MATH7510 3-Dimensional Manifolds

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Contents

CHAPTER 1

Overview

I'm typing these very sketchy notes simply for the purpose of summarizing the topics covered in class. The contents and orders in the notes will be mostly based on the lectures, but may be subject to changes for reading purposes. Some of the proofs might be skipped with references provided. There will inevitably be many typos and wrong comments, so I am very welcome to any corrections and comments.

1. 3-Manifolds

When topologists study manifolds, there are three types to be considered - topological, peicewise-linear (PL), and smooth manifolds. And it is natural to study the relations among them. A pleasant feature of 3-manifolds, in contrast to higher dimensions, is that there is no essential difference between smooth, piecewise linear (PL), and topological manifolds.

- The triangulation of 3-manifolds was first proved by Moise [**M2**] in 1951 and then by Bing [**B1**], in 1957. Every topological 3-manifold can be triangulated as a simplicial complex whose combinatorial type is unique up to subdivision. And every triangulation of a 3-manifold can be taken to be a smooth triangulation in some differential structure on the manifold, unique up to diffeomorphism. Thus every topological 3-manifold has a unique smooth structure, and the classifications up to diffeomorphism and homeomorphism coincide. The reference we will use here is [**H1**] by AJS Hamilton, which uses Kirby's "torus trick," which comes up in other contexts.
- Any PL structure is uniquely smoothable, which was proved in [**T3**], where Theorem 3.10.8 shows the existence of smoothing, and Theorem 3.10.9 shows the uniqueness of smoothing.

2. Decomposition of 3-manifolds

There are two ways of decomposing 3-manifolds.

- The prime decomposition.
- The tours decomposition.

3. Homotopy Properties

The homotopy of three-manifolds can be "realized nicely" in many ways.

- The Loop Theorem and Dehn's Lemma, which says if an embedded 2-sided surface in M is not π_1-injective, then it is compressible.
- The Sphere Theorem, which says if a 3-manifold M has nontrivial π_2, there is an essential embedded 2-sphere in M.

4. Constructions

We will mainly focus on compact 3-manifolds (sometimes with boundaries) which we know a lot about.

- There is a construction of 3-manifolds via fiber bundle, which is morally a "twisted" fiber product of spaces, and is locally a product of spaces.
- The Prime Decomposition states that every compact orientable 3-manifold M factors as a connected sum of primes.
- The JSJ decomposition gives another way of decomosping a 3-manifold.
- The geometrization conjecture states that every closed 3-manifold can be decomposed in a canonical way into pieces that each have one of eight types of geometric structure [**T1**].

Example 4.1. (1) $\mathbb{T}^3$ is modeled by $\mathbb{E}^3$, modulo the action of the integer lattice $\mathbb{Z}^3$.

(2) Sol is one of the 8 Thurston geometries which uniformizes torus bundles which fiber over the circle with Anosov monodromy.

CHAPTER 2

Introduction to 3-Manifolds

1. Triangulation of 3-Manifolds

1.1. References. This part is presented by Chaitanya Tappu and the main references for this section are [**H1**] (referring to [**Z**]), [**T3**] Section 3.1, 3.2, 3.9 and 3.10 (about smoothing), and [**RS**]. In [**H1**].

1.2. Definitions. A **cove** of a set X is a collection of subsets of X whose union contains X. If $U = \{U_\alpha : \alpha \in A\}$ is an indexed family of subsets of X, then U is a cover of X if

$$X \subseteq \bigcup_{\alpha \in A} U_\alpha.$$

A cover of a topological space X is **open** if all its members are open sets.

A **refinement** of a cover of a space X is a new cover of the same space such that every set in the new cover is a subset of some set in the old cover. In symbols, the cover $V = \{V_\beta : \beta \in B\}$ is a refinement of the cover $U = \{U_\alpha : \alpha \in A\}$ if and only if, for any V_β in V, there exists some U_α in U such that $V_\beta \subseteq U_\alpha$.

An open cover of a space X is **locally finite** if every point of the space has a neighborhood that intersects only finitely many sets in the cover, i.e. $U = \{U_\alpha : \alpha \in A\}$ is locally finite if and only if, for any X in X, there exists some neighbourhood $V(x)$ of X such that the set

$$\{\alpha \in A : U_\alpha \cap V(x) \neq \varnothing\}$$

is finite.

Definition 1.1. A space is **paracompact** if every open cover has an open refinement that is locally finite.

Definition 1.2. A (topological) n**-manifold** M is a Hausdorff space that is locally homeomorphic to the Euclidean space $\mathbb{R}^n$, and satisfying one of the following equivalent propeties

(1) paracompact, or
(2) metrisable, or
(3) every component is second countable.

Since manifolds are locally Euclidean, they are locally compact, so there exists a cover of paracompact open sets.

Remark. In our case we consider only connected 3-manifolds. The reason that we mainly focus on connected ones is, the property of being locally Euclidean implies being locally path-connected, hence M is a disjoint union of path-components, so in order to classify 3-manifolds it suffices to consider each component.

1.3. Outline.

I. Piecewise-linear (PL) structures: triangulation, handles, polyhedron, PL maps.
II. Statement of existence and uniqueness of the piecewise-linear structures on 3-manifolds. (The same statement holds for dimension 1 and 2, which we will take for granted and keep as a black box.)
III. Statement of handle straightening theorem.
IV. Proof of II using III.
V. Proof of III (Kirby torus trick).

1.4. Piecewise-linear Structure. The philosophy of studying smooth manifolds contains two directions, we can study the (abstract) global topology, or study the locally properties via multivariable calculus. Analogously, we can study PL manifolds via some thing like "PL calculus".

In order to introduce the notion of PL structure, we need to define the notion of simplices. A simplex is a generalization of the notion of a triangle or tetrahedron to arbitrary dimensions.

Definition 1.3. Given $n+1$ points $\{v_0, \cdots, v_n\}$ in $\mathbb{R}^N$, they are called **affinely independent** if

$$\begin{cases} t_0 + \cdots + t_n = 0, \\ t_0 v_0 + \cdots t_n v_n = 0 \end{cases} \implies t_0 = \cdots = t_n = 0.$$

In other words, the vectors $\{v_1 - v_0, \cdots, v_n - v_0\}$ are linearly independent.

Definition 1.4. Let $\{v_0, \cdots, v_n\}$ be $n+1$ points in $\mathbb{R}^N$ which are affinely independent. Then, the affine n-simplex determined by them is the set of points

$$C = \left\{ v = t_0 v_0 + \cdots t_n v_n \;\middle|\; \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i \right\}.$$

$(t_0, \cdots, t_n)$ is called the **barycentric coordinates** of the point v.

In other words, an affine k-complex is a k-dimensional polytope which is the convex hull of its $k+1$ vertices.

Definition 1.5. A **polyhedron** $|K|$ is a collection of affine simplices K such that

- if $\sigma \in K$, then every face of σ is in K.
- if $\sigma \neq \tau \in K$ and $\sigma \cap \tau \neq \emptyset$, then σ and τ intersect as exact one face.
- K is locally finite.

A **subdivision** L of K is a simplicial complex with same polyhedron $|L| = |K|$ such that every simplex in L is contained in a simplex of K.

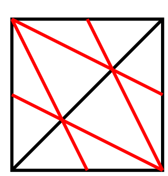


Figure 1. Example of subdivision

A non-example of a polyhedron is the closed unit disk $D^2 \subseteq \mathbb{R}^2$ with vertices the center and every vertex on the boundary, and edges are all the radius connecting the center and a point on the boundary. This is not locally finite.

Example 1.6. (1) $\mathbb{R}^2$ is a union of triangles.

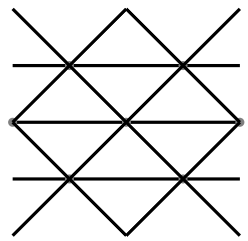
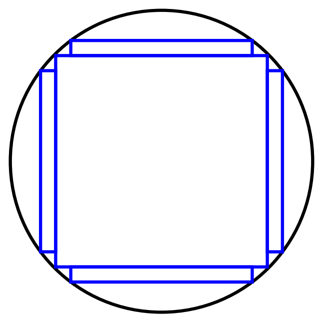


FIGURE 2. Decomposition of $\mathbb{R}^2$

(2) Any open set U in $\mathbb{R}^n$ is a polyhedron, where you can take a countable union of n-cubes contained in U expanding towards the boundary of U.



Definition 1.7. A homeomorphism $f : U \to V$ where $U, V \subseteq \mathbb{R}^N$ are open, is called **piecewise-linear** if U has a polyhedron structure, i.e. we can decompose $U = \bigcup_{i \in I} P_i$ into a (countable) union of affine simplices, such that the restriction of f on each simplex $f|_{P_i}$ is a linear map. We can also similarly define piecewise linear maps.

Exercise 1.8. Give a PL map from the decomposition of $\mathbb{R}^2$ as in the Picture 1 to the decomposition below.

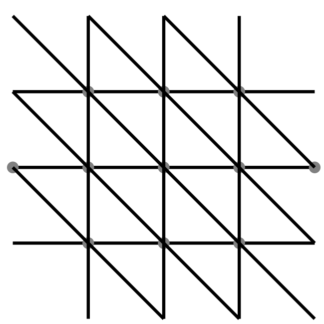


FIGURE 3. Another decomposition of $\mathbb{R}^2$

Proposition 1.9. *(1) If $f : U \to V.g : V \to W$ are two PL morphisms, so is $g \circ f : U \to W$.*
(2) Given $f : U \to \mathbb{R}^n$ and a cover $\{U_\alpha\}$ of U, if $f|_{U_\alpha}$ is PL for every α, then f is PL.
(3) If $f : U \to V$ is PL, so is f^{-1}.

Definition 1.10. A **PL structure** on M^m is a maximal atlas of PL compatible charts $(U_\alpha, \varphi_\alpha)$, i.e. $\{U_\alpha\}$ is an open cover of M and $\varphi : U_\alpha \to \mathbb{R}^m$ is a homeomorphism onto its image, such that all $\varphi_\alpha \circ \varphi_\beta$ are PL homeomorphisms. Similarly, we can define PL maps between PL manifolds.

THEOREM 1.11. *(1) Any topological 3-manifold M^3 has a PL structure.*
(2) If Σ, Σ' are two PL structures on M such that there is a PL homeomorphism $M_\Sigma \to M_{\Sigma'}$, then this map is (topological) isotopic to the identity map on M.

To prove this theorem, we will use the following fact (which we will not prove).

Fact 1.12. The same holds for dimension 1 and 2.

Question 1.13. *How can we build a PL manifold, and how to describe a PL atlas?*

We can build a PL manifold by gluing simplices, and in practice, we will not work with maximal atlas. In order to describe an atlas, we will use triangulation.

1.5. Triangulation.

Definition 1.14. A **triangulation** of a topological space X is a simplicial complex K along with a homeomorphism $|K| \to X$.

An m-dimensional gluing is obtained from a collection of m-dimensional simplices by identifying (via affine linear homeomorphisms) pairs of facets such that each pair of facets is paired with exactly one other facet.

In particular, after subdivision all simplices can be PL embedded into some Euclidean space which results in a triangulated topological space.

Question 1.15. *In general, when is a polyhedron a manifold?*

An (sufficient) answer to this question is when every link is a topological sphere. We will not give a formal definition of link here, but by showing some examples.

In the following picture, the link of the point v is the red polygon. and star is the cone of the link. In general, links can be defined for any dimensional simplex. The link of a p-simplex

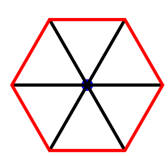


FIGURE 4. Link associated to a point

is an $(m-p-1)$-dimensional topological sphere. And the topological type of links and stars are not changed after subdivision.

PROOF. The open star is an open cone of the link which is a topological sphere, so open star are homeomorphic to open balls in $\mathbb{R}^m$. □

REMARK. This condition is not necessary. S^5 has a triangulation in which vertex links need not to be 4-spheres.

THEOREM 1.16. *Any triangulation of M^1, M^2, M^3 has spherical links.*

Definition 1.17. A **PL triangulation** is a triangulation such that links are spheres.

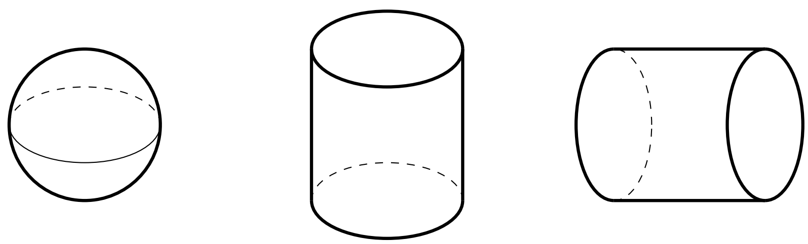
THEOREM 1.18. *For a PL triangulation manifold M, the collection of all open stars forms a PL atlas.*

Conversely, any PL manifold M can be PL triangulated.

Exercise 1.19. Prove that any PL manifold M can be PL triangulated by mimicking the proof of triangulation of 2-manifolds.

1.6. Handle. Triangulation of manifolds involves simplices that have lower dimensions. We will use the construction of handle to construct triangulations.

Definition 1.20. A **k-handle** in an m-dimensional space is $B^k \times B^n, (k+n=m)$, which is a product of closed balls.
An **open k-handle** in an m-dimensional space is $B^k \times (B^n)^o \cong B^k \times \mathbb{R}^n, (k+n=m)$.



0-handle 1-handle 2-handle

FIGURE 5. Handle

This can be thought as "thicked k-simplex" inside m-dimensional space.

A PL structure on a k-handle can be obtained by the homeomorphisms $B^k \cong [-1,1]^k$ and $B^n \cong [-1,1]^n$. We will write $rB^n = [-r,r]^n$ for $r > 0$.

THEOREM 1.21 ([**H1**], 3-dimensional Handle Straightening Theorem). *Given a topological embedding $h : B^k \times B^n \to B^3, k+n=3$, which is PL in a neighborhood of $\partial B^k \times \mathbb{R}^n = S^{k-1} \times \mathbb{R}^n$, then there is an isotopy h_t from $h_0 = h$ such that*

(1) h_1 is PL on $B^k \times B^n$, and
(2) h_t is isotopy relative to $(\partial B^k \times \mathbb{R}^n) \cup (B^k \times \mathbb{R}^n \backslash 2B^n)$, i.e the isotopy is constant away from the core and near the lower active boundary.

Proof of this theorem requires 3-dimensional PL topology and Kirby torus trick. We will delay the proof of this theorem later, and now we will use it to construct the (unique) triangulation on any 3-manifold.

1.7. Existence and Uniqueness of PL structure. The idea of the proof is as follows.

- Cover M^3 with topological charts $\{U_i\}$.
- Use (U_1, φ_1) to get a PL structure on U_1.
- For (U_2, φ_2), φ_2 and φ_1 are not compatible, but we can try to isotope φ_2 such that $\varphi_2 \circ \varphi_1^{-1}$ is PL.
- Repeat.

1.7.1. *Existence.* Cover M^3 with locally finite and paracompact collection of charts $\{(U_i, \varphi_i)\}$. Since ∂M is a 2-manifold, it has a unique PL structure. Choose $U_0 \cong \partial M \times [0,1]$ which also has a PL structure. Choose s refinement $\{W_i\}$ of $\{U_i\}$ such that $\overline{W_i} \subset U_i$.

We will construct by induction on n the number of charts. Suppose we have a PL structure on $\mathcal{U} = \cup_{i=0}^n U_i$. Since cover $\{U_i\}$ is localy finite, U_{n+1} intersect finitely many $\{U_i, i \in I\}$. Refine the cover by setting

$$\begin{cases} V_i = U_i, & i \notin I, \\ \overline{W_i} \subset V_i, \overline{V_i} \subset U_i, & i \in I. \end{cases}$$

Because $\{U_i\}$ is paracompact, $\mathcal{V} = \overline{V_{n+1}} \bigcap (\cup_{i=1}^n \overline{V_i})$ is compact.

We can define a PL structure on $\mathcal{U} \cap U_{n+1}$ by taking the PL structure on U_{n+1} and take all the finitely many simplices that intersect $\mathcal{V}$, then apply handle straightening theorem to 0-handles in $\mathcal{V}$ for the topological embedding φ_{n+1}, then do 1-handles 2-handles, and 3-handles to get $\varphi'_{n+1} \simeq \varphi_{n+1}$ with φ'_{n+1} PL.

Define a PL structure on $(\cup_{i=0}^n V_i) \bigcup V_{n+1}$ using φ'_{n+1}. Then φ'_{n+1} agrees with φ_n outside a neighborhood of $\mathcal{V}$. Note

$$\cup_{i=0}^{n+1} W_i \subset \cup_{i=0}^{n+1} V_i$$

is contained in the domain of the PL structure.

Next, rename V_i to U_i and repeat, since $\{W_i\}$ cover M, we get a PL structure on M.

1.7.2. *Uniqueness.* If M_Σ and $M_{\Sigma'}$ are two PL structures on M, then triangulate M_Σ and subdivide it enough such that every simplex lies in one chart of $M_{\Sigma'}$. Apply handle straightening theorem to this triangulation, we get a PL homeomorphism $h' : M_\Sigma \to M_{\Sigma'}$ which is isotopic to id_M.

Exercise 1.22. Use the same strategy to prove

(1) If $f : U \to \mathbb{R}^n$ is continuous and $\{U_\alpha\}$ is a cover of U such that every f_{U_α} is PL, then f is PL.
(2) If M has a PL structure, then M has a PL triangulation.

2. Prime Decomposition

All 3-manifolds in this section are assumed to be connected, orientable, and compact, possibly with boundary, unless otherwise stated or constructed.

2.1. Prime Decomposition.

The main result we would like to show in this section is the following theorem.

THEOREM 2.1 (Kneser's Theorem). *Every compact orientable 3 manifold M factors as a connected sum of primes, $M = P_1 \# \cdots \# P_n$, and this decomposition is unique up to insertion or deletion of S^3 summands.*

Implicit in the prime decomposition theorem is the fact that S^3 is prime, which is a consequence of Alexander's theorem, our first topic to cover.

2.2. Alexander's Theorem.

THEOREM 2.2 (Alexander's Theorem). *Every embedded 2-sphere in $\mathbb{R}^3$ bounds an embedded 3-ball.*

REMARK. In general, a topological sphere need not bound a topological ball. For instance, the Alexander horned sphere does not bound a topological ball.

The idea of the proof is to cut the given sphere along horizontal planes to produce simpler spheres and apply an induction argument. The proof will also use the analogous result in one lower dimension, that a smooth circle in $\mathbb{R}^2$ bounds a smooth disk. This can be proved by a similar but simpler inductive argument[1].

PROOF. (a) (Cut the sphere) Let $S \subseteq \mathbb{R}^3$ be an embedded closed surface, with height function $h : S \to \mathbb{R}$ given by the z coordinate. By a small ambient isotopy we may assume h is a Morse function (i.e. it contains no degenerate critical points), and the finitely many critical points of h all have distinct critical values.

Let $a_1 < \cdots < a_n$ be noncritical values of h such that each interval

$$(-\infty, a_1), (a_2, a_3), \cdots, (a_n, \infty)$$

contains just one critical value. For each i, $h^{-1}(a_i)$ is a disjoint union of circles, each of which bounds a disc in the plane $z = a_i$.

(b) (Surgery) Let C be a circle of $h^{-1}(a_i)$ which is innermost in the plane $z = a_i$, so the disk D bounded by C is disjoint from other circles. Then we use D to surger S along C.

For some small $\epsilon > 0$ we first remove from S the open annulus A consisting of points near C between the two planes $z = a_i \pm \epsilon$, then we cap off the resulting pair of boundary circles of $S - A$ by adding to $S - A$ the disks in $z = a_i \pm \epsilon$ which these circles bound. The result of this surgery is thus a new embedded surface, with perhaps one more component than S, if C separated S.

This surgery process can now be iterated, taking at each stage an innermost remaining circle of $h^{-1}(a_i)$, and choosing ϵ small enough so that the newly introduced horizontal cap disks intersect the previously constructed surface only in their boundaries.

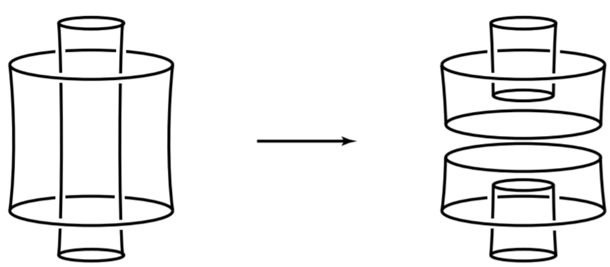


FIGURE 6. Surgery

(c) (Model for each component) After surgering all the circles of $h^{-1}(a_i)$ for all i, the original surface S becomes a disjoint union of closed surfaces S_j, each consisting of a number of horizontal caps together with a connected subsurface S_j' of S containing at most one critical point of h.

Lemma 2.3. *Each S_j is isotopic to one of seven models: the four shown in Figure 7 plus three more obtained by turning these upside down. Hence each S_j bounds a ball.*

[1]In this dimension it is even true that a topologically embedded circle in $\mathbb{R}^2$ bounds a topological disk, the **Schoenflies theorem**, whose proof is more difficult since a simple inductive argument is not possible.

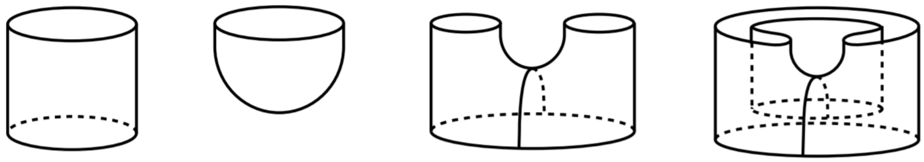


FIGURE 7. Models of components

(d) (Gluing) Since every circle in a sphere separates the sphere into two components, each surgery splits one sphere into two spheres. Reversing the sequence of surgeries, we then start with a collection of spheres S_j bounding balls. The inductive assertion is that at each stage of the reversed surgery process, we have a collection of spheres each bounding a ball. For the inductive step we have two balls A and B bounded by the spheres ∂A and ∂B resulting from a surgery. Letting the ϵ for the surgery go to 0 isotopes A and B so that $\partial A \cap \partial B$ equals the horizontal surgery disk D. There are two cases, up to changes in notation
   i $A \cap B = D$, with presurgery sphere denoted $\partial(A + B)$, or
   ii $B \subset A$, with presurgery sphere denoted $\partial(A + B)$.
   Since B is a ball, the lemma below implies that A and $A \pm B$ are diffeomorphic. Since A is a ball, so is $A \pm B$, and the inductive step is completed.

□

Lemma 2.4. *Given an n-manifold M and a ball $B^{n-1} \subset \partial M$, let the manifold N be obtained from M by attaching a ball B^n via an identification of a ball $B^{n-1} \subset \partial B^n$ with the ball $B^{n-1} \subset \partial M$. Then M and N are diffeomorphic.*

2.3. Existence and Uniqueness of Prime Decomposition.

2.3.1. *Connected Sum.* Let M be a 3 manifold and $S \subset M$ a surface which is properly embedded, i.e., $S \cap \partial M = \partial S$, a transverse intersection. Deleting a small open tubular neighborhood $N(S)$ of S from M, we obtain a 3 manifold $M|S$ which is obtained from M by splitting along S.

Now suppose that M is connected and S is a sphere such that $M|S$ has two components, M_1' and M_2'. Let M_i be obtained from M_i' by filling in its boundary sphere corresponding to S with a ball. In this situation we say M is the **connected sum** $M_1 \# M_2$.

The connected sum operation is commutative and associative, and has S^3 as an identity.

A connected 3-manifold M is called **prime** if $M = P \# Q$ implies $P = S^3$ or $Q = S^3$.

Alexander's theorem implies that S^3 is prime, since every 2-sphere in S^3 bounds a 3-ball. The latter condition, stronger than primeness, is called **irreducibility**: M is irreducible if every 2-sphere $S^2 \subset M$ bounds a ball $B^3 \subset M$. The two conditions are in fact very nearly equivalent:

Proposition 2.5. *The only orientable prime 3-manifold which is not irreducible is $S^1 \times S^2$.*

The idea is to find a nonseparating sphere S in an orientable prime 3-manifold M, and use the tubular neighborhood of S in M and non-separateness of S to glue boundaries of tubular neighborhood of S together and form a copy of $S^1 \times S^2$ in M which contains a ball, which shows any orientable prime 3-manifold M contains $S^1 \times S^2$ as a direct summand, and show $S^1 \times S^2$ is prime by construct a ball in every separating sphere in $S^1 \times S^2$ using covering space of $S^1 \times S^2$.

Now we are ready to prove the theorem.

2.3.2. *Existence.* If M contains a nonseparating S^2, this gives a decomposition $M = N \# S^1 \times S^2$. We can repeat this step of splitting off an $S^1 \times S^2$ summand as long as we have nonseparating spheres.

The process cannot be repeated indefinitely since each $S^1 \times S^2$ summand gives a $\mathbb{Z}$ summand of $H_1(M)$, which is a finitely generated abelian group since M is compact. Thus we are reduced to proving existence of prime decompositions in the case that each 2-sphere in M separates.

Each 2-sphere component of ∂M corresponds to a B^3 summand of M, so we may also assume ∂M contains no 2-spheres.

We shall prove the following assertion, which clearly implies the existence of prime decompositions.

There is a bound on the number of spheres in a system S of disjoint spheres satisfying:

$(*)$ *No component of $M|S$ is a punctured 3-sphere, i.e., a compact manifold obtained from S^3 by deleting finitely many open balls with disjoint closures.*

Observation: Given a system S satisfying $(*)$ and one sphere S_i in S, if we replace S_i by S_i' or S_i'' in the following picture, then at least one of the replacement S' or S'' satisfies $(*)$.

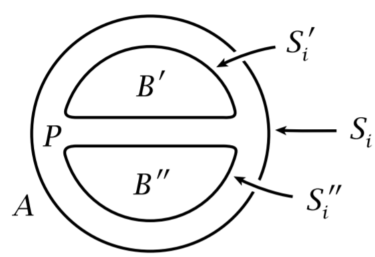


FIGURE 8. Modification of S

Now we will prove the bound.

Choose a smooth triangulation $\mathcal{T}$ of M. This has only finitely many simplices since M is compact. The given system S can be perturbed to be transverse to all the simplices of $\mathcal{T}$. This perturbation can be done inductively over the skeleta of $\mathcal{T}$: First make S disjoint from vertices, then transverse to edges, meeting them in finitely many points, then transverse to 2-simplices, meeting them in finitely many arcs and circles.

For a 3-simplex τ of $\mathcal{T}$, we can make the components of $S \cap \tau$ all disks using the same method as in the observation. In particular, note that no component of the intersection of S with a 2-simplex of $\mathcal{T}$ can be a circle.

Next, for each 2 simplex σ we eliminate arcs α of $Scap\sigma$ having both endpoints on the same edge of σ. By an isotopy of S supported near α we then push the intersection arc α across σ, eliminating α from $S \cap \sigma$.

Now consider the intersection of S with a 2-simplex σ. With at most four exceptions the complementary regions of $S \cap \sigma$ in σ are rectangles with two opposite sides on $\partial\sigma$ and the other two opposite sides arcs of $S \cap \sigma$, as in Figure 9. Thus if $\mathcal{T}$ has t 2-simplices, then all but at most $4t$ of the components of $M|S$ meet all the 2-simplices of $\mathcal{T}$ only in such rectangles.

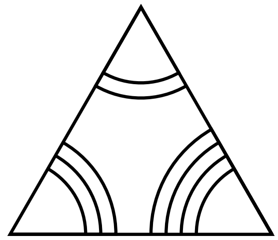


FIGURE 9. Intersection of S with 2-simplex

Let R be a component of $M|S$ meeting all 2-simplices only in rectangles. For a 3-simplex τ, each component of $R\cap\partial\tau$ is an annulus A which is a union of rectangles. The two circles of ∂A bound disks in τ, and A together with these two disks is a sphere bounding a ball in τ, a component of $R\cap\tau$ which can be written as $D^2\times I$ with $\partial D^2\times I = A$. The I-fiberings of all such products $D^2\times I$ may be assumed to agree on their common intersections, the rectangles, to give R the structure of an I-bundle. Since ∂R consists of sphere components of S, R is either the product $S^2\times I$ or the twisted I-bundle over $\mathbb{R}P^2$. The possibility $R = S^2\times I$ is excluded by $(*)$. Each I-bundle R is thus the mapping cylinder of the covering space $S^2\to\mathbb{R}P^2$. This is just $\mathbb{R}P^3$ minus a ball, so each I-bundle R gives a connected summand $\mathbb{R}P^3$ of M, hence a $\mathbb{Z}_2$ direct summand of $H_1(M)$. Thus the number of such components R of $M|S$ is bounded.

Since the number of other components was bounded by $4t$, the number of components of $M|S$ is bounded. Since every 2-sphere in M separates, the number of components of $M|S$ is one more than the number of spheres in S. This finishes the proof of the existence of prime decompositions.

2.3.3. *Uniqueness.* suppose the nonprime M has two prime decompositions

$$M = P_1\#\cdots\#P_k\#l(S^1\times S^2)$$

and

$$M = Q_1\#\cdots\#Q_m\#n(S^1\times S^2)$$

where the P_i's and Q_i's are irreducible and not S^3.

Let S be a disjoint union of 2-spheres in M reducing M to the P_i's, i.e., the components of $M|S$ are the manifolds $P_1,\cdots,P_k$ with punctures, plus possibly some punctured S^3's. Such a system S exists: take for example a collection of spheres defining the given prime decomposition $M = P_1\#\cdots\#P_k\#l(S^1\times S^2)$ together with a nonseparating $S^1\times S^2$. Note that if S reduces M to the P_i's, so does any system S' containing S.

Similarly, let T be a system of spheres reducing M to the Q_i's. If $S\cap T\neq\emptyset$, we may assume this is a transverse intersection, and consider a circle of $S\cap T$ which is innermost in T, bounding a disk $D\subset T$ with $D\cap\partial S=\partial D$. Using D, surger the sphere S_j of S containing ∂D to produce two spheres S_j' and S_j'', which we may take to be disjoint from S_j, so that S_j, S_j', and S_j'' together bound a 3-punctured 3-sphere P. the enlarged system $S\cup S_j'\cup S_j''$ reduces M to the P_i's. Deleting S_j from this enlarged system still gives a system reducing M to the P_i's since this affects only one component of $M|S\cup S_j'\cup S_j''$, by attaching P to one of its boundary spheres, which has the net effect of simply adding one more puncture to this component.

The new system S' meets T in one fewer circle, so after finitely many steps of this type we produce a system S disjoint from T and reducing M to the P_i's. Then $S\cup T$ is a system reducing M both to the P_i's and to the Q_i's. Hence $k = m$ and the P_i's are just a permutation of the Q_i's.

Finally, to show $l = n$, we have

$$M = N\#l(S^1 \times S^2) = N\#n(S^1 \times S^2),$$

so

$$H_1(M) = H_1(N) \oplus \mathbb{Z}^l = H_1(N) \oplus \mathbb{Z}^n,$$

hence $l = n$.

3. Loop and Sphere theorems

3.1. The Loop Theorem. Let M be a 3-manifold with boundary, not necessarily compact or orientable.

THEOREM 3.1 (The Loop Theorem). *If there is a map $f : (D^2, \partial D^2 \to (M, \partial M)$ with $f|_{\partial D^2}$ not null-homotopic in ∂M , then there is an embedding with the same property.*

Before proving the theorem, we will see a corollary of the loop theorem.

Corollary 3.2 (Dehn's Lemma)**.** *If an embedded circle in ∂M is null-homotopic in M, it bounds a disk in M.*

PROOF. First delete from ∂M everything but a neighborhood of the given circle C, producing a new manifold M' with $\partial M'$ an annulus. (If C had a Möbius band neighborhood in ∂M , it would be an orientation-reversing loop in M, which is impossible since it is nullhomotopic.) Apply the loop theorem to M' to obtain a disk $D^2 \subset M$ with ∂D^2 nontrivial in $\partial M'$, hence isotopic to C. □

3.1.1. *Proof of the loop theorem.* The first half of the proof consists of covering space arguments which reduce the problem to finitely many applications of the relatively simple special case that f is at most two-to-one.

Choose a triangulation of M and apply the simplicial approximation theorem (for maps of pairs) to homotope the given f to a map f_0 which is simplicial in some triangulation of the domain D^2 and which still satisfies the same hypotheses as f. We now construct a diagram as follows.

$$\begin{tikzcd}
& D_n \arrow[r, hook] & V_n \arrow[r, hook] & M_n \arrow[dl, "p_n"] \\
& & \vdots & \vdots \arrow[dl, "p_2"] \\
& D_1 \arrow[r, hook] & V_1 \arrow[r, hook] & M_1 \arrow[dl, "p_1"] \\
D^2 \arrow[r, "f_0"] \arrow[ur, "f_1"] \arrow[uuur, "f_n"] & D_0 \arrow[r, hook] & V_0 \arrow[r, hook] & M_0
\end{tikzcd}$$

where the maps are constructed as follows.

- Let D_0 be $f_0(D^2)$, a finite subcomplex of M, and let V_0 be a neighborhood of D_0 in M that is a compact 3-manifold deformation retracting onto D_0[2]. In particular, V_0 is connected since D_0 is connected.
- If V_0 has a connected 2-sheeted cover $p_1 : M_1 \to V_1$, then f_0 lifts to a map $f_1 : D^2 \to M_1$ since D^2 is simply connected. Let $D_1 = f_1(D^2)$ and let V_1 be a neighborhood of D_1 in M_1.
- If V_1 has a connected 2-sheeted cover we can repeat the process to construct a third row, and so on up the tower.
- To see that we must eventually reach a stage where the tower cannot be continued further, consider the covering

$$p_i : (p_i)^{-1}(D_{i-1}) \to D_{i-1}.$$

  Both these spaces have natural simplicial structures with simplices lifting the simplices of D_0. The nontrivial deck transformation τ_i of the covering space $(p_i)^{-1}(D_{i-1}) \to D_{i-1}$ is a simplicial homeomorphism. Since M_i is connected, so is $(p_i)^{-1}(D_{i-1})$ since M_i deformation retracts to $(p_i)^{-1}(D_{i-1})$ by lifting a deformation retraction of V_{i-1} to D_{i-1}. The set $(p_i)^{-1}(D_{i-1}) = D_i \cup \tau_i(D_i)$ is connected, so $D_i \cap \tau_i(D_i)$ must be nonempty. This means that τ_i must take some simplex of D_i to another simplex of D_i, distinct from the first simplex since τ_i has no fixed points. These two simplices of D_i are then identified in D_{i-1}, so D_{i-1} is a quotient of D_i with fewer simplices than D_i. Since the number of simplices of D_i is bounded by the number of simplices in the original triangulation of the source D^2, it follows that the height of the tower is bounded.

Now we arrive at a V_n having no connected 2-sheeted cover. So $\pi_1 V_n$ has no subgroup of index two, so there is no nontrivial morphism $\pi_1 V_n \to \mathbb{Z}_2$, hence $H^1(V_{n;\mathbb{Z}_2} = 0$. In the exact sequence

$$H_2(V_n, \partial V_n; \mathbb{Z}_2) \longrightarrow H_1(\partial V_n; \mathbb{Z}_2) \longrightarrow H_1(V_n; \mathbb{Z}_2)$$

By Poincaré duality, the first term is isomorphic to $H^1(V_n; \mathbb{Z}_2) = 0$, and the third term is isomorphic to $H_1(V_n; \mathbb{Z}_2)$ by universal coefficients theorem, thus is also zero. Therefore $H_1(\partial V_n; \mathbb{Z}_2) = 0$, which implies that all components of the compact surface ∂V_n are 2-spheres.

In the component of ∂_{V_n} containing $f_i(D_i)$, let

$$F_i = (p_1 \circ \cdots \circ p_i)^{-1}(\partial M).$$

This will be a compact surface if the neighborhoods V_i are chosen reasonably. Let $N_i \subset \pi_1 F_i$ be the kernel of $(p_1 \circ \cdots \circ p_i)_* : \pi_1 F_i \to \pi_1(\partial M)$. Since $[f_i|\partial D^2] \notin N_i$ by the initial hypothesis on f, so N_i is a proper normal subgroup of $\pi_1 F_i$.

At the top of the tower the surface F_n is a planar surface, being a subsurface of a sphere. Hence $\pi_1 F_n$ is normally generated by the circles of ∂F_n. Since $N_n \neq \pi_1 F_n$, there is some circle of ∂F_n must represent an element of $\pi_1 F_n \backslash N_n$. This circle bounds a disk in ∂V_n. Let $g_n : D^2 \to V_n$ be this embedding of a disk, with its interior pushed into V_n so as to be a proper embedding. We have $[g_n|_{\partial D^2}] \notin N_n$. We can take g_n to be a smooth embedding and work from now on in the smooth category.

[2]The classical construction of such a V_0 is to take the union of all simplices in the second barycentric subdivision of M that meet D_0.

Next, use induction to construct $g_i : D^2 \to V_i$ with $[g_i|_{\partial D^2}] \notin N_i$ to get an embedding $g_0 : D^2 \to V_0 \subset M$ with $[g_0|_{\partial D^2}] \notin N_0$, which will show that $g_0|\partial D^2$ is not nullhomotopic in ∂M, and the proof will be complete.

We will produce g_{i-1} from g_i as follows. Since p_i is a 2-sheeted cover, we can perturb g_i so that the immersion $p_i g_i$ has only simple double curves, either circles or arcs, where two sheets of $p_i g_i(D^2)$ cross transversely. We will modify g_i so as to eliminate each of these double curves in turn.

Suppose C is a double circle. A neighborhood $N(C)$ of C in $p_i g_i(D^2)$ is a bundle over C with fiber the letter X. Thus $N(C)$ is obtained from $X \times [0, 1]$ by identifying $X \times \{0\}$ with $X \times \{1\}$ by some homeomorphism given by a symmetry of X. A 90-degree rotational symmetry is impossible since p_i is two-to-one. A 180-degree rotational symmetry is also impossible since it would force the domain D_2 to contain a Möbius band. For the same reason a symmetry which reflects the X across one of its crossbars. So the remianing two cases are the identity symmetry, with $N(S) = X \times S^1$, and a reflection of X across a horizontal or vertical line. Note that a reflection can occur only if M is nonorientable.

In the case $N(S) = X \times S^1$, there are two circles in D^2 mapping to C. These circles bound disks D_1 and D_2 in D^2. If D_1 and D_2 are nested, say $D_1 \subset D_2$, then we redefine g_i on D_2 to be $g_i|_{D_1}$, smoothing the resulting corner along C . This has the effect of eliminating C from the self-intersections of $p_i g_i(D_2)$, along with any self-intersections of the immersed annulus $p_i g_i(D_2 - D_1)$. If D_1 and D_2 are disjoint we modify g_i by interchanging its values on the disks D_1 and D_2, then smoothing the resulting corners along C, as shown in Figure 10(a). Thus we eliminate the circle C from the self-intersections of pigi(D_2) without introducing any new self-intersections.

FIGURE 10. Smoothing

In the other case, we modify the immersed disk $p_i g_i(D^2)$ in the way shown in Figure 10(b), replacing each cross-sectional X of $N(C)$ by two disjoint arcs. This replaces an immersed annulus by an embedded annulus with the same boundary, so the result is again an immersed disk with fewer double circles.

Iterating these steps, all double circles can be eliminated without changing $g_i|_{\partial D^2}$.

Now consider the case of a double arc C. The two possibilities for how the two corresponding arcs in D^2 are identified to C are shown in Figure 11.

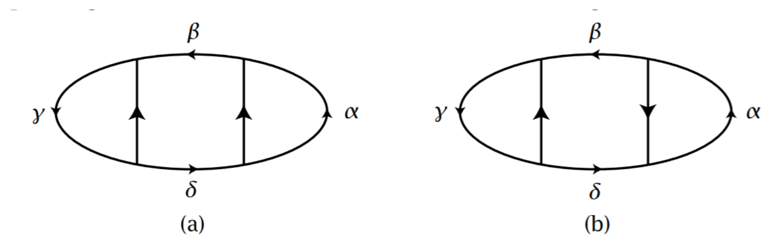


FIGURE 11. Arcs

In either case, Figure 12 shows two ways to modify g_i to produce new immersions g_i' and g_i'' eliminating the double arc C.

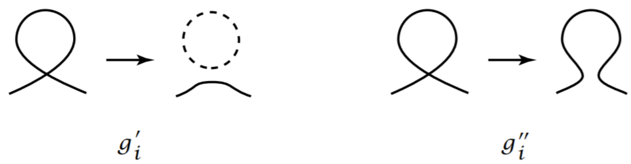


FIGURE 12. Smoothing for arcs

We claim that either $[g_i'|_{\partial D^2}] \notin N_i$ or $[g_i''|_{\partial D^2}] \notin N_i$. This can be seen by breaking the loop $g_i|_{pD^2}$ up into four pieces $\alpha, \beta\, gamma, \delta$ as indicated in Figure 11.

- In the case (a), we have
$$\alpha\beta\gamma\delta = (\alpha\gamma)\delta^{-1}(\alpha\beta^{-1}\gamma\delta^{-1})^{-1}(\alpha\gamma)\delta$$
where $\alpha\gamma = g_i'|_{\partial D^2}$ and $\alpha\beta^{-1}\gamma\delta^{-1} = g_i''|_{\partial D^2}$.
- In the case (b), we have
$$\alpha\beta\gamma\delta = (\alpha\gamma^{-1})(\gamma\delta)^{-1}(\alpha\gamma^{-1})^{-1}(\alpha\delta\gamma\beta)(\gamma\delta)$$
where $\alpha\gamma^{-1} = g_i'|_{\partial D^2}$ and $\alpha\delta\gamma\beta = g_i''|_{\partial D^2}$.

So in either case, if $g_i'|_{\partial D^2}$ and $g_i''|_{\partial D^2}$ were both in the normal subgroup N_i, so would $g_i|_{\partial D^2}$ be in N_i.

We choose the new g_i to be either g_i' or g_i'', whichever one preserves the condition $[g_i|_{\partial D^2}] \notin N_i$. Repeating this step, we eventually get g_i with $p_i g_i$ an embedding g_{i-1}, with $[g_{i-1}|_{\partial D^2}] \notin N_{i-1}$.

4. Torus decomposition

Previously we've seen the existence and uniqueness of prime decomposition of compact connected orientable 3-manifold, and since the only orientable prime 3-manifold which is not irreducible is $S^1 \times S^2$, it is natural to consider what are the irreducible (i.e. every 2-sphere in the manifold bounds a ball) 3-manifolds, and can we decompose them into further (canonical) pieces which are smaller and can be explicitly describe. The answer to this question is positive and the corresponding decomposition is called **torus decomposition**, where the splitting are along tori rather than the spheres.

The **torus decomposition** [3] is a topological construct given by the following theorem:

THEOREM 4.1 (The torus decomposition). *For a compact irreducible orientable 3-manifold M there exists a collection $T \subset M$ of disjoint compressible tori such that each component of $M|T$ is either atoroidal or a Seifert manifold, and a minimal such collection T is unique up to isotopy.*

To prove this theorem, we need to know the definition of incompressible tori, atoroidal manifold and Seifert-fibered manifold and their properties. So we will start with these preliminaries.

[3]Also known as the JSJ decomposition. The acronym JSJ is for William Jaco, Peter Shalen, and Klaus Johannson.

4.1. Incompressible Surfaces. A properly embedded (i.e. $\partial S = S \cap \partial M$) connected surface $S \subset M^3$ is called **2-sided** if its normal bundle is trivial, and **1-sided** if its normal bundle is nontrivial[4].

A 2-sided connected surface S other than S^2 or D^2 is called **incompressible** if for each disk $D \subset M$ with $D \cap S = \partial D$ (such a disk is called a compressing disk for S), there is a disk $D' \subset S$ with $\partial D = \partial D'$. Thus, surgery on S in M cannot produce a simpler surface, but only splits off an S^2 from S, leaving a diffeomorphic copy of S as the other piece resulting from the surgery.

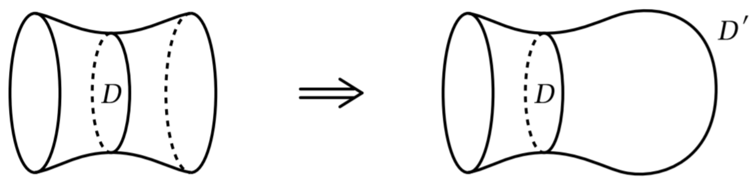


FIGURE 13. Incompressible surface

Here are some preliminary facts about incompressible surfaces.

(1) A connected 2 sided surface S which is not a sphere or disk is incompressible if the map $\pi_1(S) \to \pi_1(M)$ induced by inclusion is injective. This is because if $D \subset M$ is a compressing disc, then ∂D is null-homotopic in M, hence also in S if the map $\pi_1(S) \to \pi_1(M)$ is injective, so by Proposition
(2) There are no incompressible surfaces in $\mathbb{R}^3$ or, equivalently, in S^3.
(3) A 2-sided torus T in an irreducible M is compressible iff T either bounds a solid torus $S^1 \times D^2 \subset M$ or lies in a ball in M.
(4) If $S \subset M$ is a finite collection of disjoint incompressible surfaces, then M is irreducible iff $M|S$ is irreducible.
(5) If $S \subset M$ is a finite collection of disjoint, properly embedded surfaces that are either incompressible or spheres or disks, then a surface $T \subset M|S$ is incompressible in M iff it is incompressible in $M|S$.

4.2. Existence of Torus Decompositions.

4.2.1. *Atoroidal.* A properly embedded surface $S \subset M$ is **∂-parallel** if it is isotopic, fixing ∂S, to a subsurface of ¶M. An irreducible manifold M is **atoroidal** if every incompressible torus in M is ∂-parallel.

4.2.2. *Existence of Torus Decompositions.* In a compact connected irreducible M there exists a finite collection T of disjoint incompressible tori such that each component of $M|T$ is atoroidal.

This followes from the following proposition.

Proposition 4.2. *For a compact irreducible M there is a bound on the number of components in a system $S = S_1 \cup \cdots \cup S_n$ of disjoint closed connected incompressible surfaces $S_i \subset M$ such that no component of $M|S$ is a product $T \times I$ with T a closed surface.*

PROOF. (1) perturb and split to start with the same setup as in prime decompositions, leaving only I-bundles to handle.

[4]The 'sides' of S then correspond to the components of the complement of S in a tubular neighborhood.

(2) Replace I-bundles with one-sided surfaces S'.
(3) Show that

$$H_3(M, S'; \mathbb{Z}_2) \longrightarrow H_2(S'; \mathbb{Z}_2) \longrightarrow H_2(M; \mathbb{Z}_2)$$

is exact, and first and last terms do not depend on S', meaning middle term is bounded, showing number of components are bounded.

$\square$

Example 4.3 (Non-uniqueness of the decomposition)**.** Let $M_i, i = 1, 2, 3, 4$ be a solid torus whose boundary torus is decomposed as the union of two annuli A_i and A_i' each winding $q_i > 1$ times around the S^1 factor of M_i. The union of these four solid tori, with each A_i' glued to A_{i+1} (subscripts mod 4), is the manifold M.

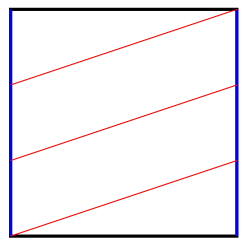


FIGURE 14. annuli A_i and A_i'

We need to introduce Seifert manifolds and include them as possible components of $M|S$ to obtain uniqueness.

4.3. Seifert Manifolds.

4.3.1. *Seifert Manifolds.* A **model Seifert fibering** of $S^1 \times D^2$ is a decomposition of $S^1 \times D^2$ into disjoint circles, called **fibers**, constructed as follows.

- Starting with $[0, 1] \times D^2$ decomposed into the segments $[0, 1] \times \{x\}$,
- identify the disks $\{0\} \times D^2$ and $\{1\} \times D^2$ via a $2\pi p/q$ rotation, for $p/q \in \mathbb{Q}$ with p and q relatively prime.
- The segment $[0, 1] \times \{0\}$ then becomes a fiber $S^1 \times \{0\}$, while every other fiber in $S^1 \times D^2$ is made from q segments $[0, 1] \times \{x\}$.

A **Seifert fibering** of a 3-manifold M is a decomposition of M into disjoint circles, the fibers, such that each fiber has a neighborhood diffeomorphic, preserving fibers, to a neighborhood of a fiber in some model Seifert fibering of $S^1 \times D^2$. A **Seifert manifold** is one which possesses a Seifert fibering.

4.3.2. *Fibers.* Ther **multiplicity** of each fiber circle C in a Seifert fibering of a 3-manifold M is the number of times a small disk transverse to C meets each nearby fiber.

For example, in the model Seifert fibering of $S^1 \times D^2$ with $2\pi p/q$ twist, the fiber $S^1 \times \{0\}$ has multiplicity q while all other fibers have multiplicity 1.

Fibers of multiplicity 1 are **regular** fibers, and the other fibers are **multiple**.

4.3.3. *Incompressible Surfaces in Seifert Manifolds.*

Definition 4.4. A non-disk S is ∂-incompressible if for each disk $D \subset M$ such that $\partial D \cap S$ is an arc γ in ∂D and the rest of ∂D lies in ∂M, there is a disk $D' \subset S$ with $\gamma \subset \partial D'$ and $\partial D' - \gamma \subset \partial S$.

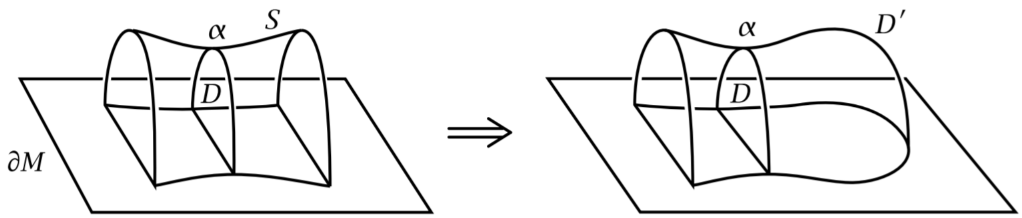


FIGURE 15. ∂-incompressible

Such a D is called a ∂-compressing disk for S.

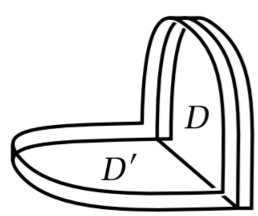
Definition 4.5. A properly embedded surface is **essential** if is it is both incompressible and ∂-incompressible.

Example 4.6. The only essential surfaces in $M = S^1 \times D^2$ are meridian disks.

- Let S be a connected essential surface in M.
- Isotope S so that all boundary circles in ∂S are either meridian circles or transverse to all meridian circles.
- Perturb S so it is transverse to a medridian disk D_0.
- Eliminate circles in $S \cap D_0$.
- There is no arc in $S \cap D_0$.
- Thus we have $S \cap D_0 = \emptyset$. So boundaries ∂S consists of only meridian circles.
- Isotope S (fixing boundaries) to meridian disks.

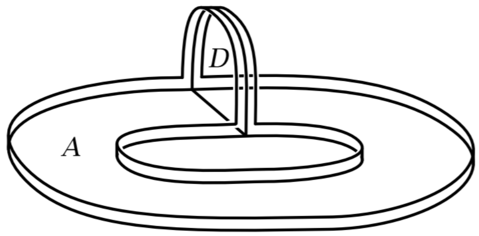
Lemma 4.7. *Let S be a connected incompressible surface in the irreducible 3-manifold M, with ∂S contained in torus boundary components of M. Then either S is essential or it is a ∂-parallel annulus.*

PROOF.
- Suppose S is ∂-compressible, with a ∂-compressing disk D meeting S in an arc α which does not cut off a disk from S. Let β be the arc $D \cap \partial M$.
- β lies in an annulus component A of $T|\partial S$.



- The endpoints of β lies in defferent circles of ∂S.
- Let N be a neighborhood of $\partial A \cup \alpha$ in S. The circle $\partial N - \partial S$ bounds a disc near $D \cup A$, lying in the complememnt of A, so the boundary of this disc also bounds a disck in S, so S is an anuulus.
- Surgering the torus $S \cup A$ via D yield a sphere, which bounds a ball in M.
- $S \cup A$ bounds a solid torus and S is ∂-parallel, being isotopic to A rel ∂S.

$\square$



Proposition 4.8. *If M is a connected compact irreducible Seifert-fibered manifold, then any essential surface in M is either* **vertical***, i.e. a union of regular fibers, or* **horizontal***, i.e. transverse to all fibers.*

Proof. We sall proceed in the following steps.

- Form a circle bundle M_0 by deleting neighbourhoods of multiple fibres C_i in M (if there is no multiple fiber then delete an interior regular fiber), then find arcs γ in the base space B_0 of M_0 so that they split B_0 in to a disc. Let A be the preimages of γ in M_0, which is a disjoint union of annuli. Then split along annuli A we can split M_0 into a solid torus M_1.
- Components of $S \cap A$ are either vertical circles or horizontal arcs.
  - circles of ∂S are either vertical or horizontal, since circles bounding disks can be eliminated using incompressibility of S and irreducibility of M.
  - trivial arcs with both endpoints on the same component of $\partial M_0 - \partial M$ can be eliminated using isotopy of S along a ball which is the union of disc D in S and D' in $\partial M_0 - \partial M$, which is of the baseball shape as below, and eliminate 2 intersection points of S with some fiber C_i.

  - trivial arcs with both endpoints on the same component of ∂M can be eliminated using ∂-incompressibility of S and irreducibility of M.
- Let $S_1 = S|A$, may assume all components of S_1 are either ∂-parallel annuli with vertical boundary or discs with horizontal boundary (meridian discs).
  - S_1 is incompressible.
  - components of S_1 as either ∂-parallel annuli or meridian disks.
  - eleminate components of S_1 that are ∂-parallel annuli with horizontal boundary using the baseball trick.
- Extend this isotopy to the original surface S to conclude that it must be horizontal or vertical.

$\square$

Vertical surfaces are circle bundles , hence they are either annuli, tori, or Klein bottles.

For a horizontal surfaces S the projection $\pi : S \to B$ onto the base space is a branched covering with a branch point of multiplicity q_i for each intersection of S with a multiple fiber if multiplicity q_i. For this branched covering, there is a useful formula relating the Euler characteristics of S and B,

$$\chi(B) - \chi(S)/n = \sum_i (1 - q/q_i).$$

Assume S is connected and 2-sided, since $S \to B$ is onto, S meets all fibers in M, and $M|S$ is an I-bundle. The associated ∂I-subbundles consists of two copies of S, so the I-bundle is the mapping cylinder of a 2-sheeted covering projection $S \coprod S \to T$ for some surface T.

(1) If $M|S$ is connected, so is T, $S \coprod S \to T$ is the trivial covering $S \coprod S \to S$, so $M|S = S \times I$ and hence M is a circle bundle over S.
(2) If $M|S$ has two components, each is a twisted I-bundle over a component T_i of T, the mapping cylinder of a nontrivial 2-sheeted covering $S \to T_i$. The parallel copies of S in the mapping cylinders together with T_1, T_2, are the leaves of a foliation of M, which are the fibers of a projection map $p : M \to I$. This structure is not exactly a fiber bundle, we will call it a **semi-bundle**.

A semi-bundle is the union of two twisted I-bundles $p^{-1}[0, \frac{1}{2}]$ and $p^{-1}[\frac{1}{2}, 1]$ glued together by a homeomorphism of $p^{-1}(\frac{1}{2})$. For example, the Klein bottle is a semi-bundle with fiber S^1, and it is the union of two Möbius bands.

Proposition 4.9. *A compact connected Seifert manifold M is irreducible unless it is $S^1 \times S^2, S^1 \tilde{\times} S^2$, or $\mathbb{R}P^3 \# \mathbb{R}P^3$.*

4.4. Uniqueness of Torus Decompositions.

Lemma 4.10. *An essential annulus in a compact connected Seifert manifold M can be isotoped to be vertical, after possibly chaing the Seifert fibering if $M = S^1 \times S^1 \times I$, $S^1 \times S^1 \tilde{\times} I$ (the twisted I-bundle over the torus), $S^1 \tilde{\times} S^1 \times I$ (the Klein bottle cross I), or $S^1 \tilde{\times} S^1 \tilde{\times} I$ (the twisted I-bundle over the Klein bottle).*

Lemma 4.11. *Let M be a compact connected Seifert manifold with orientable boundary ∂M. Then the restriction of any two Seifert fibering of M are isotopic unless $M = S^1 \times D^2$ or one of the four exceptional cases in Lemma 4.10.*

Lemma 4.12. *If M is a compact, connected, orientable, irreducible, and atoroidal, and M contains an incompressible and ∂-incompressible annulus meeting torus components of ∂M, then M is a Seifert manifold.*

Proof. If A is an annulus as in the hypothesis, there are three possibilities for how A can meet ∂M.

- Let N be a neighbourhood of A's union with the relevant components of ∂M. Then N is a circle bundle with A vertical.
- Working case-by-case, the possibilities for components of $\partial N - \partial M$ can either be isotoped into N, or they are solid tori.
- This means M is N with solid tori attached to tori of $\partial N - \partial M$. We can extend the circle bundle structure of N to these tori to get a Seifert fibering of M.

□

Now we are ready to prove the unqiueness of torus decomposition.

- Let T and T' be two minimal collections splitting M into manifolds M_j and M_j' respectively.
- We can assume no torus in T is isotopic to one in T' since if this were true, we could split along this torus and work by induction.
- We may assume $T \cap T' = \emptyset$.
  - Eliminate circles bounding disks in $T \cap T'$ and ∂-parallel annuli in $M_j \cap T'$.
  - Circles that do not bound disks induce fiberings on T_i which agree on M_j and M_k, meaning T_i could be deleted from the collection (a contradiction).
- Now all T_i and T_i' are vertical and all M_j and M_j' are Seifert-fibered.
- The T_i and T_i' together cut M into pieces we call N_p.
- We can isotope N_p so its fiberings agree on T_i except in certain exceptional cases:
  - $N_p = S^1 \times D^2$, which cannot occur because T_i would be its compressible boundary.
  - $N_p = S^1 \times S^1 \times I$ which cannot occur because its boundary components are T_i and either T_j or T_i', which can be isotoped to agree and contradict minimality.
- The only remaining case is $M_j \cap M_j' = S^1 \tilde{\times} S^1 \tilde{\times} I$, which has only one boundary component so $N_p = M_j \subset M_j'$, so we could change the fibering of M_j to match the restriction of M_j'.
- The same applies to M_k. We conclude that fiberings of M_j and M_k agree on T_i since they agree with the fibering from M_j', meaning we can omit T_i from T, contradicting minimality.

5. Seifert Manifolds

We've seen the in the previous section that Seifert manifolds play a special role in the torus decomposition, so in this section we will discuss about examples and classifications of orientable Seifert manifolds (up to diffeomorphisms.

5.1. Examples.

5.2. Classification of Seifert Manifolds.

Proposition 5.1. *(a) Every orientable Seifert-fibered manifold is isomorphic to one of the models $M(\pm g, b; \alpha_1/\beta_1, \cdots \alpha_k/\beta_k)$.*

(b) Seifert fiberings $M(\pm g, b; \alpha_1/\beta_1, \cdots \alpha_k/\beta_k)$ and $M(\pm g, b; \alpha_1'/\beta_1', \cdots \alpha_k'/\beta_k')$ are isomorphic by an orientation preserving diffeomorphism iff the following two conditions are satisfied:

(i) After possibly permuting indices, $\alpha_i/\beta_i \equiv \alpha_i'/\beta_i' \mod 1$ for each i.

(ii) $\sum_i \alpha_i/\beta_i = \sum_i \alpha_i'/\beta_i'$ when $b = 0$.

THEOREM 5.2. *Seifert fiberings of orientable Seifert maniflds are unique up to isomorphism, with the execption of the following fiberings:*

(a) $M(0, 1; \alpha/\beta)$, the vatious model Seifert fiberings of $S^1 \times D^2$.

(b) $M(0, 1; \frac{1}{2}, \frac{1}{2}) = M(-1, 1;)$, two fiberings of $S^1 \tilde{\times} S^1 \tilde{\times} I$.

(c) $M(0, 0; \alpha_1/\beta_1, \alpha_2/\beta_2)$, various fiberings of $S^3, S^1 \times S^2$, and lens spaces.

(d) $M(0, 1; \frac{1}{2}, -\frac{1}{2}, \alpha/\beta) = M(-1, 0; \beta/\alpha)$ for $\alpha, \beta \neq 0$.

(e) $M(0, 1; \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) = M(-2, 0;)$, two fiberings of $S^1 \tilde{\times} S^1 \tilde{\times} S^1$.

CHAPTER 3

Special Classes of 3-Manifolds

1. Fiber Bundles

A **fiber bundle** is a structure (E, B, π, F), where E, B, and F are topological spaces and $\pi : E \to B$ is a continuous surjection satisfying a local triviality condition: for every $x \in B$, there is an open neighborhood $U \subset B$ of x such that there is a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ (where $U \times F$ is the product space) in such a way that π agrees with the projection onto the first factor. That is, the following diagram should commute:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi", "\cong"'] \arrow[dr, "\pi|_U"] & & U \times F \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

where $\varphi : \pi^{-1}(U) \to U \times F$ is a homeomorphism. The set of all $\{(U_i, \varphi_i)\}$ is called a local trivialization of the bundle. For any $p \in B$, the preimage $\pi^{-1}(\{p\})$ is homeomorphic to F and is called the fiber over p. The space B is called the **base space** of the bundle, and E the **total space**. A fiber bundle is often denoted

$$F \hookrightarrow E \xrightarrow{\pi} B.$$

When $E = M$ is a compact 3-manifold, We may discuss possible fiber bundle constructions according to dimensions of the fiber and base space. Note we have

$$3 = \dim M = \dim F + \dim B.$$

So we only have the following 3 cases (we always assume B is connected)

(1) When $\dim F = 0$, this is just covering spaces.
(2) When $\dim F = 1, \dim B = 2$, since M is compact, both B, F are compact, thus we may take $F = S^1$ and $B = \Sigma$ is a closed surface, and M is a circle bundle.

$$S^1 \hookrightarrow M \longrightarrow \Sigma$$

For instance, if Σ be a Riemannian surface, we can take $M = UT(\Sigma)$ to be the unit tangent bundle of Σ, and when $\chi(\Sigma) \neq 0$, this is not a product.
(3) When $\dim F = 1, \dim B = 2$, this is a surface bundle over circle S^1.

$$\Sigma \hookrightarrow M \longrightarrow S^1$$

and in particular we have

$$\pi^{-1}(S^1 \backslash \{p\}) \cong \Sigma \times (0, 1)$$

so in this case $M \cong M_\varphi$ is a mapping cone for some $\varphi : \Sigma \to \Sigma$. When $\Sigma = S^2$, again we have short exact sequence of homotopy groups

$$1 \longrightarrow \pi_1\Sigma \longrightarrow \pi_1(M_\varphi) \longrightarrow \mathbb{Z} \longrightarrow 1$$

thus $\pi_1(M) = \pi_1(\Sigma) \rtimes_{\varphi_*} \mathbb{Z}$

For surface bundle over circle, when Σ is not connected, and M_φ is connected, say

$$\varphi : \Sigma_1 \cup \Sigma_2 \longrightarrow \Sigma_1 \cup \Sigma_2$$

If M_φ is connected, φ msut cyclicly permute the corners of Σ. If there is no such corners, then Σ_1 is one of them. and for $\psi = \varphi|_{\Sigma_1}$, $M_\psi \cong M_\varphi$.

Remark. If M is a surface bundle, then $b_1 = \dim_{\mathbb{Q}} H^1(M;\mathbb{Q}) \geq 1$.

There are many 3-manifolds (including the hyperbolic ones) with $b_1 = 0$.

1.1. Obstruction of circle bundles. To consider examples of compact 3-manifold which is not a sphere bundle over Σ, the obstruction is by considering the long exact sequence of homotopy groups associated with the sphere bundle

$$0 = \pi_2(S^1) \longrightarrow \pi_2(M) \longrightarrow \pi_2(\Sigma) \longrightarrow \pi_1(S^1) \longrightarrow \pi_1(M) \longrightarrow \pi_1(\Sigma) \longrightarrow 1$$

We only consider connected manifolds so all π_0 are trivial. Since $\pi_1(S^1) = \mathbb{Z}$, we have the following criterion.

Lemma 1.1. *(1) If M is a circle bundle over $\Sigma \neq S^2$, the long exact sequence 1.1 splits*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(M) \longrightarrow \pi_1(\Sigma) \longrightarrow 1$$

so $\pi_1(M)$ has infinitely many centers.
(2) If $\Sigma = S^2$, then $\pi_1(M)$ has cyclic elements.

Thus to show a 3-manifold M is not a sphere bundle, it suffices to show $\pi_1(M)$ has trivial center and contains no cyclic elements, which is true for hyperbolic M.

2. Virtual 3-Manifold Theory

Definition 2.1. Let X be a space. X **virtually has property** P if there exists a finite-sheeted cover $\tilde{X} \to X$ such that $\tilde{X}$ has property P.

This definition is a generalization of virtual groups: a group G virtually has property P if there is a finite index $\tilde{G} \to G$ such that $\tilde{G}$ has property P.

Example 2.2. (1) $\mathbb{Z}/2 \times \mathbb{Z}/2$ is virtually cyclic.
(2) M_φ with cyclic φ is virtually as product.
(3) Finite groups are virtually trivial.

Theorem 2.3 (Agol). *Hyperbolic (closed) 3-manifolds are virtually surface bundles.*

3. Dehn's Surgery and Lickorish-Wallace Theorem

CHAPTER 4

Thurston Norm

CHAPTER 5

Thurston's geometrization conjecture

1. Geometric Structure

Definition 1.1. A **model geometry** is a simply connected smooth manifold X together with a transitive action of a Lie group G on X with compact stabilizers.

Definition 1.2. A model geometry is called **maximal** if G is maximal among groups acting smoothly and transitively on X with compact stabilizers.

Definition 1.3. A **geometric structure** on a manifold M is a diffeomorphism from M to X/Γ for some model geometry X, where Γ is a discrete subgroup of G acting freely on X.

This is a special case of a complete (G,X)-structure. If a given manifold admits a geometric structure, then it admits one whose model is maximal.

2. Thurston's geometrization conjecture

Thurston's geometrization conjecture states that certain three-dimensional topological spaces each have a unique geometric structure that can be associated with them. It is an analogue of the uniformization theorem for two-dimensional surfaces, which states that every simply connected Riemann surface can be given one of three geometries (Euclidean $\mathbb{E}^2$, spherical S^2, or hyperbolicv $\mathbb{H}^2$).

In three dimensions, it is not always possible to assign a single geometry to a whole topological space. Instead, the geometrization conjecture states that every closed 3-manifold can be decomposed in a canonical way into pieces that each have one of eight types of geometric structure.

- Spherical geometry S^3.
- Euclidean geometry $\mathbb{E}^3$.
- Hyperbolic geometry $\mathbb{H}^3$.
- The geometry of $S^2 \times \mathbb{R}$.
- The geometry of $\mathbb{H}^2 \times \mathbb{R}$.
- The geometry of $\widetilde{\mathrm{SL}}(2, \mathbf{R})$.
- Nil geometry.
- Sol geometry.

Theorem 2.1 (Thurston, [T1]). *Every closed 3-manifold can be decomposed in a canonical way into pieces that each have one of eight types of geometric structure.*

The conjecture was proposed by William Thurston (1982), and implies several other conjectures, such as the Poincaré conjecture and Thurston's elliptization conjecture.

Remark. A closed 3-manifold has a geometric structure of at most one of the 8 types above, but finite volume non-compact 3-manifolds can occasionally have more than one type of geometric structure.

The possible models for circle bundles are the ones other than hyperbolic or Sol geometry.

2.1. Examples.

2.1.1. *The connected sum of two copies of 3-dimensional projective space.* $\mathbb{R}P^3 \# \mathbb{R}P^3$ has the geometry $S^2 \times \mathbb{R}$, which is the only example of a non-trivial connected sum with a geometric structure.

2.1.2. *Torus bundles.* Every torus bundle has geometries, but can have nontrivial JSJ decomposition.

2.1.3. *Hopf Fibration.* The Hopf fibration is $S^1 \hookrightarrow S^3 \xrightarrow{p} S^2$ where p is defined by

$$p(z_0, z_1) = (2z_0 z_1^*, |z_0|^2 - |z_1|^2).$$

3. Mapping Tori

3.1. Mapping Class Groups. The mapping class group $MCG(M)$ of a manifold M is interpreted as the group of isotopy-classes of automorphisms of M.

If M is a topological manifold, the mapping class group is the group of isotopy-classes of homeomorphisms of M. If M is a smooth manifold, the mapping class group is the group of isotopy-classes of diffeomorphisms of M.

Whenever the group of automorphisms of an object X has a natural topology, the mapping class group of X is defined as $\mathrm{Aut}(X)/\mathrm{Aut}_0(X)$, where $\mathrm{Aut}_0(X)$ is the path-component of the identity in $\mathrm{Aut}(X)$. [1]

The mapping class groups of irreducible orientable sufficiently large 3-manifolds (not necessarily Haken, since the boundary may be compressible) are finitely presented, and enjoy strong homological finiteness properties [**M1**] [**M$^+$**].

For compact oriented 3-manifolds whose universal cover satisfies the Poincaré Conjecture. If the mapping class group of each irreducible summand of M is finitely presented, then the mapping class group of M is finitely presented [**HM**].

Mapping class groups of 3-manifolds are closely related to mapping class groups of 2-manifolds. For example, let Σ be a surface, any homeomorphism $\varphi : \Sigma \to \Sigma$ gives a 3-manifold the mapping cone

$$M_\varphi := (\Sigma \times [0,1]) \,/\, ((x,0) \sim (\varphi(x), 1))$$

which is a surface bundle over S^1. The homotopy type of M_φ depends only on the class $[\varphi] \in MCG(\Sigma)$, which are classified by the following theorem.

THEOREM 3.1 (Thurston's classification theorem). *Given a homeomorphism $f : \Sigma \to \Sigma$, there is a map g isotopic to f such that at least one of the following holds:*

- *g is periodic, i.e. some power of g is the identity;*
- *g preserves some finite union of disjoint simple closed curves on Σ (in this case, g is called reducible); or*
- *g is pseudo-Anosov.*

3.2. The Geometries of Mapping Torus.

3.2.1. *Sphere.* When $\Sigma = S^2$, mapping tori of the identity map and antipode map have the geometry $S^2 \times \mathbb{R}$.

[1]In the compact-open topology, path components and isotopy classes coincide, i.e., two maps f and g are in the same path-component iff they are isotopic

3.2.2. *Torus.* When $\Sigma = \mathbb{T}^2$,

- the mapping torus of a Dehn twist [2] (reducible) of a 2-torus has the Nil geometry, and
- the mapping torus of a finite order automorphism of the 2-torus has the Euclidean geometry $\mathbb{E}^3$, and
- the mapping torus of an Anosov [3] map of the 2-torus has the Sol geometry.

3.2.3. *Negative Euler characteristic.* When Σ has negative Euler characteristic,

- the mapping torus of a reducible automorphism has no geometry, and
- the mapping torus of a finite order automorphism of the 2-torus has the geometry $\mathbb{H}^2 \times \mathbb{R}$, and
- the mapping torus of a pseudo-Anosov automorphism of the 2-torus has the hyperbolic geometry $\mathbb{H}^3$.

[2] By extending to the torus the twisting map $\left(e^{i\theta}, t\right) \mapsto \left(e^{i(\theta+2\pi t)}, t\right)$ of the annulus, through the homeomorphisms of the annulus to an open cylinder to the neighborhood of γ_a, yields a Dehn twist of the torus by a.

[3] an automorphism of the 2-torus given by an invertible 2 by 2 matrix whose eigenvalues are real and distinct, such as $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

CHAPTER 6

Further Topics

1. The Decision Problems for 3-Manifold Groups

1.1. Word Problem. The classical group-theoretic decision problems were formulated by Max Dehn in his work on the topology of surfaces about a century ago. And we will start with the word problem.

THEOREM 1.1. *The word problem is uniformly solvable in the class of fundamental groups of 3-manifolds* M^3.

Here our 3-manifold M's are compact, with or without boundaries. More precisely, there is an algorithm with input a finite presentation $\langle A|R\rangle$ of a 3-manifold group, and a word w on these generators A, and it tells whether the word w represents the identity or not. You can also tell whether two words represents the same element. The word uniform means this algorithm does not depend on the manifold or the presentation of the fundamental group of this 3-manifold.

A more topological interpretation of such an algorithm is as follows. Suppose we are given a 3-manifold (as a simplicial complex), and a loop γ based at a given basepoint, the algorithm determines whether the loop γ is null-homotopic.

We will see three proofs of this theorem.

1.1.1. *Proof 1.* The first proof is based on geometrization which gives residual finiteness (Hampel 1987), and gives the result.

Definition 1.2. A group Γ is **residual finite** if for any nontrivial element $g \in \Gamma$, there is a surjection $\varphi : \Gamma \to \Lambda$ to a finite quotient with $\varphi(g) \neq 1$.

Now we will show how residual finiteness gives us the result.

Given an input $\langle A|R\rangle$, and a word w, we shall try to check two things in parallel, both $w = 1$ and $w \neq 1$.

If we want to check $w = 1$, i.e.

$$w = \prod_{i=1}^{N} u_i r_i^{\epsilon_i} u_i^{-1}, r_i \in R, \epsilon_i = \pm 1$$

is the product of conjugates of defining relations, we just need to enumerate all such products and test equality in the free group of the generators. If $w = 1$, this method would eventually establish the desired result.

At the same we try to prove $w \neq 1$, which can be done using residual finiteness condition, i.e. we enumerate all morphisms $\varphi : \Gamma to \Lambda$ with Λ finite, and check whether $\varphi(w)$ dies or not. If $w \neq 1$, this will eventually find a morphism φ such that $\varphi(w) \neq 1$, and this will prove the result.

Next, we will discuss why geometrization implies residual finiteness. We may assume M is orientable, since taking orientable double covering reflects and preserves residual finiteness.

First, use Kneser's prime decomposition to write

$$M = M_1 \# \cdots \# M_n$$

then

$$\pi_1 M = \pi_1 M_1 * \cdots * \pi_1 M_n$$

and an element in a free product group is residual finite if and only if it is residual finite in each component (one direction requires some care). Thus we may assume M is prime, and this means we can further assume M is irreducible, since the only exceptions are not problems for understanding residual finiteness.

Next we cut our manifold M along incompressible tori via JSJ decomposition. Every cutting incompressible torus gives an injection $\mathbb{Z}^2 \hookrightarrow \Gamma = \pi_1 M$, and the components we have now are hyperbolic or Seifert fibered. We can use this decomposition to get a graph of groups as follows.

The $\mathbb{Z}^2$ copies are attached to the edges and the fundamental groups of the JSJ components are attached to the vertices, and you also get how the edge groups maps into the vertex groups. Then the fundamental group of M is the pushout of this graph of groups. The idea for this to work is, residual finiteness of the whole thing is inherited from the residual finiteness of the edge and vertex groups.

In general however, it is not true that this works for any graph of groups. A bad example is given by

$$\mathbb{Z}\bullet \circlearrowleft^{\times 2}_{\times 3} \ \mathbb{Z}$$

where the edge $\mathbb{Z}$ injects into the vertex $\mathbb{Z}$ by two morphisms, and in this case our group is

$$\langle a, t | t^{-1} a^2 t = a^3 \rangle$$

which is not residual finite. But in this particular setting, Hampel shows it works. This is related to how the $\mathbb{Z}^2$ pieces sits inside the complimentary bits. In this case, when you split up the incompressible tori to get the components, some of the eight geometries do not really show up at this stage (they appear in the prime decomposition stage). You won't get Nil or Sol, and this is important. The detailed proofs involves working with each cases by matrices and see how each $\mathbb{Z}^2$ sits inside, and build residual finiteness from there.

1.1.2. *Proof 2.* Let M be an orientable 3-manifold with no prime factors admit Nil or Sol geometry. Then $\pi_1 M$ is **automatic**. In this case, we have an efficient solvable word problem, and it also gives us quadratic Dehn functions.

The idea of automaticity is to use a directed graph with labeled edges to describe a language of some alphabet $\mathcal{S}$ (i.e. a subset of the free monoid generated by $\mathcal{S}^{\pm}$), and the language given by an automaton gives a normal form of the group we are interested in. That is, you can get a unique representation of your elements in your group as a word of generators. When your normal forms are of the shortest length, the language of normal forms are called regular.

Appendices

A. Surface Theory

THEOREM A.1 ([**B2**], Theorem 5). *Let h be a homeomorphic embedding of $S^{n-1} \times [0,1] \to S^n$. Then the closure of either complementary domain of $h(S^n\ times \frac{1}{2})$ is an n-cell.*

Proposition A.2. *A null-homotopic embedded circle in a surface bounds a disk in the surface.*

PROOF. Let S be the surface, assumed to be connected. Let $\gamma \subset S$ be the null-homotopic embedded circle. We must find an embedded disc in S with boundary γ.

If S is the sphere or the plane, this is just the Schőnflies Theorem.

Otherwise, consider a universal covering map $f : X \to S$ and let $\tilde{\gamma} \subset X$ be a homeomorphic lift of γ. From the classification of surfaces we see that X itself is the sphere or the plane, and so $\tilde{\gamma} = \partial D$ for some embedded disc $D \subset X$. The idea is now to prove that the restriction $f|_D : D \to S$ is injective, and it will then follow that $f(D)$ is an embedded disc with $\partial(f(D)) = \gamma$.

Consider the deck transformation action of $\pi_1 S$ on X. To prove that $f|_D$ is injective, it is equivalent to prove that for each deck transformation T, if T is not the identity then $D \cap (T \cdot D) = \emptyset$.

Suppose that $D \cap (T \cdot D) = \emptyset$. There are two cases.

In the first case, suppose that $\partial D \cap (T \cdot \partial D) = \emptyset$. The Schőnflies Theorem tells us that D is one of the two components of $X - \partial D = X - \tilde{\gamma}$, and similarly $T \cdot D$ is one of the two components of $X - \partial(T \cdot D) = X - (T \cdot \tilde{\gamma})$. It follows that either $D \subset T \cdot D$ or $T \cdot D \subset D$. In either case, the Brouwer Fixed Point Theorem implies that T has a fixed point. Since T is a deck transformation, it follows that T is the identity.

In the second case, suppose that $\partial D \cap (T \cdot \partial D) \neq \emptyset$, and so for some $x, y \in \partial D = \tilde{\gamma}$ we have $T \cdot x = y$. If $x = y$ then T is the identity, as in the previous case. If $x \neq y$ then we have $f(x) = f(y)$ and so $f|_{\tilde{\gamma}}$ is not injective, contradicting that $\tilde{\gamma}$ is a homeomorphic lift of γ. $\square$

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