow \mathbf{Pr}\,(\mathcal{C})$.

Goal. Define derived stacks as functors $\mathbf{sCommAlg}_k \to \mathbf{sSet}$.

We approach this in two steps:

1. define nonderived (higher) stacks as functor $\mathbf{CommAlg}_k \to \mathbf{sSet}$
2. extend this to $\mathbf{sCommAlg}_k$.

Definition 8.1. A *sieve* over $X \in Ob\,(\mathcal{C})$ is a presheaf $u \in \mathbf{Pr}\,(\mathcal{C})$ which comes with $u \hookrightarrow h_X$ (subfunctor of h_X, i.e.$u\,(Y) \subseteq h_X\,(Y)\,, \forall Y \in Ob\,(\mathcal{C})$, and for all arrows $f : Y' \to Y$ of $\mathcal{C}$, $u\,(f)$ is the restriction of h_f to $u\,(Y)$).

Note u may or may not be representable,

Definition 8.2. A *Grothendieck topology* $\mathscr{T}$ on $\mathcal{C}$ consist of the data, for any $X \in Ob\,(\mathcal{C})$, there is a family of sieves $\mathrm{cov}\,(X)$ over X (covereing sieves) satisfying

1. $h_X \in \mathrm{cov}\,(X)$.

2. For any $f : Y \to X$ and $u \in \operatorname{cov}(X)$, $f^*(u) = h_Y \times_{h_X} u \in \operatorname{cov}(Y)$.

$$\begin{tikzcd} f^*(u) \arrow[r] \arrow[d] & u \arrow[d, hook] \\ h_Y \arrow[r, "h_f"] & h_X \end{tikzcd}$$

3. For any $u \in \operatorname{cov}(X)$, and any sieve $v \subset h_X$ over X, if for any $Y \in Ob(\mathcal{C})$ and any $f \in u(Y) \subset h_X(Y) = \operatorname{Hom}_{\mathcal{C}}(Y, X)$, $f^*(v) \in \operatorname{cov}(Y)$, then $v \in \operatorname{cov}(X)$.

Remark 8.1. The reason to use presheaves instead of objects in $\mathcal{C}$ to define covering is that $\mathcal{C}$ is not cocomplete (even coproducts may not exist). For example, let X be a topological space and $\mathcal{C} = \mathcal{O}(X)$ the category of open sets in X. If $u_1, u_2 \in Ob(\mathcal{C})$, then $u_1 \coprod u_2 \notin Ob(\mathcal{C})$, $\{u_1, u_2\}$is a covering of u if $u_1 \bigcup u_2 = u$.

Question. How to associate to $\{u_1, u_2\}$ a subfunctor $v \subset h_u$?

$$v = \operatorname{coeq}\left\{ h_{u_1 \cap u_2} \underset{i_2}{\overset{i_1}{\rightrightarrows}} h_{u_1} \coprod h_{u_2} \right\} \hookrightarrow h_u$$

is a covering sieve.

Definition 8.3. A *sheaf* on a Grothendieck site $(\mathcal{C}, \mathscr{T})$ is a presheaf $F : \mathcal{C}^{op} \to \mathbf{Set}$ satisfying the sheaf axiom: for any $X \in Ob(\mathcal{C})$ and $u \in \operatorname{cov}(X)$, the inclusion $u \hookrightarrow h_X$ induces a bijection of sets

$$F(X) \cong \operatorname{Hom}_{\mathbf{Pr}(\mathcal{C})}(h_X, F) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}}(u, F).$$

Fact 8.1. *The inclusion functor $i : \mathbf{Sh}(\mathcal{C}) \hookrightarrow \mathbf{Pr}(\mathcal{C})$ has a left adjoint the shifification functor $a : \mathbf{Pr}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{C})$ which is exact in the sense that it preserves all finite limits. For $F \in Ob(\mathbf{Pr}(\mathcal{C}))$, aF is called the associated sheaf of F.*

Notation. Let $\mathbf{sPr}(\mathcal{C})$ be the category of simplicial presheaves, i.e. simplicial objects in $\mathbf{Pr}(\mathcal{C})$.

$$\begin{aligned} \mathbf{sPr}(\mathcal{C}) &= \mathbf{Fun}(\Delta^{op}, \mathbf{Pr}(\mathcal{C})) \\ &\cong \mathbf{Fun}(\Delta^{op}, \mathbf{Fun}(\mathcal{C}^{op}, \mathbf{Set})) \\ &\cong \mathbf{Fun}(\Delta^{op} \times \mathcal{C}^{op}, \mathbf{Set}) \\ &\cong \mathbf{Fun}(\mathcal{C}^{op}, \mathbf{Fun}(\Delta^{op}, \mathbf{Set})) \\ &= \mathbf{Fun}(\mathcal{C}^{op}, \mathbf{sSet}) \end{aligned}$$

Thus any object $F \in Ob(\mathbf{sPr}(\mathcal{C}))$ can be viewed as a functor $F : \mathcal{C}^{op} \to \mathbf{sSet}$.

Definition 8.4. Let $f : F \Rightarrow G$ be a morphism of presheaves in $\mathbf{sPr}(\mathcal{C})$. We say that

- f is a *global weak equivalence* if for any $X \in Ob(\mathcal{C})$, $(f_X : FX \to GX) \in \mathrm{WE}(\mathbf{sSet})$.
- f is a *global fibration* if for any $X \in Ob(\mathcal{C})$, $(f_X : FX \to GX) \in \mathrm{Fib}(\mathbf{sSet})$.
- f is a *global cofibration* if it has LLP with respect to all acyclic fibration.

This makes $\mathbf{sPr}(\mathcal{C})$ a proper cofibrantly generated model category.

It does not depend on $\mathscr{T}$ at all. We want to refine this such that it depends on $\mathscr{T}$. For this, we need to introduce homotopy sheaves of $F \in Ob(\mathbf{sPr}(\mathcal{C}))$.

Given $F : \mathcal{C}^{op} \to \mathbf{sSet}$, define the presheaf $\tilde{\pi}_0(F) \in \mathbf{Pr}(\mathcal{C})$ by

$$\begin{tikzcd} \tilde{\pi}_0(F) : \mathcal{C}^{op} \arrow[r, "F"] & \mathbf{sSet} \arrow[rr, "\pi_0"] \arrow[dr, "|-|"'] & & \mathbf{Set} \\ & & \mathbf{Top} \arrow[ur, "\pi_0"'] & \end{tikzcd}$$

For $X \in Ob(\mathcal{C})$ choose $p \in F(X)_0$ a 0-simplex in $F(X)$ and for $i \geq 1$, define

$$\begin{array}{rccc} \tilde{\pi}_i(F,p) : & (\mathcal{C}/X)^{op} & \longrightarrow & \mathbf{Gr}(\subseteq \mathbf{Set}) \\ & \left(Y \xrightarrow{f} X\right) & \longmapsto & \pi_i(|F(Y)|, f^*(p)) \end{array}$$

where

$$\begin{array}{rccc} f^* : & F(X) & \to & F(Y) \\ & p & \mapsto & f^*(p) \end{array}$$

Denote $\pi_0(F) \coloneqq a\tilde{\pi}_0(F), \pi_i(F) \coloneqq a\tilde{\pi}_i(F,p)$ and refine to these as homotopy sheaves of F.

Definition 8.5. (local model structure on $\mathbf{sPr}(F)$)) Given $f : F \Rightarrow G$ in $\mathbf{sPr}(\mathcal{C})$ we say

- f is a *local weak equivalence* if
  1. $\pi_0(f) : \pi_0(F) \to \pi_0(G)$ is an isomorphism of sheaves, and
  2. For any $X \in Ob(\mathcal{C})$, $p \in F(X)_0, i \geq 1$, and f induces isomorphisms $\pi_i(F,p) \xrightarrow{\sim} \pi_i(G, f^*(p))$.
- f is a *local cofibration* if it is a global cofibration.
- f is a *local fibration* if it has RLP with respect to all local cofibrations.

Theorem 8.2. *[R. Jardine, A. Joyal]* $\mathbf{sPr}(\mathcal{C})$ *equipped with local weak equivalences, local cofibrations, local fibrations, is a model category. We call it the local model structure on* $\mathbf{sPr}(\mathcal{C})$.

Hypercovering

Question. What are fibrant objects?

The key is the notion of hypercovering [Verdier].

Definition 8.6. A *hypercovering* of $X \in Ob(\mathcal{C})$ is a simplicial presheaf $H \in Ob(\mathbf{sPr}(\mathcal{C}))$ with morphisms $p : H \to h_X$ (covering map) such that

1. For any $n \geq 0$, the presheaf $H_n : \mathcal{C}^{op} \xrightarrow{H} \mathbf{sSet} \xrightarrow{(-)_n} \mathbf{Set}$ is a disjoint union of representable presheaves, i.e. there exists $X_{n,i} \in Ob(\mathcal{C})$ such that $H_n = \coprod_i h_{X_{n,i}}$.

2. For any $n \geq 0$, the map of presheaves

$$H_n \cong \mathrm{Hom}(\Delta[n], H) \longrightarrow \mathrm{Hom}(\partial\Delta[n], H) \times_{\mathrm{Hom}(\partial\Delta[n], h_X)} \mathrm{Hom}(\Delta[n], h_X)$$

   induces an epimorphism of associated sheaves.

Remark 8.2. $\Delta[n]$ is the standard n-simplex

$$\begin{aligned} \Delta[n] : \quad \Delta^{op} &\to \mathbf{Set} \\ [k] &\mapsto \mathrm{Hom}_\Delta([k], [n]) \end{aligned}$$

and

$$\begin{aligned} \mathrm{Hom}(\Delta[n], H) : \quad \mathcal{C}^{op} &\to \mathbf{Set} \\ X &\mapsto \mathrm{Hom}(\Delta[n], H(X)) \end{aligned}$$

Remark 8.3. Condition 2 can be equivalently restated as local lifting property: For any $Y \in Ob(\mathcal{C})$, and any commutative diagram

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] & H(Y) \arrow[d, "p"] \\ \Delta[n] \arrow[r] & h_X(Y) = \mathrm{Hom}(Y, X) \end{tikzcd}$$

there is a covering sieve $u \in \mathrm{cov}(Y)$ such that for any $(f : U \to Y) \in u(Y) \subset \mathrm{Hom}(Y, X)$ there is a morphism $\Delta[n] \to H(u)$ such that

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] & H(u) \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dotted] & h_X(Y) \end{tikzcd}$$

This is a local analogue of the lifting property characterizing acyclic fibration of $\mathbf{sSet}$. In particular, $p : H \to h_X$ is a local weak equivalence.

Remark 8.4. If $n \geq 2$, condition 2 can be restated as: $H_n \to \mathrm{Hom}(\partial\Delta[n], H)$ induces an epimorphism of associated sheaves. Note this statement is independent of X nor covering map.

Let $F \in Ob(\mathbf{sPr}(\mathcal{C}))$. For any $X \in Ob(\mathcal{C})$ and hypercovering $p : H \to h_X$, define an coaugmented cosimplicial diagram of simplicial sets

$$\begin{array}{rccc} F(H_*): & \Delta & \to & \mathbf{Set} \\ & [n] & \mapsto & F(H_n) \end{array}$$

as follows.

For $n \geq 0$, $(p : H \to h_X) \Longleftrightarrow \{p_n : H_n \to h_X\}$ where $h_X \in \mathbf{Pr}(\mathcal{C}) \hookrightarrow \mathbf{sPr}(\mathcal{C})$.

Applying $\mathrm{Hom}(-, F)$ and use condition (1)

$$F(X) \cong \mathrm{Hom}(h_X, F) \xrightarrow{p_n^*(X)} \mathrm{Hom}(H_n, F) = \mathrm{Hom}\left(\coprod h_{X_{n,i}}, F\right) \cong \prod_{F,i} \mathrm{Hom}\left(h_{X_{n,i}}, F\right) \cong \prod F(X_{n,i}) =: F(H_n).$$

Thus, we have a map of cosimplicial diagram in $\mathbf{sSet}$

$$F(X) \to F(H_*)$$

so we have

$$\alpha : F \Rightarrow \underset{[n]\in\Delta}{\mathrm{hocolim}} F(H_n). \tag{11}$$

Theorem 8.3. *[DHI] An object $F \in Ob(\mathbf{sPr}(\mathcal{C}))$ is fibrant if and only if*

1. *For any $X \in Ob(\mathcal{C})$, $F(X)$ is fibrant in $\mathbf{sSet}$, i.e. Kan complex.*

2. *For any $X \in Ob(\mathcal{C})$ and any hypercovering $p : H \to h_X$ over X, 11 is a weak equivalence of simplicial sets.*

Definition 8.7. A simplicial presheaf $F \in Ob(\mathbf{sPr}(\mathcal{C}))$ is called a *stack* if it satisfies 2.

By theorem, the fibrant objects of $\mathbf{sPr}(\mathcal{C})$ are all stacks.

Notation/Terminology.

1. $\mathrm{Ho}(\mathbf{sPr}(\mathcal{C}))$ is the category of stacks (because every object is isomorphic to a fibrant one). Stack, morphisms, isomorphisms of stacks are all referred to $\mathrm{Ho}(\mathbf{sPr}(\mathcal{C}))$.

2. For any $F, F' \in \mathrm{Ho}(\mathbf{sPr}(\mathcal{C}))$, $[F, F'] := \mathrm{Hom}_{\mathrm{Ho}}(F, F')$.

Proposition 8.1. *A presheaf $F : \mathcal{C}^{op} \to$ **Set** (viewed as a constant simplicial presheaf) is a stack if and only if it is a sheaf (in the usual sense).*

Proof. This is because homotopy limits becomes limits and condition 2 becomes the usual sheaf axiom. □

Moral. Condition 2 is homotopy (analogue of) sheaf axiom.

Put differently, a presheaf is fibrant in $\mathbf{sPr}(\mathcal{C})$ if and only if it is a sheaf. Thus homotopy theory of $\mathbf{sPr}(\mathcal{C})$ knows about the relation between sheaves and presheaves.

Corollary 8.1. *Let F be an object in $\mathbf{Pr}(\mathcal{C}) \hookrightarrow \mathbf{sPr}(\mathcal{C})$, and $aF \in \mathbf{Sh}(\mathcal{C})$ is the associated sheaf, then for any $G \in Ob(\mathbf{sPr}(\mathcal{C}))$,*

1. $[G, F] \cong [G, aF] \cong Hom_{\mathbf{sPr}(\mathcal{C})}(F, aF)$.
2. *Moreover, if G is an object in $\mathbf{Pr}(\mathcal{C}) \hookrightarrow \mathbf{sPr}(\mathcal{C})$, then*

$$[G, F] \cong [aG, aF] \cong Hom_{\mathbf{sPr}(\mathcal{C})}(aG, aF) \cong Hom_{\mathbf{Pr}(\mathcal{C})}(G, F)$$

where the last isomorphism follows from the fact that $\mathbf{Sh}(\mathcal{C})$ is a full subcategory of $\mathbf{Pr}(\mathcal{C})$.

Corollary 8.2. *The natural functor*

$$\mathbf{Sh}(\mathcal{C}) \hookrightarrow \mathbf{Pr}(\mathcal{C}) \hookrightarrow \mathbf{sPr}(\mathcal{C}) \xrightarrow{loc} Ho(\mathbf{sPr}(\mathcal{C}))$$

is full embedding. Moreover, it has left adjoint

$$\begin{array}{rccccc} \pi_0 : & Ho(\mathbf{sPr}(\mathcal{C})) & \to & \mathbf{Pr}(\mathcal{C}) & \xrightarrow{a} & \mathbf{Sh}(\mathcal{C}) \\ & F & \mapsto & \tilde{\pi}_0(F) & \mapsto & a\tilde{\pi}_0(F) \end{array}$$

so the category of stacks is just extension of the category of usual sheaves.

Warning. In general, sheaves of sets are exactly stacks, sheaves of simplicial sets are far from being stacks.

Example 8.3. Take a sheaf of groups $H : \mathcal{C}^{op} \to \mathbf{Gr}$ then

$$\begin{array}{rccc} BG : & \mathcal{C}^{op} & \to & \mathbf{sSet} \\ & X & \mapsto & N_* G(X) = \{G(X)^n\}_{n \geq 0} \end{array}$$

is a sheaf of simplicial sets, but not a stack.

Stacks are generalization of sheaves.

8.2 Example: Generalized Manifolds

We will consider simplicial presheaves on smooth manifolds following [FH13] and Joyce 2011.

Let $\mathcal{C} = \mathbf{Man}$ be the category of finite dimensional smooth manifolds over $\mathbb{R}$ with smooth maps. $\mathcal{C}$ has a natural Grothendieck topology, where coverings are usual open coverings of manifolds, , i.e. a covering of X is a collection $\mathcal{U} = \{U_i\}_{i\in I}$ of open subsets such that $\bigcup_{i\in I} U_i = X$. Associated to $\mathcal{U}$ a sieve

$$u = \mathrm{eq}\left\{ \coprod_{i,j} h_{U_i\cap U_j} \rightrightarrows \coprod_{i\in I} h_{U_i} \right\}.$$

We identify X as representable sheaves

$$\begin{aligned} h_X : \quad \mathbf{Man}^{op} &\to \mathbf{Set} \\ Y &\mapsto h_X(Y) = \mathrm{Hom}(Y, X) \end{aligned}$$

so $\mathbf{Pr}(\mathbf{Man})$ is the category of generalized manifolds in the sense that every $F : \mathbf{Man}^{op} \to \mathbf{Set}$ is determined by its action on test objects (manifolds) $X \mapsto F(X)$. The point is to extend differential geometric constructions from usual to generalized manifolds.

1. Differential forms on generalized manifolds.

For $p \geq 0$, and fix a manifold X, consider the set $\Omega^p(X)$ of smooth p-forms on X. In local coordinates, $\Omega^p(X) \ni \omega = \sum f_{i_1,\cdots,i_p}(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_p}$. Since for any map $f : X \to Y$ between manifolds, the pull-back map gives $f^p : \Omega^p(Y) \to \Omega^p(X)$, $\Omega^p : \mathbf{Man}^{op} \to \mathbf{Set}$ is a presheaf on $\mathbf{Man}$ (generalized manifold).

For $p = 0$, $\Omega^0(X) = C^\infty(X)$ is a sheaf of sets, so is $\Omega^p(X), \forall p \geq 1$. Thus $\Omega^p : \mathbf{Man}^{op} \to \mathbf{Set}$ is a stack on $\mathbf{Man}$.

2. G-connection

Let G be a Lie group, a group object in $\mathbf{Man}$.

Let $\pi : P \to X$ be a principal G-bundle on X. Recall π is a locally trivial fibre bundle such that P is equipped with a right G-action

$$P \times G \to P$$

such that each fibre $\pi^{-1}(x), x \in X$ is preserved under the action and the restriction of the G-action to

$\pi^{-1}(x)$ is free and transitive, i.e. for any $p \in \pi^{-1}(x)$, the action map

$$\begin{array}{ccc} G & \to & \pi^{-1}(x) \\ g & \mapsto & p \cdot g \end{array}$$

is a diffeomorphism.

A *connection* in P is a natural way to relate different fibres. More precisely, a G-connection is a direct sum decomposition of the tangent space T_pP for each $p \in P$: $T_pP \cong H_p \oplus V_p$ (which is invariant under the action of G) where

$$V_p = T_p\pi^{-1}(\pi(p)) \hookrightarrow T_pP$$

is the subspace of *vertical vectors*, contained canonically in T_pP. Thus a connection is determined by the choice of subspace H_p (*horizontal vectors*) for each $p \in P$, i.e. $p \mapsto H_p$.

Definition 8.8. [Ehresmann] A G-connection a distribution on the total space (i.e. a subbundle of T_*P) which is G-invariant and transverse to fibers of π.

Note for each $p \in P$, the differential (at $e \in G$) of the natural map

$$\begin{array}{cccc} i_p: & G & \to & \pi^{-1}(\pi(p)) \\ & g & \mapsto & p \cdot g \end{array}$$

identifies

$$\mathfrak{g} = T_eG \cong T_p\pi^{-1}(\pi(p)) =: V_p$$

A G-connection is thus determined by the projection

$$\Theta_p : T_pP \to V_p \cong \mathfrak{g}$$

such that $\Theta_p \circ d_pi_p = \mathrm{Id}$, i.e. $\Theta_p \in \Omega^1(P) \otimes \mathfrak{g} =: \Omega^1(P, \mathfrak{g})$ is a $\mathfrak{g}$-valued 1-form (connection form) on P.

Formally, Θ is characterized by 2 properties.

1. Θ is G-invariant form, i.e. $\Theta \in \Omega^1(P, \mathfrak{g})^G$, i.e. for $g \in \mathfrak{g}$, $R_g^*(\Theta) = \mathrm{Ad}_{g^{-1}}(\Theta)$ where R_g^* is induced by the right G-action
$$\begin{array}{cccc} R_g: & P & \to & P \\ & p & \mapsto & p \cdot g \end{array}$$
2. For any $p \in P$, $i_p^*(\Theta) = \Theta_{MC}$ where i_p^* is induced by $i_p : G \to \pi^{-1}(\pi(p)) \hookrightarrow P$ and Θ_{MC} is the

Maurer-Cartan form.

(If G is a matrix group, then $\Theta_{MC} = g^{-1}dg$.)

Goal. To classify principal G-bundles with G-connections.

Example 8.4. Fix a Lie group G. Define

$$\begin{array}{rccc} F: & \mathbf{Man}^{op} & \to & \mathbf{Set} \\ & X & \mapsto & F(X) \end{array}$$

by

$$F(X) = \{\text{equivalenc classes of } G\text{-connections on } X\} = \{(\pi : P \to X, \Theta)\} / \sim$$

where $(\pi, \Theta) \sim (\pi', \Theta')$ if there exists isomorphism $\varphi : P \to P'$ such that

$$\begin{tikzcd} P \arrow[rr, "\varphi", "\sim"'] \arrow[dr, "\pi"'] & & P' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

such that $\Theta = \varphi^* (\Theta')$.

F is a presheaf but not a sheaf. Indeed, consider $X = \mathbb{S}^1$ and take covering $\{U_1 = \mathbb{S}^1 \setminus \{i\}, U_2 = \mathbb{S}^1 \setminus \{-i\}\}$, then $U_1 \cap U_2 = I_1 \cup I_2$ a disjoint union of two open intervals. If F is a sheaf, then we have a Cartesian diagram

$$\begin{tikzcd} F(\mathbb{S}^1) \arrow[r] \arrow[d] & F(U_1) \arrow[d] \\ F(U_2) \arrow[r] & F(I_1 \cup I_2) \end{tikzcd}$$

Since (flat) G-bundles are determined by equivalence classes of holonomy representations $\rho : \pi_1(X) \to G$ under the adjoint action. The equivalence classes of holonomy representations in this case are

$$\begin{tikzcd} G/G \arrow[r] \arrow[d] & \{\bullet\} \arrow[d] \\ \{\bullet\} \arrow[r] & \{\bullet, \bullet\} \end{tikzcd}$$

which is not Cartesian.

Stack G-connection

Idea. Instead of classifying connections up to isomorphisms we will classify connections together with isomorphisms.

Define a presheaf of groupoids $B_\nabla G$ and $E_\nabla G$ by constructing two categories for each $X \in Ob\,(\mathbf{Man})$

$Ob\,(B_\nabla G\,(X))$**:** (π, Θ) where $\pi : P \to X$ is a principal G-bundle, $\Theta \in \Omega^1\,(P) \otimes \mathfrak{g}$ is a G-connection.

$Mor(B_\nabla G\,(X))$**:** commutative diagram

$$\begin{tikzcd} P \arrow[rr, "\varphi", "\sim"'] \arrow[dr, "\pi"'] & & P' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

such that $\Theta = \varphi^*\,(\Theta')$.

$Ob\,(E_\nabla G\,(X))$**:** $(\pi, \Theta, s : X \to P)$ where $s : X \to P$ is a global section (trivialization), $\pi s = \mathrm{Id}$.

$Mor(B_\nabla G\,(X))$**:** commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "s"'] \arrow[dr, "s'"] & \\ P \arrow[rr, "\varphi", "\sim"'] \arrow[dr, "\pi"'] & & P' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

such that $\Theta = \varphi^*\,(\Theta')$.

Definition 8.9. The simplicial presheaves $B_\nabla G$ and $E_\nabla G$ are defined by

$$\begin{array}{rccc} B_\nabla G : & \mathbf{Man}^{op} & \longrightarrow & \mathbf{sSet} \\ & X & \longmapsto & \mathcal{N}_*\,(B_\nabla G\,(X)) \end{array}$$

and

$$\begin{array}{rccc} E_\nabla G : & \mathbf{Man}^{op} & \longrightarrow & \mathbf{sSet} \\ & X & \longmapsto & \mathcal{N}_*\,(E_\nabla G\,(X)) \end{array}$$

Note that there is a natural map of simplicial presheaves

$$p : E_\nabla G \to B_\nabla G$$

induced by the forgetful functor of presheaves of groupoids $(\pi, \Theta, s) \mapsto (\pi, \Theta)$.

Theorem 8.4. *[FH13]*

1. *Both $B_\nabla G$ and $E_\nabla G$ are stacks (i.e. satisfying homotopy sheaf axiom).*

2. *$E_\nabla G$ is weak equivalent to the sheaf $\Omega^1 \otimes \mathfrak{g}$, viewed as a discrete simplicial presheaf*

$$E_\nabla G \simeq \Omega^1 \otimes \mathfrak{g}.$$

Proof. For (2), the weak equivalence is induced (at the level of groupoids) by the two functors

$$\alpha : E_\nabla G \rightleftarrows \Omega^1 \otimes \mathfrak{g} : \beta$$

defined by

$$\begin{array}{rccc} \alpha_X : & E_\nabla G\,(X) & \longrightarrow & \Omega^1\,(X) \otimes \mathfrak{g} \\ & (\pi, \Theta, s) & \longmapsto & s^*\,(\Theta) \end{array}$$

and

$$\begin{array}{rccc} \beta_X : & \Omega^1\,(X) \otimes \mathfrak{g} & \longrightarrow & E_\nabla G\,(X) \\ & \omega & \longmapsto & (\pi_\omega, \Theta_\omega, s_\omega) \end{array}$$

where $\pi_\omega = \pi_X : X \times G \to X$ the canonical projection gives a trivial G-bundle and

$$\begin{array}{rccc} s_\omega : & X & \longrightarrow & X \times G \\ & x & \longmapsto & (x, e) \end{array}$$

and $\Theta_\omega = \pi_G^* \omega + \pi_G^* \Theta_{MC}$ where Θ_{MC} is the Maurer-Cartan 1-form on G.

Note $\alpha_X \cdot \beta_X = \mathrm{Id}_X$ and

$$\begin{array}{rccccc} \beta_X \cdot \alpha_X : & E_\nabla G\,(X) & \to & \Omega^1\,(X) \otimes \mathfrak{g} & \to & E_\nabla G\,(X) \\ & (\pi, \Theta, s) & \mapsto & \omega = s^*\,(\Theta) & \mapsto & (\pi_\omega, \Theta_\omega, s_\omega) \end{array}$$

we have a commutative diagram

$$\begin{tikzcd} E_\nabla G\,(X) \arrow[r, "\beta_X \cdot \alpha_X"] \arrow[d, "\wr"'] & E_\nabla G\,(X) \arrow[d, "\wr"', "\varphi"] \\ E_\nabla G\,(X) \arrow[r, "\mathrm{Id}"] & E_\nabla G\,(X) \end{tikzcd} \qquad \begin{tikzcd} (\pi, \Theta, s) \arrow[r, mapsto] \arrow[d, mapsto] & (\pi_\omega, \Theta_\omega, s_\omega) \arrow[d, mapsto] \\ (\pi, \Theta, s) \arrow[r, mapsto] & (\pi, \Theta, s) \end{tikzcd}$$

where

$$\begin{tikzcd} X \times G \arrow[rr, "\varphi", "\sim"'] \arrow[dr, "\pi_\omega"'] & & P \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

Thus $E_\nabla G \simeq \Omega^1 \otimes \mathfrak{g}$. □

Differential Forms on Stacks

Idea. If we want to extend a natural geometric construction on a manifold X, we have to express this construction in terms of h_X and replace h_X by arbitrary F.

Note if we identify a smooth manifold X with its corresponding presheaf $h_X : \mathbf{Man}^{op} \to \mathbf{Set}$, then by Yoneda lemma,

$$\mathrm{Hom}_{\mathbf{Pr(Man)}}\left(h_X, \Omega^p\right) \cong \Omega^p\left(X\right).$$

This suggests that for any $F : \mathbf{Man}^{op} \to \mathbf{Set}$, we may define

$$\Omega^\bullet\left(F\right) := \mathrm{Hom}_{\mathbf{Pr(Man)}}\left(F, \Omega^\bullet\right).$$

For example, take $F = \Omega^q$ for fixed $q \geq 0$, and look at $\Omega^p\left(\Omega^q\right) = \mathrm{Hom}_{\mathbf{Pr(Man)}}\left(\Omega^q, \Omega^p\right)$ differential p-forms on differential q-forms.

Explicitly, an element, $\tau \in \Omega^p\left(\Omega^q\right)$ is a natural transformation of from q-forms to p-forms on manifolds, in the sense, for any $f : X \to Y$,

$$\begin{tikzcd} \Omega^q\left(Y\right) \arrow[r, "\tau_Y"] \arrow[d, "\Omega^q(f)"'] & \Omega^p\left(Y\right) \arrow[d, "\Omega^p(f)"] \\ \Omega^q\left(X\right) \arrow[r, "\tau_X"] & \Omega^p\left(X\right) \end{tikzcd}$$

Example 8.5. (Topology) Take $p = q$ and consider $\tau = \omega^p : \Omega^p \Rightarrow \Omega^p$ given by $\omega^p_X = \mathrm{Id}_{\Omega^p(X)}$, then $\tau \in \Omega^p\left(\Omega^p\right)$.

Take $q = 1$ and consider $\Omega^p\left(\Omega^1\right)$.

Theorem 8.5. *[FH13]*

1. *The space* $\Omega^p\left(\Omega^1\right) = \begin{cases} \mathit{span}_{\mathbb{R}}\left\{\overbrace{d\omega^1 \wedge \cdots \wedge d\omega^1}^{p/2}\right\}, & p \text{ even,} \\ \mathit{span}_{\mathbb{R}}\left\{\omega^1\underbrace{d\omega^1 \wedge \cdots \wedge d\omega^1}_{(p-1)/2}\right\}, & p \text{ odd.} \end{cases}$

2. *The de Rham complex of Ω^1 looks as*

$$\Omega^\bullet\left(\Omega^1\right)=\left[\mathbb{R}^1 \xrightarrow{0} \mathbb{R}^1 \xrightarrow{1} \mathbb{R}^1 \xrightarrow{0} \mathbb{R}^1 \xrightarrow{1} \cdots\right]$$

so

$$H^p_{DR}\left(\Omega^1,\mathbb{R}\right)=\begin{cases}\mathbb{R}^1, & p=0,\\ 0, & p\neq 0.\end{cases}$$

Definition 8.10. For $\mathcal{F}\in\mathrm{Ho}\left(\mathbf{sPr}\left(\mathbf{Man}\right)\right)$ any simplicial presheaf, define de Rham complex

$$\Omega^\bullet\left(\mathcal{F}\right):=\left[\mathcal{F},\Omega^\bullet\right]=\left\{\left[\mathcal{F},\Omega^0\right]\xrightarrow{d}\left[\mathcal{F},\Omega^1\right]\xrightarrow{d}\cdots\right\}$$

where $[-,-]:=\mathrm{Hom}_{\mathrm{Ho}(\mathbf{sPr}(\mathbf{Man}))}\left(-,-\right)$.

Since each Ω^p is a sheaf, hence a fibrant object in $\mathbf{sPr}\left(\mathbf{Man}\right)$, by corollary 8.1,

$$\begin{aligned}\Omega^\bullet\left(\mathcal{F}\right) &= \left[\mathcal{F},\Omega^\bullet\right]\\ &= \mathrm{Hom}_{\mathbf{sPr}(\mathbf{Man})}\left(\mathcal{F},\Omega^\bullet\right)\\ &\cong \mathrm{eq}\left\{\mathrm{Hom}_{\mathbf{Pr}(\mathbf{Man})}\left(\mathcal{F}_0,\Omega^\bullet\right)\underset{d_1^*}{\overset{d_0^*}{\rightrightarrows}}\mathrm{Hom}_{\mathbf{Pr}(\mathbf{Man})}\left(\mathcal{F}_1,\Omega^\bullet\right)\right\}\\ &= \mathrm{Ker}\left[\left(d_1^*-d_0^*\right):\Omega^\bullet\left(\mathcal{F}_0\right)\to\Omega^\bullet\left(\mathcal{F}_1\right)\right]\end{aligned}$$

where

$$\mathcal{F}:=\left\{\mathcal{F}_0\underset{d_1}{\overset{d_0}{\leftleftarrows}}\mathcal{F}_1 \leftleftarrows\!\!\!\!\leftarrow \mathcal{F}_2\cdots\right\}$$

and

$$\Omega^\bullet:=\left\{\Omega^\bullet\leftleftarrows\Omega^\bullet\leftleftarrows\!\!\!\!\leftarrow\Omega^\bullet\cdots\right\}$$

From this definition it's immediate that $\Omega^\bullet\left(\mathcal{F}\right)$ is a homotopy invariant construction (while we cannot see this from the usual definition).

Universal G-connection

Let $F:\mathbf{Man}^{op}\to\mathbf{Set}$ be a presheaf on $\mathbf{Man}$ and G be a Lie group.

A (left) G-action on F is defined by

$$a:G\times F\longrightarrow F$$

which on test manifold

$$a_X : \mathrm{Hom}\,(X,G) \times F\,(X) \longrightarrow F\,(X)$$

is the action of the over group $\mathrm{Hom}\,(X,G)$ on $F\,(X)$.

Associated to a G-action of F is a presheaf of action groupoid $G \ltimes F$ defined objectwise by the nerve of the category associated to a manifold X with

Objects: $Ob\,(G \ltimes F\,(X)) \coloneqq F\,(X)$

Morphisms: $Mor\,(G \ltimes F\,(X)) = \mathrm{Hom}\,(X,G) \times F\,(X)$ with

$$\begin{array}{rccc} s: & Mor\,(G \ltimes F\,(X)) & \longrightarrow & Ob\,(G \ltimes F\,(X)) \\ & (g,\xi) & \longmapsto & \xi \end{array} \qquad \begin{array}{rccc} t: & Mor\,(G \ltimes F\,(X)) & \longrightarrow & Ob\,(G \ltimes F\,(X)) \\ & (g,\xi) & \longmapsto & a_X\,(g,\xi) = g\cdot\xi \end{array}$$

Thus

$$\begin{array}{rccc} G \ltimes F: & \mathbf{Man}^{op} & \longrightarrow & \mathbf{sSet} \\ & X & \longmapsto & \mathcal{N}_*\,(G \ltimes F\,(X)) \end{array}$$

Apply this to $F \coloneqq \Omega^1 \otimes \mathfrak{g}$, this has the obvious right G-action

$$a : \left(\Omega^1 \otimes \mathfrak{g}\right) \times G \longrightarrow \left(\Omega^1 \otimes \mathfrak{g}\right)$$

defined objectwise by

$$\begin{array}{rccc} a_X: & \left(\Omega^1\,(X) \otimes \mathfrak{g}\right) \times \mathrm{Hom}\,(X,G) & \longrightarrow & \left(\Omega^1\,(X) \otimes \mathfrak{g}\right) \\ & (\omega, g) & \longmapsto & g^*\,(\Theta_{MC}) + \mathrm{Ad}_{g^{-1}}\,(\omega) \end{array}$$

Thus

$$B^{\mathrm{triv}}_{\nabla}G \coloneqq G \ltimes \left(\Omega^1 \otimes \mathfrak{g}\right) \in Ob\,(\mathbf{sPr}\,(\mathbf{Man}))\,.$$

There is a natural map of presheaves of groupoids

$$\psi : B^{\mathrm{triv}}_{\nabla}G \to B_{\nabla}G$$

defined on objects as

$$\begin{array}{rccc} \psi_X: & \Omega^1\,(X) \otimes \mathfrak{g} & \longrightarrow & \{(\pi,\Theta)\} \\ & \omega & \longmapsto & (\pi_X : X \times G \to X, \Theta_\omega = \pi_G^*\,(\omega) + \pi_G^*\,(\Theta_{MC})) \end{array}$$

Proposition 8.2. *ψ is a weak equivalence.*

Then we have a commutative diagram

$$\begin{tikzcd} \Omega^1 \otimes \mathfrak{g} \arrow[r, two heads, "\mathrm{can}"] \arrow[d, "\beta"', "\simeq"] & B_{\nabla}^{\mathrm{triv}} G \arrow[d, "\psi", "\simeq"'] \\ E_{\nabla} G \arrow[r, "p"] & B_{\nabla} G \end{tikzcd}$$

Hence $E_{\nabla}G \xrightarrow{p} B_{\nabla}G$ is a generalized principal G-bundle.

Note β is G-equivariant and G acts on $E_{\nabla}G$ freely.

Recall any map

$$\begin{aligned} \alpha : \quad E_{\nabla}G &\longrightarrow \Omega^1 \otimes \mathfrak{g} \\ (\pi, \Theta, s) &\longmapsto s^* (\Theta) \end{aligned}$$

is equivalent to

$$\alpha \in \left[E_{\nabla}G, \Omega^1 \otimes \mathfrak{g}\right] \cong \left[E_{\nabla}G, \Omega^1\right] \otimes \mathfrak{g} \cong \Omega^1 \left(E_{\nabla}G\right) \otimes \mathfrak{g},$$

so $\Theta^{\mathrm{un}} := \alpha$ can be viewed as a G-connection on $E_{\nabla}G \xrightarrow{p} B_{\nabla}G$.

Theorem 8.6. *Θ^{un} is the universal G-connection in the strong sense: given any (π, Θ), there is a unique classifying map $\left(f, \bar{f}\right)$ such that*

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\pi"'] & E_{\nabla} G \arrow[d, "p"] \\ X \arrow[r, "\bar{f}"] & B_{\nabla} G \end{tikzcd}$$

and $\Theta = f^* \left(\Theta^{un}\right)$.

Remark 8.5. This is to be compared with classical classification of principal G-bundles (without connection).

For a Lie group G, there exists two infinite-dimensional spaces EG and BG defined (up to homotopy)

$$\begin{tikzcd} P \cong f^* (EG) \arrow[r] \arrow[d] & EG \arrow[d, "p"] \\ X \arrow[r, "f"] & BG \end{tikzcd}$$

where f is also defined up to homotopy.

One point in common: classifying spaces are not usual manifolds. (infinite-dimensional/generalized).

Theorem 8.7.

1. $\Omega^{\bullet}(E_{\nabla}G) \cong \mathbb{W}(\mathfrak{g}) = \Lambda(\mathfrak{g}^*) \otimes Sym(\mathfrak{g}^*)$.

2. $\Omega^{\bullet}(B_{\nabla}G) \cong Sym(\mathfrak{g}^*)^G$.

3. *There is a canonical map* $Sym(\mathfrak{g}^*)^G \hookrightarrow \Lambda(\mathfrak{g}^*) \otimes Sym(\mathfrak{g})$ *called the universal Chern-Weil map.*

Problem 8.1. The representation homology can be defined in smooth category, then it has a new argument ("coefficients") given by simplicial presheaves on **Man**.

Given $X \in Ob(\mathbf{Top}_{0,*}), G \in Ob(\mathbf{LieGr}), \mathcal{F} \in \mathbf{sPr}(\mathbf{Man})$, define $HR_*(X, G, \mathcal{F})$ such that when $\mathcal{F} = \Omega^0 = \mathcal{O}(G)$, then $HR_*(X, G, \mathcal{O}) = HR_*(X, G)$.

$HR_*(X, G, \mathcal{F})$ is a cochain differential graded algebra.

Solution. $HR_*(X, G, \mathcal{F}) \coloneqq B(\mathbb{G}(X, \mathcal{G}, \mathcal{F}G))$ where B is the May bar construction and $\mathbb{G}(X) : \mathcal{G}^{op} \to$ **sSet**,

$$\mathcal{F}G : \mathcal{G} \xrightarrow{\mathbb{G}} \mathbf{Man}^{op} \xrightarrow{\mathcal{F}} \mathbf{sSet}\ .$$

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