Classifying Space and Bar Construction

Taught by Yuri Berest

Notes by Yun Liu
yl2649@cornell.edu

Cornell University
Fall, 2022

Last updated April 25, 2024.

Contents

Contents by Lecture

1 Introduction

1.1 Outline

1. Nerves of discrete groups.
2. Classifying space of compact Lie groups and finite loop spaces.
3. Nerves of categories.

2 Category Theory

2.1 Small Object Argument

Correction Let $\mathcal{M}$ be a model category and $I \subset \mathsf{Mor}(\mathcal{M})$ a set of maps.

Definition 2.1. I-fibration = the class of maps in $\mathcal{M}$ which has RLP with respect to I.
I-cofibrations = LLP with respect to I-fib.

Definition 2.2 (Hir, 11.1.2)**.** A model category $\mathcal{M}$ is cofibrantly generated if

1. there is a set of generating cofibrations morphisms $I \subset \mathsf{Mor}(\mathcal{M})$
   (a) I permits SOA.
   (b) $\mathsf{RLP}(I) = \mathsf{Fib}(\mathcal{M}) \cap \mathcal{W}$.
2. there is a set of generating acyclic cofibrations J such that
   (a) J permits SOA.
   (b) $\mathsf{RLP}(J) = \mathsf{Fib}(\mathcal{M})$.

Theorem 2.3 (Recognition Theorem)**.** *Assume $(\mathcal{M}, \mathcal{W})$ satisfies (MC1), (MC2), (MC3). Assume I, J*

1. *SOA.*
2. $J - \mathsf{cof} \subseteq I - \mathsf{cof} \cap W$
3. $I - \mathsf{Fib} \subseteq J - \mathsf{Fib} \cap W$
4. *either (2) or (3) is equality.*

Then $(\mathcal{M}, I, J, \mathcal{W})$ is a cofibrantly generated model category with

- *weak equivalences $\mathcal{W}$,*
- *fibrations* = $\mathsf{RLP}(J)$,
- *cofibrations* = $\mathsf{LLP}(\mathsf{Fib} \cap \mathcal{W})$.

Theorem 2.4 (Promoting Model Structure)**.** *Let $(\mathcal{M}, I, J, \mathcal{W})$ be a cofibrantly generated model category. Let $\mathcal{N}$ be a bicomplete category such that there is a pair of adjoint functors*

$$F : \mathcal{M} \rightleftarrows \mathcal{N} : U$$

Define $F(I) = \{F(f) | f \in I\}$ and $F(J) = \{F(f) | f \in J\}$. Assume

1. $F(I)$ *and* $F(J)$ *permits soa.*
2. U *maps relative* $F(J)$*-cell complexes to weak equivalences in* $\mathcal{M}$.

then $(\mathcal{N}, F(I), F(J), \mathcal{W}_{\mathcal{N}})$ where $\mathcal{W}_{\mathcal{N}} = \{f \in \mathsf{Mor}(\mathcal{N}) | UF \in \mathcal{W}_{\mathcal{M}}\}$ is cofibrantly generated model category. Moreover (F, U) is a Quillen pair.

Notation Let $\mathcal{M}$ be a model category and $I \subset \mathrm{Mor}(\mathcal{M})$, $\mathcal{C}$ a small category. $\mathcal{M}^{\mathcal{C}} = \mathrm{Fun}(\mathcal{C}, \mathcal{M})$.

$F_I^{\mathcal{C}} = \{\varphi \in \mathrm{Mor}(\mathcal{M}^{\mathcal{C}}) | \varphi_c \in I, \forall c \in \mathrm{Ob}(\mathcal{C})\}$.

Theorem 2.5 (Hir, 11.6.17). *Let $(\mathcal{M}, I, J, \mathcal{W})$ be a cofibrantly generated model category, and let $\mathcal{C}$ be any small category, then $(\mathcal{M}^{\mathcal{C}}, F_I^{\mathcal{C}}, F_J^{\mathcal{C}}, F_{\mathcal{W}}^{\mathcal{C}})$ is a cofibrantly generated model category with objectwise weak equivalences and objectwise fibrations.*

Sketch of proof. Based on Theorem 1 and 2.

Step 1. Let $i : \mathcal{C}^\delta \to \mathcal{C}$ be the discrete subcategory of $\mathcal{C}$, with

- objects are objects in $\mathcal{C}$
- morphisms are $\mathrm{Hom}_{\mathcal{C}^\delta}(c, c) = \{\mathrm{Id}_c\}, \mathrm{Hom}_{\mathcal{C}^\delta}(c, d) = \emptyset, c \neq d$.

then $\mathcal{M}^{\mathcal{C}^\delta} = \prod_{c \in \mathrm{Ob}(\mathcal{C})} \mathcal{M}$ is cofibrantly generated model category.

Step 2. The forgetful functor $U = i^* : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}^{\mathcal{C}^\delta}$ has a left adjoint, the free diagram functor

$$F : \mathcal{M}^{\mathcal{C}^\delta} \longrightarrow \mathcal{M}^{\mathcal{C}}$$

$F(X) := \coprod_{c \in \mathrm{Ob}(\mathcal{C})} X(c) \otimes F_*^c : \mathcal{C} \to \mathcal{M}$ so that for any $d \in \mathrm{Ob}(\mathcal{C})$,

$$F(x)(d) = \coprod_{c \in \mathrm{Ob}(\mathcal{C})} \coprod_{\mathrm{Hom}(c,d)} X(d)$$

where for $S : \mathcal{C} \to \mathbf{Set}$,

$$S \otimes \mathcal{M} : \mathcal{C} \longrightarrow \mathcal{M}$$
$$c \longmapsto S(c) \otimes A = \textstyle\coprod_{c \in \mathrm{Ob}(\mathcal{C})} A$$

here

$$S = F_*^c : \mathcal{C} \longrightarrow \mathbf{Set}$$
$$d \longmapsto \mathrm{hom}_{\mathcal{C}}(c, d)$$

Step 3. Check conditions of Theorem 2 for (F, U). □

2.2 Quillen functors and Quillen Equivalences

Lemma 2.6. *A Quillen pair*

$$F : \mathcal{M} \rightleftarrows \mathcal{N} : U$$

is an adjunction such that one of the following equivalent conditions hold.

1. F *preserves cofibrations and* U *preserves fibrations.*
2. F *preserves cofibrations and acyclic cofibrations.*
3. G *preserves fibrations and acyclic fibrations.*

Definition 2.7. A **Quillen equivalence** is a Quillen pair

$$F : \mathcal{M} \rightleftarrows \mathcal{N} : U$$

such that for any $X \in \mathrm{Ob}(\mathcal{M}^{cof})$ and $A \in \mathrm{Ob}(\mathcal{M}^{fib})$, $f : F(X) \to A$ is a weak equivalence in $\mathcal{N}$ if and only if $f^\# : X \to U(A)$ is a weak equivalence in $\mathcal{M}$, where

$$\hom_{\mathcal{N}}(FX, A) \overset{\cong}{\to} \hom_{\mathcal{M}}(X, UA)$$

$$f \longmapsto f^\#$$

Remark. A map $\mathcal{M} \to N$ is a left Quillen functor.

Definition 2.8 (Dugger)**.** A Quillen homotopy between (left) Quillen functors $F_1, F_2 : \mathcal{M} \to \mathcal{N}$, is a natural transformation $\varphi : F_1 \Rightarrow F_2$ such that for any $X \in \mathrm{Ob}(\mathcal{M}^{cof})$, $\varphi_X : F_1(X) \to F_2(X)$ are weak equivalences.

Two functors $F, F' : \mathcal{M} \to \mathcal{N}$ are **Quillen homotopic** if there is a chain of Quillen homotopies

$$F = F_0 \Leftarrow F_1 \Rightarrow \cdots \Leftarrow F_n = F'$$

Derived Functors $F : \mathcal{M} \to \mathcal{N}$ a functor between model categories.

Total left derived functors are right Kan extensions

Alternative Characterization Left derived functor

$LF : \mathcal{M} \to \mathrm{Ho}(\mathcal{N})$ is a universal homotopical functor if there is a natural tranformation $\varepsilon : LF \Rightarrow \gamma_{\mathcal{M}} \circ F$ which is terminal among all homotopical functors to $\gamma_{\mathcal{M}} \circ F$

$LF = \mathbb{F} \circ \gamma_{\mathcal{M}}$.

Theorem 2.9 (Quillen Adjunction Theorem)**.** *1. Any Quillen pair*

$$F : \mathcal{M} \rightleftarrows \mathcal{N} : U$$

induces an adjunction of homotopy categories

$$\mathbf{L}F : \mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{N}) : \mathbf{R}U$$

where $\mathbf{L}F(\overline{X}) = F(QX)$ *where* $QX \xrightarrow{\sim} X$ *is a cofibrant replacement and* $\mathbf{R}G(\overline{A}) = G(RA)$ *where* $A \xrightarrow{\sim} RA$ *is a fibrant replacement.*

2. If (F, U) *is a Quillen equivalence, then*

$$\mathbf{L}F : \mathcal{M} \rightleftarrows \mathcal{N} : \mathbf{R}U$$

is an equivalence of categories.

Next time.

1. Universal model categories.
2. Model approximation.

We say a map $L : \mathcal{M} \to \mathcal{N}$ between model categories to be a Quillen pair

$$L : \mathcal{M} \rightleftarrows \mathcal{N} : R.$$

3 Model Categories Generated by Small Categories

3.1 Motivation

Let $\mathcal{C}$ be a small category and let $\mathrm{Pre}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{op}, \mathbf{Set})$ be the category of presheaves on $\mathcal{C}$.

Remark. When X is a topological space, $\mathcal{C} = \mathrm{Open}(X)$ the poset of open sets of X, then any functor $F : \mathcal{C}^{op} \to \mathrm{Set}$ is equivalent to a presheaf on X.

The category $\mathrm{Pre}(\mathcal{C})$ has the UMP of being "free" cocomplete categories generated by $\mathcal{C}$.

Proposition 3.1. 1. *There is a natural fully faithful functor*

$$h : \mathcal{C} \hookrightarrow \mathrm{Pre}(\mathcal{C})$$

$$X \longmapsto (h_X : Y \mapsto \mathrm{Hom}_{\mathcal{C}}(Y, X))$$

satisfying the following universal property: given any cocomplete category $\mathcal{D}$ with

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\gamma"] \arrow[d, hook] & \mathcal{D} \\ \mathrm{Pre}(\mathcal{C}) \arrow[ur, "\mathrm{Re}"] & \end{tikzcd}$$

there is a functor $\mathrm{Re} : \mathrm{Pre}(\mathcal{C}) \to \mathcal{D}$ such that $\mathrm{Re} \circ h \xrightarrow{\sim} \gamma$ which is unique up to isomorphism.

2. $\mathrm{Re} : \mathrm{Pre}(\mathcal{C}) \to \mathcal{D}$ *has a right adjoint* Sing.

Proof. Note every presheaf on $\mathcal{C}$ can be canonically expressed as a colimit of representable presheaves in $\mathrm{Pre}(\mathcal{C})$.

Recall the Kan extension construction

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\gamma"] \arrow[d, "G"] & \mathcal{D} \\ \mathcal{E} \arrow[ur] & \end{tikzcd}$$

when $\mathcal{D}$ is cocomplete,

$$\mathrm{Lan}_G(\gamma)(e) = \operatorname*{colim}_{G \downarrow e}(G \downarrow e \xrightarrow{\pi} \mathcal{C} \xrightarrow{\gamma} \mathcal{D})$$

where $G \downarrow e$ is the slice category. In particular when G is fully faithful, $\mathrm{Lan}_G(\gamma) \circ G \cong \gamma$.

For any presehaf $\mathcal{F} \in \mathrm{Pre}(\mathcal{C})$,

$$\mathcal{F} \cong \operatorname*{colim}_{\mathcal{C}\downarrow\mathcal{F}}(\mathcal{C} \downarrow \mathcal{F} \xrightarrow{\pi} \mathcal{C} \xrightarrow{h} \mathrm{Pre}(\mathcal{C})).$$

Similarly,

$$\mathrm{Re}(\mathcal{F}) = \operatorname*{colim}_{\mathcal{C}\downarrow\mathcal{F}}(\pi^*\gamma).$$

This can be viewed as a realization of formal colimits in $\mathcal{D}$.

The right adjoint of Re is given by

$$\begin{aligned} \mathrm{Sing} : \mathcal{D} &\longrightarrow \mathrm{Pre}(\mathcal{C}) \\ d &\longmapsto (\mathrm{Sing}(d) : X \mapsto \mathrm{Hom}_{\mathcal{D}}(\gamma X, d)) \end{aligned}$$

□

Example 3.2.

1. geometric realization
2. nerve of small categories. categorization

Remark. This construction extends to an equivalence of categories

$$\mathcal{D}^{\mathcal{C}} \simeq \mathrm{Adj}(\mathrm{Pre}(\mathcal{C}), \mathcal{D}).$$

Goal: a generalization (homotopical completion of $\mathcal{C}$) by adding all homotopy colimits.

3.2 Universal Homotopy Categories

Given a small category $\mathcal{C}$ and two model categories $\mathcal{M}$ and $\mathcal{N}$ together with two functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\gamma"] \arrow[d, "r"] & \mathcal{M} \\ \mathcal{N} & \end{tikzcd}$$

we define the factorization of γ through r to be a triple (L, R, η) where

1. $L : \mathcal{N} \rightleftarrows \mathcal{M} : R$ is a Quillen pair.
2. $\eta : L \circ r \Rightarrow \gamma$ is a natural weak equivalence.

The category of factorizations $\mathrm{Fact}_{\mathcal{N}}(\gamma)$ is defined by

- objects are triples (L, R, η), and

- morphisms are natural transformations $\alpha : L \to L'$ such that

$$\begin{tikzcd} L \circ r \arrow[rr] \arrow[dr] & & L' \circ r \arrow[dl] \\ & \gamma & \end{tikzcd}$$

Definition 3.3. Let $\mathcal{UC} = s\mathsf{Pre}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{op}, \mathbf{sSet})$ be the category of simplicial presheaves equipped with projective model structure.

There is a natural functor

$$\begin{array}{ccccc} r : \mathcal{C} & \xhookrightarrow{h} & \mathsf{Pre}(\mathcal{C}) & \longrightarrow & s\mathsf{Pre}(\mathcal{C}) \\ X & \longmapsto & h_X & \longmapsto & h_X : [n] \mapsto h_X \end{array}$$

Theorem 3.4 (Dugger)**.** *$(\mathcal{UC}, r : \mathcal{C} \to \mathcal{UC})$ is the universal model category generated by $\mathcal{C}$ in the sense*

1. *any functor $\gamma : \mathcal{C} \to \mathcal{M}$ from $\mathcal{C}$ to a model category $\mathcal{M}$ factors*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\gamma"] \arrow[d, "r"'] & \mathcal{M} \\ \mathcal{UC} \arrow[ur] & \end{tikzcd}$$

up to natural weak equivalences $\eta : \mathsf{Re} \circ r \xrightarrow{sim} \gamma$.

2. *Moreover, then factorization is homotopically unique in the sense that $\mathtt{Fact}_{\mathcal{UC}}(\mathcal{UC})$ is contractible, i.e. $\mathtt{BFact}_{\mathcal{UC}}(\mathcal{UC}) \simeq \mathbf{pt}$.*

Remark. Notice that a category being contractible is a natural generalization of the property of being unique up to unique isomorphism.

Remark. Given a factorization is equivalent to give a cosimplicial resolution of diagrams $\mathcal{C} \xrightarrow{\gamma} \mathcal{M}$.

Next step is to construct a set of maps S in $\mathcal{UC}$ such that

$$\mathsf{Re} : \mathcal{UC}/S = \mathcal{L}_S(\mathcal{UC}) \rightleftarrows \mathcal{M} : \mathsf{Sing}$$

is a Quillen equivalence.

Example 3.5 (Homotopy Simplicial Groups)**.** Let $\mathcal{G} \subset \mathbf{Gr}$ be the full subcategory of finite generated free groups, then $\mathbf{Gr} \simeq \mathsf{Pre}^{\otimes}(\mathcal{G})$ product-preserving presheaves. Then $\mathbf{sGr} \simeq \mathbf{s}\mathsf{Pre}^{\otimes}(\mathcal{G})$.

Definition 3.6. A homotopy simplicial group $\Gamma : \mathcal{G}^{op} \to \mathbf{sSet}$ is one such that

$$\Gamma(\langle n\rangle) \simeq \prod_n \Gamma(\langle 1\rangle)$$

Theorem 3.7 (Badzioch)**.** *There is a Quillen equivalence* $\mathbf{sGr}^{h}\mathbf{sGr}$.

3.3 Bousfield-Kan Formula for Homotopy Colimits

Recall given a small category $\mathcal{C}$ together with $\mathsf{Pre}(\mathcal{C})$ the universal category gnerated by $\mathcal{C}$

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, hook, "h"] & \mathcal{D} \arrow[dl, shift left, "\mathsf{Sing}"] \\ \mathsf{Pre}(\mathcal{C}) \arrow[ur, shift left, "\mathsf{Re}"] & \end{tikzcd}$$

Key fact: denote $\mathcal{C} \downarrow \mathcal{F} = h \downarrow \mathcal{F}$, then

$$\pi^* h : \mathcal{C} \downarrow \mathcal{F} \xrightarrow{\pi} \mathcal{C} \xrightarrow{h} \mathsf{Pre}(\mathcal{C})$$

we have $\mathcal{F} \cong \mathrm{colim}_{\mathcal{C}\downarrow\mathcal{F}}(\pi^* h)$.

Consider

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\gamma"] \arrow[d, hook, "h"] & \mathcal{M} \arrow[dl, shift left, "\mathsf{Sing}"] \\ \mathcal{U}\mathcal{C} \arrow[ur, shift left, "\mathsf{Re}"] & \end{tikzcd}$$

Remark. In general, we cannot ask for η to ve a natural isomorphism. If γ takes X to $\gamma(X)$ not cofibrant, then the above diagram cannot commute. Note $r(X)$ is cofibrant in $\mathcal{U}\mathcal{C}$, thus $\mathsf{Re}(\gamma(X))$ is cofibrant.

We want a universal expression of simplicial presheaves in terms of representable presheaves rX. The answer is BK formula for homotopy colimits.

3.3.1 Classical Nerves

The simplex category Δ

nonstrictly order preserving maps

generators: coface codegeneracy

Simplicial relations

Simplicial Sets presentation X_* with face and degeneracy maps.

Example 3.8 (Standard n-simplex)**.** $\Delta[n]_*$ corepresents a functor

$$(-)_n : \mathbf{sSet} \longrightarrow \mathbf{Set}$$

$$X_* \longmapsto X_n$$

3.3.2 Categorical Realization

$$\Delta \hookrightarrow \mathbf{Cat}$$
$$[n] \mapsto \overrightarrow{n}$$

$$\mathfrak{c} : \mathbf{sSet} \rightleftarrows \mathbf{Cat} : \mathcal{N}$$

description of $\mathcal{N}_*$:
description of $\mathfrak{c}$: the free category generated by the graph at level ≤ 2

Exercise 3.9. $\mathfrak{c}$ is a left inverse, i.e. $\mathcal{N}_*$ is fully faithful.

3.3.3 Homotopy Inverse of $\mathcal{N}_*$

Construction Category of elements (BK construction) Given $\mathcal{C}$ a small category and $X : \mathcal{C} \to \mathbf{Set}$ $\mathcal{C}_{\mathcal{F}} = \mathcal{C} \int \mathcal{F}$

Definition 3.10. $\mathrm{hocolim}^{\mathbf{sSet}}_{\mathcal{C}}(\mathcal{F}) \coloneqq \mathcal{N}_*(\mathcal{C}_{\mathcal{F}})$

Definition 3.11. For $\mathcal{C} \in \mathrm{Ob}(\mathbf{Cat})$, $B\mathcal{C} \coloneqq |\mathcal{N}_*\mathcal{C}|$.

Definition 3.12. $\mathrm{hocolim}^{\mathbf{Top}}_{\mathcal{C}}(\mathcal{F}) \coloneqq B(\mathcal{C}_{\mathcal{F}})$.

Exercise 3.13. $X_* : \Delta^{op} \to \mathbf{sSet}$. There is a weak homotopy equivalence $|X| \simeq \mathrm{hocolim}_{\Delta^{op}}(X_*) \eqqcolon \|x\|$.

next time: ΔC, ΔG,
Borel fiber sequence

$$\|x\| \longrightarrow \|x\|^{cy} \simeq ES^1 \times_{S^1} |X| \longrightarrow BS^1$$

Example 3.14.

$$Ad : \underline{G} \longrightarrow \mathbf{Set}$$

$$* \longrightarrow G$$

$$g \in G \longmapsto Ad_g : G \to G$$

$\mathrm{hocolim}^{\mathbf{Top}}_{G}(Ad) \simeq \mathcal{L}(BG)$
cyclic nerve of G

3.4 BK formula

There is a Quillen adjunction

$$|-| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : \mathcal{S}ing$$

There is an equivalence of homotopy categories

$$\mathrm{Ho}(\mathbf{sSet}) \simeq \mathrm{Ho}(\mathbf{Top})$$

Example 3.15 (Classifying Space, or simplicial nerves)**.** Consider the composite

$$\mathsf{B} : \mathbf{Cat} \xhookrightarrow{\mathcal{N}_*} \mathbf{sSet} \xrightarrow{|-|} \mathbf{Top}$$

$$\mathcal{C} \longmapsto \mathcal{N}_*\mathcal{C} \longmapsto \mathsf{B}\mathcal{C}$$

There are two generalizations

1. Is there a similar construction when we replace **Top** by $G-$**Top**? What is such ΔG?
   There is one example for $G = S^1$ with ΔC the cyclic categories. And this leads to the concept of crossed simplicial groups.

endieck, Cisinski (07'), Maltsionitis

How to characterize small categories $\mathcal{A}$ that behave like Δ such that $\mathrm{Ho}(\mathrm{Pre}(\mathcal{A})) \simeq \mathrm{Ho}(\mathbf{Top})$?
Apart from Δ, there is also cubical category $\square$. (See Voevodsky's lectures in Cambridge about cubical sets, IN Institute.)

3.4.1 Homotopy Inverse of Nerves

Recall the category of elements $\mathcal{C}_F$ in $F : \mathcal{C} \to \mathbf{Set}$.

Example 3.16. Let $X : \Delta^{op} \to \mathbf{Set}$ be a simplicial set. Then Δ^{op}_X is the category with

- objects are $([n], x)$ where $[n] \in \mathrm{Ob}(\Delta^{op})$ and $x \in X_n$, and
- morphisms $f : ([m], y) \to ([n], x)$ are $f \in \mathrm{Mor}_{\Delta^{op}}([m],[n])$ such that $X(f)(y) = x$.

There is an alternative construction. Consider

$$h : \Delta \hookrightarrow \mathbf{sSet}$$

$$[n] \mapsto \Delta[n]_*$$

Define $\Delta X \coloneqq h \downarrow X = \Delta \downarrow X$ the category of simplicies of X, then

- objects are $([n], x)$ where $[n] \in \mathrm{Ob}(\Delta)$ and $x : \Delta[n]_* \to X$, or equivalently, objects are elements in $X_n, \forall n \geq 0$.
- morphisms $\varphi : ([n], x) \to ([m], y)$ are $\varphi \in \mathrm{Mor}_\Delta([n], [m])$ such that $y \circ \Delta\varphi = x$.

By Yoneda lemma, we can see that $\Delta_X^{op} \cong (\Delta X)^{op}$.

Exercise 3.17. Consider a small category $\mathcal{C}$ and take $\Delta\mathcal{C} := \Delta(\mathcal{N}_*\mathcal{C})$, consider $\Delta : \Delta \to \mathbf{Cat}$ then

$$\Delta\mathcal{C} \cong \Delta \downarrow \mathcal{C}.$$

Define

$$W_\infty := \{F : \mathcal{C} \to \mathcal{D} | B\mathcal{C} \simeq B\mathcal{D}\}$$

(the most fundamental localizers)

then there are equivalences of homotopy categories

$$\overline{\mathcal{N}} : \mathrm{Ho}(\mathbf{Cat}) \simeq \mathrm{Ho}(\mathbf{sSet}) \simeq \mathrm{Ho}(\mathbf{Top})$$

Theorem 3.18 (Latch,1987)**.** *The functor*

$$\begin{aligned} i_\Delta : \mathbf{sSet} &\to \mathbf{Cat} \\ X &\longmapsto \Delta X \end{aligned}$$

is a homotopy inverse of $\overline{\mathcal{N}}$ (but not adjoint to $\mathcal{N}_*$).

Idea Look at analogue of i_Δ for Δ replaced by $\mathcal{A}$.

Notation: Let $\mathcal{A}$ be a small category and write $\mathcal{A} - \mathbf{Set} = \mathrm{Pre}(\mathcal{A})$.

$$\begin{aligned} i_\mathcal{A} : \mathcal{A} - \mathbf{Set} &\longrightarrow \mathbf{Cat} \\ X &\longmapsto i_\mathcal{A}(X) := h^\mathcal{A} \downarrow X \end{aligned}$$

The functor $i_\mathcal{A}$ has a right adjoint

$$i_\mathcal{A} : \mathcal{A} - \mathbf{Set} \rightleftarrows \mathbf{Cat} : i_\mathcal{A}^*$$

Let $\varepsilon_\mathcal{A} : i_\mathcal{A}^* i_\mathcal{A} \Rightarrow \mathrm{Id}_{\mathbf{Cat}}$ be the counit and $\varepsilon_\mathcal{C} : i_\mathcal{A}^* i_\mathcal{A} \to \mathcal{C}$.

Definition 3.19 (Grothendieck)**.** $\mathcal{A}$ is called a **test category** if

1. $B\mathcal{A} \simeq \mathbf{pt}$ is contractible.
2. For any $\mathcal{C} \in \mathrm{Ob}(\mathbf{Cat})$ and for any $c \in \mathrm{Ob}(\mathcal{C})$, $B(\varepsilon_\mathcal{C} \downarrow \mathcal{C}) \simeq \mathbf{pt}$.

Theorem 3.20 (Cisinski). *For any test category $\mathcal{A}$, the category $\mathcal{A}-\mathbf{Set}$ has a closed model category structure with*

- *Cof = injective maps of $\mathcal{A}^{\mathrm{op}}$-diagrams of sets.*
- *We =* $f : X \to Y$ *in* $\mathcal{A}-\mathbf{Set}$ *such that* $B(i_{\mathcal{A}} f : B(i_{\mathcal{A}} X) \xrightarrow{\sim} B(i_{\mathcal{A}} Y)$ *in* **Top**.

There is a Quillen equivalence

$$i_{\mathcal{A}} : \mathcal{A}-\mathbf{Set} \rightleftarrows \mathbf{Cat} : i_{\mathcal{A}}^*$$

See [J].

Definition 3.21. The fundamental localizers $W = \{F : \mathcal{C} \to \mathcal{D}\}$ (the classes of weak equivalences) such that

1. 2-of-3 property
2. retraction property
3. $\{\mathcal{C} \to \mathbf{pt}\} \in W$ whenever $\mathcal{C}$ has a terminal objects.

Conjecture 3.22. $W_\infty = \bigcap\{$*all fundamental localizers* $W\}$

4 Cyclic Objects

4.1 Cyclic Category

Definition 4.1. ΔC has the same objects as Δ

1. objects $[n], n \geq 0$.
2. there is an embedding $\Delta \hookrightarrow \Delta C$ such that Δ is a subcategory of ΔC.
3. $\mathrm{Aut}_{\Delta C}([n]) = C_{n+1} = \mathbb{Z}/(n+1)\mathbb{Z}$
4. unique factorization (crossed relations)

crossed group category, Berger and M... 2014

Exercise 4.2. $\Delta C \cong \Delta^{op}$. Connes duality

Cyclic Homology $HC_*(k[\Gamma]) \cong H_*(ES^1 \times_{S^1} \mathcal{L}(B\Gamma), k)$.

4.1.1 Crossed Group Categories

Recall we have the following equivalence of homotopy categories

$$\mathrm{Ho}(\mathbf{sSet}) \simeq \mathrm{Ho}(\mathbf{Top})$$

and we 'd like to generalize to see if there is any equivalence of homotopy categories

$$\mathrm{Ho}(\mathbf{Set}^{\Delta G}) \simeq \mathrm{Ho}(G - \mathbf{Top})$$

4.1.2 Dwyer-Hopkins-Kan presentation

Definition 4.3 (Duplicial Category)**.** The category K^{op} is an amalgamation of Δ^{op} and Δ by identifying (in each dimension) the degeneracy operators $s_i \in \Delta^{op}$ with all but one coface operator $d^i \in \Delta$ and the codegeneracy operators s^j with all but one face operator $d_j \in \Delta^{op}$. Explicitly, K^{op} is defined by

- objects are the same as Δ, i.e. finite natural numbers $[n], n \geq 0$,
- morphism are generated by d_i, s_j subject to the relations

$$\begin{aligned} d_i d_j &= d_{j-1} d_i, i \leq j \\ s_j s_i &= s_i s_{j-1}, i \leq j \\ d_i s_j = s_{j-1} d_i &: [n], 0 < j-1 \leq n, \\ &= \mathrm{Id} - 1 \leq j - i \leq 0, \\ & s_j d_{i-1}, j-i < -1. \end{aligned}$$

Note $d_0 s_{n+1} \neq s_n d_0$.
Observe that there are two natural functors

$$j : \Delta^{op} \hookrightarrow \mathcal{K}^{op}$$

and an isomorphism

$$m : \mathcal{K}^{op} \longrightarrow \mathcal{K}$$

$$d_i \longmapsto s_{n-i}$$

$$s_i \longmapsto d_{n-i}$$

Proposition 4.4. *1. There is a curios endomorphism of* Id : $\mathcal{K}^{op} \to \mathcal{K}^{op}$

$$\nu = \{\nu_n = d_0 s_{n+1}^{n+1} : [n] \to [n]\}$$

2. DHK presentation: $\Delta C^{op} \cong \mathcal{K}^{op}/(\nu_n, n \geq 0)$ *with a functor*

$$\mathcal{K}^{op} \longrightarrow \Delta C^{op}$$

$$d_0 s_{n+1} \mapsto t_{n+1} = \tau_n^{op}$$

$$d_i \longmapsto d_i$$

$$s_j \longmapsto s_j$$

"Analogy" Δ or Δ^{op}, $\mathcal{K} \longleftrightarrow$ "differential operators" on varieties $\longleftrightarrow$ Differential operators.
$d_i \longleftrightarrow$ "coordinates" on varieties $q_i = x_i$, Elliptic Hall Algebras
$s_j \longleftrightarrow$ "derivations" in $p^i = \frac{\partial}{x_i}$
Dold-Kan correspondence $\longleftrightarrow$ Cannings-Holland Correspondence, Derived Hall Algebra
Crossed simplicial groups $\longleftrightarrow$ Double affine Hecke algebras

Theorem 4.5. ΔC *is generated by*

1. $\mathrm{Aut}_{\Delta C}([n]) =: C_n \cong \mathbb{Z}/(n+1), n \geq 0$
2. *any morphism* $f : [n] \to [n]$ *factors uniquely as* $f = \varphi \circ \gamma$ *where* $\gamma \in C_n$ *and* $\varphi \in \mathrm{Mor}(\Delta)$.

The assignment $[n] \mapsto C_n$ extends to a functor

$$C_* : \Delta C^{op} \to \mathbf{Set}$$

$$[n] \longmapsto C_n$$

$$a \longmapsto a_*$$

where $a_* : C_n \to C_m$

Example 4.6 (Coclyclic Space)**.** $\tau_n : (x_0, x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_n, x_0)$.

Theorem 4.7. *1. Let* $X : \Delta C^{op} \to \mathbf{Top}$ *be a cyclic space, then* $|X|$ *the geometric realization of* X *as a simplicial space carries a natural* S^1*-action.*

2. It extends to a functor

$$\mathbf{Top}^{\Delta C^{op}} \longrightarrow S^1 - \mathbf{Top}$$

3. $HC_*(k[X]) \cong H_*^{S^1}(|X|, k)$.

4.2 Cyclic Homology

Let ΔC be Connes' category.

Let k be a commutative ring.

Consider the following functor

$$C_* : \Delta C^{op} \to \mathrm{Mod}_k$$
$$[n] \longmapsto C_n$$

be a cyclic module. Define $\tilde{t}_n \coloneqq (-1)^n t_n$, it comes with relations

$$\tilde{t}_n^{n+1} = \mathrm{Id}$$
$$d_i \tilde{t}_n = -\tilde{t}_{n-1} d_{i-1}, 1 \le i \le n$$
$$s_i \tilde{t}_n = -\tilde{t}_n s_{i-1}, 1 \le i \le n$$
$$d_0 \tilde{t}_n = (-1)^n d_n$$
$$s_0 \tilde{t}_n = (-1)^n t_{n+1} s_n$$

This is Chern-Simons formalism.

Definition 4.8 (Tsygan, Loday-Quillen)**.** The **cyclic bicomplex** $CC_*(C)$ associated to C_* is defined by

$$\begin{tikzcd}
 & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] \\
\cdots \arrow[r] & C_2 \arrow[r, "1-\tilde{t}"] \arrow[d, "b"] & C_2 \arrow[r, "1-\tilde{t}"] \arrow[d, "-b'"] & C_2 \arrow[r, "1-\tilde{t}"] \arrow[d, "b"] & C_2 \arrow[d, "-b'"] \\
\cdots \arrow[r] & C_2 \arrow[r, "1-\tilde{t}"] \arrow[d, "b"] & C_2 \arrow[r, "1-\tilde{t}"] \arrow[d, "-b'"] & C_2 \arrow[r, "1-\tilde{t}"] \arrow[d, "b"] & C_2 \arrow[d, "-b'"] \\
\cdots \arrow[r] & C_2 \arrow[r, "1-\tilde{t}"] \arrow[d] & C_2 \arrow[r, "1-\tilde{t}"] \arrow[d] & C_2 \arrow[r, "1-\tilde{t}"] \arrow[d] & C_2 \arrow[d] \\
 & 0 & 0 & 0 & 0
\end{tikzcd}$$

Define the total complex of CC_* as

$$\mathrm{Tot}(CC_*) \coloneqq \left(\bigoplus_{p+q=n} C_{p,q}, d = d^h + d^v \right)$$

and cyclic homology is defined as

$$HC_*(C) \coloneqq H_*(\mathrm{Tot}_{ast}(CC_*)).$$

Let $CC_*^{\{2\}}$ be the subcomplex consisting of the first two columns, then Hochschild homology is

$$HH_*(C) \coloneqq H_*(\mathrm{Tot}(CC_*^{\{2\}}))$$

Remark. If (C_*, b') is acyclic, then the natural projection

$$CC_*^{\{2\}} \xrightarrow[q-sim]{\sim} (C_*, b)$$

induces an isomorphism

$$HH_*(C) \cong H_*(C_*, b)$$

Theorem 4.9 (Connes' Periodicity Exact Sequence)**.** *For any cyclic module, there is a long exact sequence ("ISB"):*

$$\cdots \longrightarrow HH_n(C) \xrightarrow{I} HC_n(C) \xrightarrow{S} HC_{n-2}(C) \xrightarrow{B} HH_{n-1}(C) \xrightarrow{I} \cdots$$

Proof. There is the following short exact sequence of bicomplexes

$$0 \longrightarrow CC_*^{\{2\}} \hookrightarrow CC_* \longrightarrow CC_*[2,0] \longrightarrow 0$$

where $CC_*[2,0]_{p.q} \coloneqq CC_{p-2,q}$ is a bidegree shift. This induces an exact sequence in total complexes

$$0 \longrightarrow \mathrm{Tot}(CC_*^{\{2\}}) \hookrightarrow \mathrm{Tot}(CC_*) \longrightarrow CC_*[2,0] \longrightarrow 0$$

□

Remark. The relation $b(1-\tilde{t}_n) = (1-\tilde{t}_{n-1})b'$ implies that

$$C_*^\lambda(C) \coloneqq (C_*/(1-\tilde{t}_*, b)$$

is a well-defined cyclic (Connes') complex. If $k \supseteq \mathbb{Q}$, then

$$CC_* \longrightarrow C_* \twoheadrightarrow C_*^\lambda$$

gives

$$HC_*(C) \cong H_*(C_*^\lambda, b).$$

In general, there is a spectral sequence with E^1 page

$$E^1_{p,q} \coloneqq H^{EM}_p(\mathbb{Z}/(n+1), C_q) \Rightarrow HC_{p+q}(C)$$

Quillen is trying to connect this to Bott periodicity.
G. Segal (2011)

Example 4.10. Let A be an associative algebra with 1. Define

$$\begin{aligned} C_*(A) : \Delta C^{op} &\longrightarrow \mathrm{Mod}_k \\ [n] &\longmapsto A^{\otimes(n+1)} \end{aligned}$$

with

$$\begin{aligned} d_i(a_0, \cdots, a_n) &= \begin{cases} (a_0, \cdots, a_i a_{i+1}, \cdots, a_n), & 0 \le i < n, \\ (a_n a_0, a_1, \cdots, a_{n=1}), & i = n. \end{cases} \\ s_i(a_0, \cdots, a_n) &= (a_0, \cdots, a_i, 1, a_{i+1}, \cdots, a_n) \\ t_n(a_0, \cdots, a_n) &= (a_n, a_0, \cdots, a_{n-1}) \end{aligned}$$

then $HH_*(C_*(A)) = HH_*(A)$.

If A is unital, $(C_*(A), b')$ is an acyclic complex.
Consider

$$B : C_n(A) \longrightarrow C_{n+1}(A)$$

where

$$B(a_0, \cdots, a_n) = \sum_{i=0}^{n} (-1)^{ni}(1, a_i, \cdots, a_n, a_0, \cdots, a_{i-1}) = (-1)^{ni}(a_i, 1, a_{i+1}, \cdots, a_n, a_0, \cdots, a_{i-1})$$

$B^2 = b^2 = 0$ and $Bb + bB = 0$. Then

$$HC_*(A) \coloneqq HC(C_*(A)).$$

4.2.1 Quillen's Ayclisim Formalism

Theorem 4.11 (Loday-Quillen-Tsygan Theorem)**.** *Let $k \supseteq \mathbb{Q}$ be a field, and let A be a unital associative algebra. Define*

$$\mathfrak{gl}_\infty(A) \coloneqq \lim_{n\to\infty} \mathfrak{gl}_n(A)$$

where $\mathfrak{gl}_n(A) = (\mathbb{M}_n(A), [-,-])$. Then

$$H^{CE}_*(\mathfrak{gl}_\infty(A); k) \cong \lim_{n\to\infty} H^{CE}_*(\mathfrak{gl}_n(A); k) \cong \Lambda_k[HC_{*-1}(A)]$$

where Λ_k is graded symmetric algebra. This is analogue of Quillen's Q-construction.

Note $H_*^{CE}(\mathfrak{gl}_\infty(A); k)$ is a graded Hopf algebra.
Compare this with $GL_\infty(A)$ and higher K-theory construction

$$K_i(A) \coloneqq \pi_i(BGL_\infty(A)^+$$

Let $K_i(A)_{\mathbb{Q}} = K_i(A) \otimes_{\mathbb{Z}} \mathbb{Q}$, then

$$H_*(GL_\infty(A), \mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(K_*(A)_{\mathbb{Q}}).$$

This can be thought as an infinitestimal K-theory.

4.2.2 Cyclic (Co)homology and Algebra Extensions

Remark. This is the true Grothendieck understanding.

Let A be an associative k-algebra.
Consider the category of all algebra extensions $\mathbf{Alg}_k \downarrow A$ with

- objects are algebra extensions $\{f : R \twoheadrightarrow A\}$
- morphisms are $f : R \to R'$ such that

$$\begin{tikzcd} R \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & R \arrow[dl, "f'"] \\ & A & \end{tikzcd}$$

Higher Trace Functor

$$\mathbf{Alg}_k \downarrow A \longrightarrow \mathrm{Mod}_k$$

$$(R \twoheadrightarrow A) \longmapsto R/(I^{n+1} + [R, R])$$

Theorem 4.12. *For $n \geq 0$,*

$$HC_n(A) \cong \begin{cases} \lim_{\mathbf{Alg}_k \downarrow A} R/(I^{n+1} + [R, R]), & n = 2k, \\ \lim_{\mathbf{Alg}_k \downarrow A} (I^{k+1}/(I^k, I)), & n = 2k+1 \end{cases}$$

even trace

Question 4.13. *odd trace?*

4.2.3 Chern-Simons Formalism for Algebra Cochains

Reduced Bar Construction Let A be a nonunital algebra. Define

$$\overline{B}(A) := \bigoplus_{n=1}^{\infty} \overline{B}(A)_n$$

where $\overline{B}(A)_n = A^{\oplus n}$ with

$$b'(a_1, \cdots, a_n) = \sum_{i=1}^{n-1} (-1)^{i-1} (a_1, \cdots, a_i a_{i+1}, \cdots, a_n)$$

and

$$\begin{aligned} \Delta : \overline{B} &\longrightarrow \overline{B} \otimes \overline{B} \\ (a_1, \cdots, a_n) &\longmapsto \textstyle\sum_{i=1}^{n-1} (a_1, \cdots, a_i) \otimes (a_{i+1}, \cdots, a_n) \end{aligned}$$

Claim 4.14. $(\overline{B}(A), b', \Delta)$ *is a DG coalgebra with* $d(ab = d(a)b + (-1)^{|a|} a d(b)$.

On algebras, there are two natural constructions

- universal traces

$$\mathrm{Tr}^{un} : A \longrightarrow A_\sharp = A/\,[A, A]$$

Universal cotrace on DG coalgebras

$$C^\sharp := \ker\left(C \xrightarrow{\Delta - \tau\circ\Delta} C \otimes C\right)$$

we have

$$(\overline{B}A)^\sharp[1]) \cong C^\lambda(A)$$

Exercise 4.15. What is $CC_*(A)$?

Koszul dual construction
noncommutative de Rham complex

4.3 Functor Homology

Let $\mathcal{C}$ be a small category, and let k be a commutative ring (or a field). We may consider

$$\mathrm{Mod}_k(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}, \mathrm{Mod}_k)$$

as left $\mathcal{C}$-modules and

$$\mathrm{Mod}_k(\mathcal{C}^{op}) := \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Mod}_k$$

as right $\mathcal{C}$-modules, which are abelian categories with enough projective and injective objects.

Define functor tensor product

$$\begin{aligned} -\bigotimes_{\mathcal{C},k} - : \mathrm{Mod}_k(\mathcal{C}^{op}) \times \mathrm{Mod}_k(\mathcal{C}) &\longrightarrow \mathrm{Mod}_k \\ (\mathcal{N}, \mathcal{M}) &\longmapsto \mathcal{N} \bigotimes_{\mathcal{C},k} \mathcal{M} \end{aligned}$$

where

$$\mathcal{N} \bigotimes_{\mathcal{C},k} \mathcal{M} := \bigoplus_{c \in Ob(\mathcal{C})} \mathcal{N}(c) \bigotimes \mathcal{M}(c)$$

which is right exact (in each argument) and (left) balanced, which induces derived functors

$$-\bigotimes^{\mathbb{L}}_{\mathcal{C},k} - : \mathcal{D}(\mathrm{Mod}_k(\mathcal{C}^{op})) \times \mathcal{D}(\mathrm{Mod}_k(\mathcal{C})) \longrightarrow \mathcal{D}(k) = \mathcal{D}(\mathrm{Mod}_k)$$

and we can define

Definition 4.16. $\mathrm{Tor}^{\mathcal{C}}_*(\mathcal{N}, \mathcal{M}) := H_*[\mathcal{C} \overset{\mathbb{L}}{\underset{\mathcal{C},k}{\bigotimes}} \mathcal{M}]$

Example 4.17. Consider k-linearized Yoneda functor: fix $A \in Ob(\mathcal{C})$,

$$\begin{aligned} k\left[h^A\right] : \mathcal{C} &\xrightarrow{h^A} \mathrm{Mod}_k \\ X &\longmapsto \mathrm{Hom}_{\mathcal{C}}(A, X) \mapsto k\left[\mathrm{Hom}_{\mathcal{C}}(A, X)\right] \\ k\left[h_A\right] : \mathcal{C} &\xrightarrow{h_A} \mathrm{Mod}_k \\ X &\longmapsto \mathrm{Hom}_{\mathcal{C}}(X, A) \mapsto k\left[\mathrm{Hom}_{\mathcal{C}}(X, A)\right] \end{aligned}$$

There is a (linearized) Yoneda lemma: for any $A \in Ob(\mathcal{C})$, $N \in \mathrm{Mod}_k(\mathcal{C})$, $M \in \mathrm{Mod}_k(\mathcal{C})$,

$$\mathcal{N} \bigotimes_{\mathcal{C},k} k[h^A] \cong \mathcal{N}(A)$$
$$k[h^A] \bigotimes_{\mathcal{C},k} \mathcal{M} \cong \mathcal{M}(A)$$

4.3.1 Hochschild Homology

Given a simplicial module

$$\begin{aligned} E_* : \Delta^{op} &\to \mathrm{Mod}_k \\ [n] &\longmapsto E_n \end{aligned}$$

can define

$$H_*(E) \coloneqq H_*(C_*(E), d)$$

where $d = \sum_{i=0}^{n}(-1)^i d_i$.

Example 4.18. Let A be an associative k-algebra. The cyclic module

$$\begin{aligned} C_*(A) : \Delta C^{op} &\to \mathrm{Mod}_k \\ [n] &\longmapsto A^{n+1} \end{aligned}$$

can be restricted to a simplicial module.

Given an A-bimodule $\mathcal{M}$, we can introduce coefficients

$$\begin{aligned} C_*(A, M) : \Delta^{op} &\longrightarrow \mathrm{Mod}_k \\ [n] &\longmapsto M \otimes_k A^n \end{aligned}$$

where

$$d_i(m, a_1, \cdots, a_n) \coloneqq \begin{cases} (ma_1, a_2, \cdots, a_n), & i = 0, \\ (m, a_1, \cdots, a_i a_{i+1}, \cdots, a_n), & 0 < i < n, \\ (a_n m, a_1, \cdots, a_{n-1}) \end{cases}$$

then classical Hochschild homology is defined as

$$HH_*(A, M) \coloneqq H_*(C_*(A, M)).$$

4.3.2 Cyclic Homology

Question 4.19. *What are coefficients for cyclic homology?*

Answer. One possible answer is by Kaledin.

Example 4.20. Suppose $X : \Delta^{op} \to \mathbf{Set}$ is a simplicial set, we can linearize it to a functor then

$$H_*(k[x]) \cong H_*(|X|, k).$$

Proposition 4.21. *For any simplicial module $E : \Delta^{op} \to \mathrm{Mod}_k$*

$$\mathrm{Tor}_*^{\Delta^{op}}(k, E) \cong H_*(E)$$

In particular, if $E = C_*(A, M)$, then

$$\mathrm{Tor}_*^{\Delta^{op}}(k, C_*(A, M)) \cong HH_*(A, M)$$

where

$$\begin{aligned} k : \Delta &\to \mathrm{Mod}_k \\ [n] &\longmapsto k \end{aligned}$$

is a trivial cosimplcial module.

Proof. There is a natural isomorphism

$$\mathrm{Tor}_{*\mathcal{C}}(\mathcal{N},\mathcal{M}) \cong \mathrm{Tor}_*^{\mathcal{C}^{op}}(\mathcal{M},\mathcal{N})$$

we need to prove

$$\mathrm{Tor}_*^{\Delta}(E,k) \cong H_*(E)$$

We construct for $k : \Delta \to \mathrm{Mod}_k$ the projective resolution as follows.

Let

$$\Delta_n = h^{[n]} : \Delta \longrightarrow \mathbf{Set}$$

$$[m] \longmapsto \mathrm{Hom}_\Delta([n],[n])$$

and we may linearize ti to be a functor

$$K_n = k[\Delta_n] : \Delta \longrightarrow \mathrm{Mod}_k$$

$$[m] \longmapsto k[\mathrm{Hom}_\Delta([n],[n])]$$

which composes together to form a cosimplicial complex

$$K_* : \Delta \to \mathrm{Mod}_k^{\Delta^{op}} \xrightarrow{C_*} \mathrm{Ch}_k$$

$$[m] \longmapsto K_*[m] : [n] \mapsto k[\mathrm{Hom}_\Delta([n],[m])]$$

and

$$k : \Delta \to \mathrm{Mod}_k \hookrightarrow \mathrm{Ch}_k$$

$$[m] \longmapsto k \longmapsto [0 \leftarrow k \leftarrow 0]$$

with augmentation

$$\varepsilon_{[m]} : K_0([m]) \cong k[\{0,\cdots,m\}] \to k i \mapsto \delta_{i0}$$

□

Lemma 4.22. $K_*(\bullet) \to k$ *is a projective resolution of* k *as a trivial* Δ*-module.*

Proof. For each $m \geq 0$, we have

$$H_i(K_*([m])) \cong \begin{cases} k, & i = 0 \\ 0, & i > 0 \end{cases}$$

where we have isomorphisms

$$H_i(K_*([m]))$$

Each K_n is a projective object in Mod_k^{Δ}.

$$\mathrm{Tor}_*^{\Delta^{op}}(k, E) \cong \mathrm{Tor}_*^{\Delta}(E, k) \cong H_*(E \bigotimes_{\Delta} K_*) \cong H_*(E).$$

where the second last isomorphism follows from the lemma and the last isomorphism is from one previous example. □

Theorem 4.23 (Connes). *For any cyclic module,* $C : \Delta C^{op} \to \mathrm{Mod}_k$, *there is a natural isomorphism*

$$\mathrm{Tor}_*^{\Delta C^{op}}(k, C) \cong HC_*(C).$$

Sketch. For $(p, q) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} (K(p, q))_* : \Delta &\longrightarrow \mathrm{Mod}_k \\ [n] &\longmapsto k\,[\mathrm{Hom}_{\Delta}([q]\,, [n])] \end{aligned}$$

□

extend this to ΔC.

Given $g \in \mathrm{Aut}_{\Delta C}([n])$, let g act by $\circ$,

$$\begin{aligned} (K(p, q))_* : \Delta &\longrightarrow \mathrm{Mod}_k \\ g : ([n] \xrightarrow{f} [n]) &\mapsto ([q] \xrightarrow{f} [n] \xrightarrow{g} [n]) \end{aligned}$$

For each $n \geq 0$, (with p fixed)

$$\begin{aligned} (K(p, \bullet))_n : \Delta^{op} &\longrightarrow \mathrm{Mod}_k \\ [q] &\longmapsto (K(p, q))_n \end{aligned}$$

which gives rise to

$$\begin{aligned} (K(p, \bullet))_n : \Delta^{op} &\to \mathrm{Mod}_k^{\Delta C} \\ [q] &\longmapsto (K(p, q))_n \end{aligned}$$

which gives us a complexx of ΔC-module, then we can get a biresoultion of

ΔC-module.

$$\begin{tikzcd}
(K(0,2))_* \arrow[r, leftrightarrow, "N_2"] & (K(1,2))_* \arrow[d] & (K(2,2))_* \arrow[l, "1-\tilde{t}_2"'] \arrow[d] \\
(K(0,1))_* \arrow[r, leftrightarrow, "N_1"] & (K(1,1))_* \arrow[d] & (K(1,1))_* \arrow[l, "1-\tilde{t}_1"'] \arrow[d] \\
(K(0,0))_* \arrow[r, leftrightarrow, "N_0"] & (K(1,0))_* \arrow[d] & (K(0,0))_* \arrow[l, "1-\tilde{t}_0"'] \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

Next time:

$$|X| \longrightarrow ES^1 \times_{S^1} |X| \longrightarrow BS^1$$

Gysin sequence and Connes' Periodicity Sequence

4.4 Topological Interpretation of Cyclic Homology

Cyclic Spaces A cyclic space is a functor

$$X : \Delta C^{op} \to \mathbf{Top}$$
$$[n] \longmapsto X_n$$

The underlying simplicial space is given by precomposing with $\Delta^{op} \hookrightarrow \Delta C^{op}$.

We assume that X is a "good simplicial space" (in the sense of G. Segal "Categories and Cohomology Theories", Appendix A), so that geometric realization

$$|X| = \left(\coprod_{n \geq 0} X_n \times \Delta^n \right) / \sim$$

where the equivalence relation is defined as

$$\left(d_i^X(x), (t_0, \cdots, t_n)\right) \sim (x, d_\Delta^i(t_0, \cdots, t_n))$$
$$\left(s_j^X(x), (t_0, \cdots, t_n)\right) \sim (x, s_\Delta^j(t_0, \cdots, t_n))$$

The pathology is, it may happen that $f : X_* \to Y_*$ such that $f_n : X_n \to Y_n$ for all $n \geq 0$, but $|f| : |X_*| \not\simeq |Y_*|$.

Definition 4.24 (Segal)**.** A fat realization

$$\|X\| \coloneqq \coprod_{n \geq 0} (X_n \times \Delta^n) / \sim$$

where $\sim$ is only generated by the face relations.

There is a canonical comparison map

$$f_X : \|X\| \longrightarrow |X|$$

$X : \Delta^{op} \to \mathbf{Top}$ is **good** if f_X is homotopy equivalence.

Theorem 4.25 (Segal, Proposition A1)**.** *Let* $X : \Delta^{op} \to \mathrm{Top}$

1. *If all* X_n *are of homotopy type of CW complex, so is* $\|X\|$.
2. *If* $f : X_* \to Y_*$ *such that* $f_n : X_n \xrightarrow{\sim} Y_n$ *for all* $n \geq 0$*, then* $\|X\| \xrightarrow{\|f\|} \|Y\|$.
3. $\|X \times Y\| \simeq \|X\| \times \|Y\|$.

Example 4.26. For a bisimplcial set $X_{*,*} : \Delta^{op} \to \mathbf{sSet}$, $|X_{*,*}| : \Delta^{op} \to \mathbf{sSet} \xrightarrow{|-|} \mathbf{Top}$ is always a good simplicial space.

Remark (C. Berger-K. Ratlcovic, "Gabriel=Morita theorem", 2019). [Wendli 2018, P. May, Ajay and Yuri] Let $\Delta_{inj} \subseteq \Delta$ be the subcategory of injectives generated by d^i's.

$$i : \Delta^{op}_{inj} \hookrightarrow \Delta^{op}$$

induces an adjunction

$$i_! : \Delta^{\Delta^{op}_{inj}} \rightleftarrows \mathbf{Set}^{\Delta^{op}} : i^*$$

which gives $\Delta^{\Delta^{op}_{inj}}$ induced model structure with weak equivalences maps $f : X \to Y$ such that $i_! f : i_! X \xrightarrow{\sim} i_! Y$ is a weak equivalence in **sSet**.

Curious Fact There is no model structure on $\Delta^{\Delta^{op}_{inj}}$, thus Δ_{inj} is not a test category. Thus the above adjunction can never be a Quillen adjunction, but a right model approximation.

Note we always have $\|X\| \sim |i_!(X)|$, but $|i_! i^*(X)| \simeq |X|$ provided X is fibrant, i.e. a Kan complex.

Claim 4.27. $(i_!, i^*)$ *is a right model approximation of* $\mathbf{Set}^{\Delta^{op}_{inj}}$ *such that*

$$\mathrm{Ho}(\mathbf{Set}^{\Delta_{inj}}) \subseteq \mathrm{Ho}(\mathbf{sSet}).$$

Theorem 4.28. *Let* $X : \Delta C^{op} \to \mathbf{Top}$ *be a (good) cyclic space. Then*

1. $|X|$ *carries a canonical* S^1*-action.*
2. $X \mapsto |X|$ *extends to a functor*

$$|-| : GCS \longrightarrow S^1 - Sp$$

How can we define S^1-action?
Recall

$$C_* : \Delta C^{op} \to \mathbf{Set}$$

$$[n] \longmapsto C_n = \mathrm{Aut}_{\Delta^{op}}([n]) \cong \mathbb{Z}/(n+1)$$

there is another way to define $C_* \cong S^1_* = \Delta[1]_*/\partial\Delta[1]_*$ with $|C_1| \cong S^1$.

Naive idea is to define a map

$$ev : C_n \times X_n \longrightarrow X_n$$

$$(g, x) \longmapsto X(g)x$$

Exercise 4.29. This is not a simplicial map, so it does not induce a map on geometric realization unless X is a trivial cyclic space.

The modification is to introduce a new "twisted" simplicial set $F(X)$ with two properties

1. $|F(X)| \sim |C_* \times X_*| \cong S^1 \times |X|$.
2. there is a canonical map $F(X) \to X$ inducing the action map $S^1 \times |X| \to |X|$.

Lemma 4.30. *The fogetful functor*

$$F : \mathbf{Top}^{\Delta^{op}} \rightleftarrows \mathbf{Top}^{\Delta C^{op}} : U$$

with three properties

1. $F(\mathbf{pt}) = C_*$.
2. *the map* $F(X) \to X$ *comes from the counit* $\varepsilon_X : FU(X) \to X$.
3. *for any simplicial space* X*, there is canonical homeomorphism*

$$|F(X)| \cong |C_*| \times |X| \cong S^1 \times |X|.$$

Proof. Recall in ΔC^{op} we defined the two operations

1. for all $g \in C_m$,

$$g^* : \mathrm{Hom}_{\Delta^{op}}([m], [n]) \to \mathrm{Hom}_{\Delta^{op}}([m], [n])$$

$$a \longmapsto g^*(a)$$

2. for all $a \in \mathrm{Hom}_{\Delta C^{op}}([m],[n])$,

$$\begin{aligned} a_* : C_m &\to C_n \\ g &\longmapsto a_*(g) \end{aligned}$$

which are defined by the unique factorization property $a \circ g = a_*(g) \circ g^*(a)$ in ΔC^{op}:

$$\begin{tikzcd} {[n]} & {[m]} \arrow[l, "a"'] \\ {[n]} \arrow[u, "a_* g"] & {[m]} . \arrow[l, "g^* a"] \arrow[u, "g"'] \end{tikzcd}$$

$\square$

Explicitly,

$$\begin{aligned} F : \mathbf{Top}^{\Delta^{op}} &\longrightarrow \mathbf{Top}^{\Delta C^{op}} \\ Y &\longmapsto F(Y)_* : \Delta C^{op} \to \mathbf{Top} \end{aligned}$$

where $F(Y)_n = C_n \times Y_n$.

For any $f \in \mathrm{Hom}_{\Delta^{op}}([m],[n])$, write $f = h \circ a$ where $a \in \mathrm{Hom}_{\Delta^{op}}([m],[n])$ and $h \in \mathrm{Aut}_{\Delta C^{op}}([n]) = C_n$. Then

$$\begin{aligned} F(Y)_f : F(Y)_m &\longrightarrow F(Y)_n \\ (g,y) &\longmapsto (h \cdot a_*(g), Y(g^*_*(a))(y)) \end{aligned}$$

Define for any cyclic space $X : \Delta C^{op} \to \mathbf{Top}$

$$ev : F(UX) = F(X)_* \longrightarrow X_*$$

$$\begin{aligned} ev_m : C_m \times X_m &\longrightarrow X_m \\ (g,x) &\longmapsto X(g)(x) =: g_*(x) \end{aligned}$$

Need to check

$$\begin{tikzcd} F(X)_m \arrow[r, "ev_m"] \arrow[d, "F(X)_a"'] & X_m \arrow[d, "a_*"] \\ F(X)_n \arrow[r, "ev_n"] & X_n \end{tikzcd}$$

for all $a \in \mathrm{Hom}_{\Delta C^{op}}([m],[n])$, i.e. whether the following diagram

$$\begin{tikzcd} (g,x) \arrow[r, mapsto] \arrow[d, mapsto] & g_*(x) \arrow[d, mapsto] \\ (a_*(g), (g^*(a))_*(x)) \arrow[r, mapsto, "?"] & a_* g_*(x) \end{tikzcd}$$

Check

$$\begin{aligned} ev_n\left(F(X)_a(g,x)\right) &= ev_n\left(a_*(g),(g^*(a))_*(x)\right) \\ &= (a_*(g))\left(g^*(a)_*(X)\right) \\ &= \left(a_*(g)\circ g^*(a)\right)_*(x) \\ &= (a\circ g)_*(x) \\ &= a_*\left(g_*(x)\right). \end{aligned}$$

Now we can define

$$\Phi : \mathrm{Hom}_{\mathbf{Top}^{\Delta C^{op}}}(F(Y),X) \longrightarrow \mathrm{Hom}_{\mathbf{Top}^{\Delta^{op}}}(Y,UX)$$

$$\left(F(Y)\xrightarrow{\alpha}X\right) \longmapsto \left(Y\xrightarrow{i}F(Y)\xrightarrow{\alpha}X\right)$$

$$\Psi : \mathrm{Hom}_{\mathbf{Top}^{\Delta^{op}}}(Y,UX) \longrightarrow \mathrm{Hom}_{\mathbf{Top}^{\Delta C^{op}}}(F(Y),X)$$

$$(\beta : Y\to UX) \longmapsto \left(ev\circ F : F(Y)\to F(UX)=F(X)\xrightarrow{ev}X\right)$$

Check

$$\Phi\circ\Psi=\mathrm{Id},\quad \Psi\circ\Phi=\mathrm{Id}.$$

It remains to prove for any simplicial topological space X,

$$|F(X)| \xrightarrow{\cong} |C|\times|X|$$

We will define

$$p_1 : |F(X)| \twoheadrightarrow |C|$$

$$p_2 : |F(X)| \twoheadrightarrow |X|$$

and show that

$$(p_1,p_2) : |F(X)| \xrightarrow{\cong} |C|\times|X|$$

1. p_1 is induced by the map of simplicial space

$$p_{1,*} : F(X)_* \to C_*$$

$$(g,x) \longmapsto g$$

2. p_2 is defined as follows.
   Recall $\Delta^* : \Delta\to\mathbf{Top}$ can be extended to $\Delta C\to\mathbf{Top}$ where $\tau_n : [n]\to[n]$ acts by

$$e_i \mapsto \begin{cases} e_{i-1}, & 1\le i\le n \\ e_n, & i=0 \end{cases}$$

so it acts on Δ^n by

$$\tau_n(x_0, \cdots, x_n) = (x_n, x_0 \cdots, x_{n-1}).$$

Now we can define

$$p_2(g, x, u) = (x, g^*(u))$$

where g^* is the standard action of $g \in \mathrm{Aut}([n])$ on Δ^*. We need to check p_2 is well-defined. For any $f \in \mathrm{Hom}_{\Delta^{op}}([m], [n])$,

$$\begin{aligned} p_2(f(g, x); u) = p_2\left(f_*(g), g^*(f)_*(x); u\right) \\ &= (g^*(f)_*(x); (f_*(g))^*(u)) \\ &= (x, g^* f^*(f_*(g))^*(u)) \\ &= (x, (g^*(f) \circ f_*(g))^*(u)) \\ &= (x, (g \circ f)^*(u)) \\ &= (x, g^*(f^*(u))) \\ &= p_2(g, x; f^*(u)) \end{aligned}$$

3. $(p_1, p_2) : |F(X)| \to |C| \times |X|$ is well-defined, and we can define a natural map

$$h_X : |F(X)| \xrightarrow{\sim} |C \times X|$$

$$(g, x; u) \mapsto (g^{-1}, x; g^*(u))$$

h_X is a homeomorphism.

4. To prove (2) we factor (p_1, p_2) as follows:

$$(p_1, p_2) : |F(X)| \xrightarrow[\cong]{h_X} |C \times X| \xrightarrow[\cong]{(|p_1|, |p_2|)} |C| \times |X| \xrightarrow[\cong]{h_{pt} \times \mathrm{Id}} |C| \times |X|$$

If we identify $|C| \cong S^1$ via $C_* \cong S^1_*$, use evaluation map we define the action map

$$\xi_X : |C| \times |X| \xrightarrow{(p_1, p_2)} |F(X)| \xrightarrow{ev} |X|.$$

Lemma 4.31. $\xi_{|C|} : |C| \times |C| \to |C|$ *is the standard group structure on* S^1.

Proof. The diagram commutes

$$|C| \times |C| \xleftarrow{(p_1, p_2)} |F(C)| \xrightarrow{ev} |C|$$

Consider $F(C) = \{C_n \times C_n\}_{n \geq 0}$.

- there are three nondegenerate simplices in dim 1: $(1, t_1), (t_1, 1), (t_1, t_1)$.

- there are two nondegenerate simplices in dim 2: $(t_2, t_2), (t_2^2, t_2^2)$.

The triangulation for $S^1 \times S^1$ is as follows

•

For $(u, v) \in [0, 1]^2$,

$$(u, v) \in \begin{cases} \{((t_2^2, t_2^2), \Delta^2)\} & u + v \leq 1 \\ \{((t_2, t_2), \Delta^2)\} & u + v \geq 1 \end{cases}$$

Check

$$ev : F(C)_* \longrightarrow C_*$$

$$(t_2, t_2) \longmapsto t_2^2 = s_0(t_1)$$

$$(t_2^2, t_2^2) \longmapsto t_2^4 = t_2 = s_1(t_1)$$

so

$$|ev|(u, v) = \begin{cases} u + v, & \in \{t_1\} \times \Delta^2 \\ u + v - 1, & \in \{t_1\} \times \Delta^2 \end{cases}$$

identify this with multiplication of S^1.

it remains to check that $\xi \circ (1 \times \xi) = \xi(\xi \times 1)$ and ξ is well-defined.

Given any $X \in \mathbf{Top}^{\Delta C^{op}}$ define

$$\mu : F(F(X)) \longrightarrow F(X)$$

$$(g, h, x) \longmapsto (gH, X)$$

and

$$p_1' : |F(F(X))| \twoheadrightarrow |F(C)|$$

$$(g, h, x; u) \longmapsto (gh; u)$$

together with

$$p_2' : |F(F(X))| \longrightarrow |X|$$

$$(g, h, x; u) \longmapsto (x, (gh)_* u)$$

Lemma 4.32. p_1', p_2' *are well-defined and*

$$(p_1', p_2') : |F(F(X))| \xrightarrow{\cong} |F(C)| \times |X|$$

is a homeomorphism.

Same argument as in Lemma 1 for p_1, p_2.
Now consider the following commutative diagram

$$\begin{tikzcd}
{|C| \times |C| \times |X|} & {|F(C)| \times |X|} \arrow[l, "(p_1,p_2)\times 1"', "\cong"] \arrow[r, "|ev|\times 1"] & {|C|C \times |X|} \\
{|C| \times |F(X)|} \arrow[u, "1\times(p_1,p_2)", "\cong"'] \arrow[d, "1\times|ev_X|"] & {|F(F(X))|} \arrow[l, "(p_1,p_2)"', "\cong"] \arrow[r, "|\mu|"] \arrow[u, "(p_1',p_2')", "\cong"'] \arrow[d, "F(ev)"] & {|F(X)|} \arrow[u, "(p_1,p_2)", "\cong"'] \arrow[d, "|ev|"] \\
{|C| \times |X|} & {|F(X)|} \arrow[l, "(p_1,p_2)"', "\cong"] \arrow[r, "|ev|"] & {|X|}
\end{tikzcd}$$

where the top, left, bottom and right composites are what we want. □

Hodge decomposition of string topology

5 Homotopy Colimits

AFTER Cisinski 2006 global Mike Shulman, Homotopy colimits 2009.

5.1 Homotopy Kan Extension

Notation Let $\mathcal{M}$ be a cofibrantly generated model category with all (small) limits and colimits.

1. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{I}$ be small categories.
2. $\mathcal{M}^{\mathcal{A}}$ $\mathcal{A}$-diagrams with projective (Bousfield-Kan) model structure.

Now consider a functor $f : \mathcal{A} \to \mathcal{B}$ in **Cat**, there is an adjunction

$$f_! : \mathcal{M}^{\mathcal{A}} \rightleftarrows \mathcal{M}^{\mathcal{B}} : f^*$$

$$\left(\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{X} \mathcal{M}\right) \longleftarrow \left(\mathcal{B} \xrightarrow{X} \mathcal{M}\right).$$

$$(X : \mathcal{A} \to \mathcal{M}) \longmapsto (\mathrm{Lan}_f(X) : \mathcal{B} to \mathcal{M})$$

Lemma 5.1. *(5.1) is a Quillen pair.*

Proof. f^* preserves both weak equivalences and fibrations. ☐

Thus we have a derived adjunction

$$\mathbb{L}f_! : \mathrm{Ho}(\mathcal{M}^{\mathcal{A}}) \rightleftarrows \mathrm{Ho}(\mathcal{M}^{\mathcal{B}}) : f^*$$

Definition 5.2. $\mathbb{L}f_!$ is called homotopy left Kan extension. And

$$\mathbb{L}\operatorname*{colim}_{\mathcal{A}} := \mathbb{L}(\mathcal{A} \to *)$$

is the homotopy colimit functor.

Theorem 5.3 (Axiomatics after Grothendieck and Cisinski)**.** *Fix $\mathcal{M}$ as above.*

1. *2-functorality: the pullback functors f^* fits together to give a strict 2-functor*

$$\mathrm{Ho}(\overline{\mathcal{M}}) : \mathbf{Cat}^{\mathrm{op}} \longrightarrow \mathbf{CAT}$$

$$\mathcal{A} \longmapsto \mathrm{Ho}(\mathcal{M}^{\mathcal{A}})$$

In particular, this imply canonical isomorphism

$$\mathbb{L}(f \circ g)_! \cong \mathbb{L}(f)_! \circ \mathbb{L}(g)_!$$

*for composable $f, g \in$ **Cat**.*

2. $\mathrm{Ho}(\mathcal{M}^{\mathcal{A}})$ *is weakly product preserving in the sense that for any (possibly empty) set* $\{\mathcal{A}_s\}_{s\in S}$ *in* $\mathrm{Ob}(\mathrm{Cat})$,
$$\mathcal{A} = \coprod_{s\in S} \mathcal{A}_s$$
then $i_s : \mathcal{A}_s \to \mathcal{A}$ *induces* $i_s^* : \mathcal{M}^{\mathcal{A}} \to \mathcal{M}^{\mathcal{A}_s}$ *thus a map*
$$i^* : \mathrm{Ho}(\mathcal{M}^{\mathcal{A}}) \xrightarrow{\cong} \textstyle\prod_{s\in S} \mathrm{Ho}(\mathcal{M}^{\mathcal{A}_s})$$
which is an equivalence of categories.

3. *(Reflexity) For any small category* $\mathcal{A}$*, the canonical inclusion* $i : \mathcal{A}^{\delta} \to \mathcal{A}$ *of discrete subcategory gives*
$$i^* : \mathrm{Ho}(\mathcal{M}^{\mathcal{A}}) \longrightarrow \mathrm{Ho}(\mathcal{M}^{\mathcal{A}^{\delta}})$$
is conservative, i.e. for any $\alpha : X \to X'$*, if* $i^*\alpha : i^*X \to i^*X'$ *is a weak equivalence in* $\mathcal{M}^{\mathcal{A}^{\delta}}$ *then* $\alpha \in \mathrm{We}(\mathcal{M}^{\mathcal{A}})$.

4. *(Base change) Recall for* $f : \mathcal{A} \to \mathcal{B}$*, and any* $b \in \mathrm{Ob}(\mathcal{B})$*, we have a sliced category* $f \downarrow b$ *such that we have the following diagram*
$$\begin{tikzcd} f\downarrow b \arrow[r, "\pi"] \arrow[d, "p"] & \mathcal{A} \arrow[d, "f"] \\ * \arrow[r, hook, "b"] & \mathcal{B} \end{tikzcd}$$
which gives a natural weak equivalence
$$\mathbb{L}p_! \circ \pi^* \xrightarrow{\sim} b^* \circ \mathbb{L}f_!$$
and for any $X : \mathcal{A} \to \mathcal{M}$,
$$\mathbb{L}\operatorname*{colim}_{f\downarrow b}(\pi^*(X)) \simeq \mathbb{L}f_!(X)(b)$$
i.e.
$$f_!(X)(b) \cong \operatorname*{colim}_{f\downarrow b}(X \to \pi)$$

Remark. Properties (2)-(4) can be facily summarized by saying that
$$\mathrm{Ho}(\mathcal{M}^{-}) : \mathbf{Cat}^{op} \longrightarrow \mathbf{CAT}$$
$$\mathcal{A} \longmapsto \mathrm{Ho}(\mathcal{M}^{\mathcal{A}})$$
is a (weak left) Grothendieck derivator associated to cofibrantly generated model caegory $\mathcal{M}$.

Theorem 5.4. *This object can be associated to an arbitrary model category.*

Groth, Handbook of Homotopy
A deeper theorem is 5.16.

Definition 5.5 (Quillen, HAK, Chapter 1)**.** $f : \mathcal{A} \to \mathcal{B}$ is called **right homotopy cofinal** if for any object $b \in \mathrm{Ob}(\mathcal{B})$, $b \downarrow f$ is contractible, i.e. $B(b \downarrow f) \simeq \mathbf{pt}$.

Lemma 5.6. *If $f : \mathcal{A} \to \mathcal{B}$ has a left adjoint, then it is right homotopy cofinal.*

Proof.

$$g : \mathcal{B} \rightleftarrows \mathcal{A} : f$$

gives the initial object

$$(b, \eta_b : gf(b) \to b)$$

in $b \downarrow f$, thus $B(b \downarrow f) \simeq \mathbf{pt}$. □

5.2 Grothendieck Construction

ref:

1. Grothendieck SGG1 Expose IV
2. Quillen HAK Chapter 1
3. R. Thomason, Homotopy Colimits in Cat, op-lax functors

Given a strict diagram of small categories

$$F : \mathcal{C} \longrightarrow \mathbf{Cat}$$

its Grothendieck Construction $\mathcal{C} \int F$ is defined to be the following category

1. objects are (c, x) where $c \in \mathrm{Ob}(\mathcal{C})$ and $x \in \mathrm{Ob}(F(c))$, and
2. morphisms are given by

   $\mathrm{Hom}_{\mathcal{C}\int F}((c,x),(d,y)) = \{(\varphi, f) : \varphi \in \mathrm{Hom}_{\mathcal{C}}(c,d), f \in \mathrm{Hom}_{F(c)}(F\varphi(x), y)\}$

3. composition is given as follows: for $(\varphi : c \to c', f : F(\varphi)x \to x'), (\varphi' : c' \to c'', f : F(\varphi')x' \to x'')$, their composition is given by

   $$(\varphi' \circ \varphi : c \to c'', f' \circ F(\varphi')(f) : F(\varphi' \circ \varphi)x \to F(\varphi')x' \to x''$$

Notation . For $\mathcal{R} \subset \mathbf{Cat}$ a subcategory, $i_{\mathcal{R}} : \mathcal{R} \to \mathbf{Cat}$, and for $F : \mathcal{C} \to \mathcal{R}$ we write $\mathcal{C} \int \mathcal{F} \coloneqq \mathcal{C} \int i_{\mathcal{R}} \circ F$.

Example 5.7. For $\mathcal{R} = \mathbf{Gr}$, $i_G : \mathbf{Gr} \hookrightarrow \mathbf{Cat}$ where we view groups as one object categories. Given $G, N \in \mathrm{Ob}(\mathbf{Gr})$ and a group action $F : G \to \mathrm{Aut}(N)$, it gives a functor

$$\begin{aligned} F_N : \underline{G} &\longrightarrow \mathbf{Gr} \\ * &\longmapsto N \\ g \in G &\mapsto F(g) : N \to N \end{aligned}$$

In this case, $G \int F_N = G \ltimes N$.

Grothendieck's construction is a generalization of semi-direct product of groups.

Question 5.8. *What is a "deformed" Grothendieck construction?*

Example 5.9. Let $\mathcal{R} = \mathbf{Set}$ with $i_{\mathbf{Set}} : \mathbf{Set} \hookrightarrow \mathbf{Cat}$. For $F : \mathcal{C} \to \mathbf{Set}$, $\mathcal{C} \int F = \mathcal{C}_F$ is Bousfield-Kan's category of elements of F.

5.2.1 Universal Mapping Property

Let $\mathcal{C}, \mathcal{D}$ be two fixed categories, and let $F : \mathcal{C} \to \mathbf{Cat}$ be a fixed diagram.

Proposition 5.10. *There is a natural bijection between the set of all functors $G : \mathcal{C} \int F \to \mathcal{D}$ and the following data*

1. *for each $c \in \mathrm{Ob}(\mathcal{C})$ a functor $G(c) : F(c) \to \mathcal{D}$.*
2. *for every $\varphi : c \to_{\c{}}'$ in $\mathcal{C}$, a natural transformation of functors $F(c) \to \mathcal{D}$*

$$G(\varphi) : G(c) \Longrightarrow G(c') \circ F(\varphi)$$

such that

   (a) $G(\mathrm{Id})_{\mathcal{C}} = \mathrm{Id}$.

   (b) for any composite $(c \xrightarrow{\varphi} c' \xrightarrow{\varphi'} c''$, $G(\varphi' \circ \varphi) = G(\varphi') \circ G(\varphi)$ *in* $\mathcal{C}$.

Proof. Given the data aboce, we define

$$G : \mathcal{C} \int F \to \mathcal{D}$$

by

- $G(c, x) = G(c)x$ on objects.
- $G(\varphi : c \to c', F(c)x \xrightarrow{f} x')$

On the other hand, given a functor $G : \mathcal{C} \int F \to \mathcal{D}$ □

5.2.2 Fibered/cofibered Categories

Note there is a projection forgetful functor $p : \mathcal{C} \int F \to \mathcal{C}$ which simply forgets the second argument.

Fix $\mathcal{C} \in \mathbf{Cat}$,

$$\mathcal{C} \textstyle\int (-) : \mathrm{Fun}(\mathcal{C}, \mathbf{Cat}) \to \mathbf{Cat} \downarrow \mathcal{C}$$

$$F \longmapsto (p : \mathcal{C} \textstyle\int F \to \mathcal{C})$$

$$(F \Rightarrow F') \longmapsto (\mathcal{C} \textstyle\int \alpha : \mathcal{C} \int F \to \mathcal{C} \int F')$$

$$\mathcal{C} \textstyle\int \alpha : \mathcal{C} \int F \longrightarrow \mathcal{C} \int F$$

where $$(c, x) \longmapsto (c, \alpha_c(x))$$

$$(c \xrightarrow{\varphi} c', F(\varphi)x \xrightarrow{f} x') \longmapsto (c \xrightarrow{\varphi} c', \alpha_c)$$

Theorem 5.11 (Grothendieck)**.** $\mathcal{C} \int (-)$ *is fully faithful with its image being small categories, cofibered over* $\mathcal{C}$*.*

5.2.3 Fibered/Cofibered Functors

Given $\pi : \mathcal{A} \to \mathcal{B}$, for any $b \in \mathrm{Ob}(\mathcal{B})$, there are three different categories over b:

- fiber category at b: $\pi^{-1}(b)$. Objects are $a \in \mathrm{Ob}(\mathcal{A})$ such that $\pi(a) = b$. Morphisms are $f : a \to a' \in \mathrm{Mor}(\mathcal{A})$ such that $\pi(f) = \mathrm{Id}_b$.
- slice category $b \downarrow \pi$. Objects are pairs (a, f), $a \in \mathrm{Ob}(\mathcal{A})$, $f : b \to \pi(a)$.
- slice category $\pi \downarrow b$. Objects are pairs (a, g), $a \in \mathrm{Ob}(\mathcal{A})$, $g : \pi(a) \to b$.

Theorem 5.12 (Grothendieck)**.** $\mathcal{C} \int (-)$ *fully faithful with its images being small categories cofibered over* $\mathcal{C}$*.*

Consider the following two functors

$$i^* : \pi^{-1}(b) \hookrightarrow b \downarrow \pi$$

$$a \longmapsto (a, \mathrm{Id}_b)$$

$$j_* : \pi^{-1}(b) \hookrightarrow \pi \downarrow b$$

$$a \longmapsto (a, \mathrm{Id}_b)$$

Definition 5.13. $\pi : \mathcal{A} \to \mathcal{B}$ is **prefibered** if for any $b \in \mathrm{Ob}(\mathcal{B})$, i_* has right adjoint $i^!$.
$\pi : \mathcal{A} \to \mathcal{B}$ is **precofibered** if for any $b \in \mathrm{Ob}(\mathcal{B})$, j_* has left adjoint j^*.

Suppose $\pi : \mathcal{A} \to \mathcal{B}$ is prefibered, we can define a base cahnge functor

$$f^* : \pi^{-1}(b') \to \pi^{-1}(b)$$

for every $f : b \to b'$ by the following composite

$$f^* : \pi^{-1}(b') \xrightarrow{i'_*} b' \downarrow \pi \xrightarrow{f \downarrow \pi} b \downarrow \pi \xrightarrow{i^!} \pi^{-1}(b)$$

such that for any composite $b \xrightarrow{f} b' \xrightarrow{g} b''$,

$$(g \circ f)^* = f^* \circ g^* : \pi^{-1}(b'') \to \pi^{-1}(b).$$

These two are related by the following natural transformation

$$\begin{tikzcd}
\pi^{-1}(b'') \arrow[r, "i''_*"] \arrow[rrrddd, "(gf)^*"] & b'' \downarrow \pi \arrow[r, "g\downarrow\pi"] & b' \downarrow \pi \arrow[r, "(i')^!"] \arrow[rd, "\mathrm{Id}"] & \pi^{-1}(b') \arrow[d, "i'_*"] \\
& & & b' \downarrow \pi \arrow[d, "f\downarrow\pi"] \\
& & & b \downarrow \pi \arrow[d, "i^!"] \\
& & & \pi^{-1}(b)
\end{tikzcd}$$

together with natural transformation $f^* g^* \Rightarrow (gf)^*$.

Definition 5.14. $\pi : \mathcal{A} \to \mathcal{B}$ is called **fibered** if

1. it is prefibered, and
2. $\alpha_{f,g}$ is an isomorphism for all composable morphisms.

Dually, we can define cofibered functors and similar 1-1 correspondences.

5.3 Thomason's Formula

for homotopy colimits in **Cat**. (Chacholski-Sschrez, Homotopy theory of diagrams, Chapter 26)

Base Change Axiom For $f : \mathcal{A} \to \mathcal{B}$,

$$\mathrm{be} \quad \begin{tikzcd} f \downarrow b \arrow[r, "\pi"] \arrow[d, "p"] & \mathcal{A} \arrow[d, "f"] \\ * \arrow[r] & \mathcal{B} \end{tikzcd}$$

$\mathbb{L}p_!\pi^* \xrightarrow{\sim} b^*\mathbb{L}f_!$

For any $X : \mathcal{A} \to \mathcal{M}$, $\mathcal{M}$ a model category.

$$\mathbb{L}f_!(X)(b) = \mathrm{hocolim}_{f\downarrow b}(\pi^* X)$$

Lemma 5.15. *If $f : \mathcal{A} \to \mathcal{B}$ is precofibered.*

{thm:cofinal}

Theorem 5.16 (Cofinality Theorem)**.** *If $f : \mathcal{A} \to \mathcal{B}$ is right homotopy cofinal, i.e. $f \downarrow b$ is contractible, then $f^* : \mathrm{hocolim}_{\mathcal{A}}(f^*(X)) \xrightarrow{\sim} \mathrm{hocolim}_{\mathcal{B}}(X)$ for any $X : \mathcal{B} \to \mathcal{M}$.*

Example 5.17. For a compact conencted Lie group G, Fix a prime p, let $\mathscr{B}_p$ be the poset of all p-subgroups and $\mathscr{A}_p$ be the poset of all nontrivial elementary abelian subgroups. Then $f : \mathscr{A}_p \hookrightarrow \mathscr{B}_p$ is right homotopy cofinal.

Lemma 5.18. *If $f : \mathcal{A} \to \mathcal{B}$ is precofibered, then $\mathbb{L}f_!X(b) \simeq \mathrm{hocolim}_{f^{-1}(b)}(i^{ast}X)$.*

This implies the following theorem

Theorem 5.19 (Generalized Thomason Formula, Chacholski-Schmer, 2002)**.** *Let $\mathcal{M}$ be a cofibrantly generated model category and $X : \mathcal{C} \int F \to \mathcal{M}$. There is a natural weak equivalence*

$$\mathrm{hocolim}_{\mathcal{C}\in F}(X) \cong \mathrm{hocolim}_{c\in\mathcal{C}}(\mathrm{hocolim}_{F(c)}\, X(c)),$$

where $X(c) \coloneqq i_c^*(X)$ *and*

$$\begin{aligned} i_c : F(c) &\longrightarrow \mathcal{C}\int F \\ x &\longmapsto (c, x) \\ (x \xrightarrow{f} x') &\mapsto (\mathrm{Id}_c, f) \end{aligned}$$

When we take $\mathcal{M} = \mathbf{sSet}$, we have the classical Thomason's formula.

Corollary 5.20 (Thomason's Formula)**.** $\operatorname{hocolim}_{\mathcal{C}}^{\mathbf{sSet}}(\mathcal{N}_* F) \simeq \mathcal{N}_*(\mathcal{C} \int F)$ *is a weak equivalence in* **sSet**.

Corollary 5.21 (BK construction)**.** *Let* $\mathcal{M} = \mathbf{sSet}$ *and consider* $F : \mathcal{C} \to \mathbf{sSet}$, *then*

$$\operatorname{hocolim}_{\mathcal{C}} F \cong \mathcal{N}_*(\mathcal{C}_F).$$

Example 5.22. $\operatorname{hocolim}_{\Delta C^{op}}(B^{cyc}(\Gamma)) \simeq ES^1 \times_{S^1} \mathcal{L}(B\Gamma)$.

Sketch. Recall $p : \mathcal{C} \int F \to \mathcal{C}$ is canonically cofibered over $\mathcal{C}$.

$p : \mathcal{A} \to \mathcal{B}$ is precofiber if $j : p^{-1}(b) \hookrightarrow p \downarrow b$ has left adjoint.

cofibered means

1. p is precofibered, and
2. $p^{-1} : \mathcal{B} \to \mathbf{Cat}$, $b \mapsto p^{-1}(b)$ is a functor, i.e. composable morphisms maps to composable functors between categories.

Consider the following diagam

$$\begin{tikzcd} p^{-1}(b) \arrow[r, hook, "i"] \arrow[d, "j"] & \mathcal{A} \arrow[r, "X"] & \mathcal{M} \\ p \downarrow b \arrow[ur, "\pi"'] & & \end{tikzcd}$$

Lemma 5.23. *If* $p : \mathcal{A} \to \mathcal{B}$ *is precofibered, then for any* $X : \mathcal{A} \to \mathcal{M}$, *we have*

$$\mathbb{L}p_!(X)(b) \simeq \operatorname{hocolim}_{p^{-1}(b)}(i^*(X)) =: \mathbb{L}(p^{-1}(b) \to *)(i^*(X))$$

where $i : p^{-1}(b) \hookrightarrow \mathcal{A}$.

Proof. Observe $p : \mathcal{A} \to \mathcal{B}$ is cofibered implies $j : p^{-1}(b) \to p \downarrow b$ is right adjoint, thus right homotopy cofinal, or equivalently, $B(b \downarrow p) \sim \mathbf{pt}$.

$$\begin{aligned} \mathbb{L}p_!(X)(b) &\simeq \operatorname{hocolim}_{\pi \downarrow b}(\pi^*(X)) \\ &\simeq \operatorname{hocolim}_{p^{-1}(b)}(j^*\pi^* X) \\ &\simeq \operatorname{hocolim}_{p^{-1}(b)}(i^* X) \\ &= \operatorname{hocolim}_{p^{-1}(b)}(j^*(X)). \end{aligned}$$

□

Since $p : \mathcal{C} \int F \to \mathcal{C}$ is precofibered and $p^{-1}(c) = F(c)$, so by lemma

$$\operatorname{hocolim}_{\mathcal{C} \int F}(X) \simeq \operatorname{hocolim}_{c \in \mathcal{C}}(\operatorname{hocolim}_{F(c)} X(c)).$$

□

5.4 Homotopy Coends

Define $\mathcal{C}$-bifunctor to be a functor

$$F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{M}$$

5.4.1 Twisted Arrow Category/Factorization Category

See Quillen HAK.

Consider

$$\begin{aligned} \mathrm{Hom} : \mathcal{C}^{op} \times \mathcal{C} &\longrightarrow \mathbf{Set} \\ (c, d) &\longmapsto \mathrm{Hom}_{\mathcal{C}}(c, d) \end{aligned}$$

Let

$$\mathcal{F}(\mathcal{C}) = (\mathcal{C}^{op} \times \mathcal{C}) \int \mathrm{Hom} =: (\mathcal{C}^{op} \times \mathcal{C})_{\mathrm{Hom}}$$

Note $\mathcal{F}(\mathcal{C})^{op} \not\cong \mathcal{F}(\mathcal{C}^{op})$.

The category $\mathcal{F}(\mathcal{C})$ is given by

1. Objects: $\mathrm{Ob}(\mathcal{F}(\mathcal{C})) = \{(i \xrightarrow{f} j) : f \in \mathrm{Mor}(\mathcal{C})\}$.
2. Morphisms: $\mathrm{Hom}_{\mathcal{F}(\mathcal{C})}(i \xleftarrow{f} j, i' \xleftarrow{f} j') := \{(i \xrightarrow{\beta} i', j \xleftarrow{\alpha} j') : g = \beta \circ f \circ \alpha\}$.

$$\begin{tikzcd} i \arrow[r, "\beta"] & i' \\ j \arrow[u, "f"] & j \arrow[l, "\alpha"'] \arrow[u, "g"'] \end{tikzcd}$$

In opposite category $\mathcal{F}(\mathcal{C})^{op}$ we reverse all arrows.

Define $p = s \times t : \mathcal{F}(\mathcal{C}) \to \mathcal{C}^{op} \times \mathcal{C}$, then by definition p is precofibered.

$$\begin{aligned} s^{op} : \mathcal{F}(\mathcal{C})^{op} &\longrightarrow \mathcal{C} \\ (c \xrightarrow{f} d) &\longmapsto c \\ t^{op} : \mathcal{F}(\mathcal{C})^{op} &\longrightarrow \mathcal{C}^{op} \\ (c \xrightarrow{f} d) &\longmapsto d \end{aligned} \tag{5.1}$$ {eq:s-t-op}

Lemma 5.24 (Quillen). *Both functors 5.1 are right homotopy cofinal.*

Proof. 1. $s \times t : \mathcal{F}(\mathcal{C}) \to \mathcal{C}^{op} \times \mathcal{C}$ is cofibered implies $s : \mathcal{F}(\mathcal{C}) \to \mathcal{C}^{op}$ and $t : \mathcal{F}(\mathcal{C}) \to \mathcal{C}$ are cofibered, so for any $c, d \in \mathrm{Ob}(\mathcal{C})$

$$s^{-1}(c) \hookrightarrow s \downarrow c$$

$$t^{-1}(d) \hookrightarrow t \downarrow d$$

are homotopy cofinal, i.e $B(s^{-1}(c)) \simeq B(s \downarrow c)$ and $B(t^{-1}(d)) \simeq B(t \downarrow d)$, so

$$B(s \downarrow c) \simeq B(c \downarrow \mathcal{C}) \simeq \mathbf{pt}$$
$$B(t \downarrow d) \simeq B((\mathcal{C} \downarrow d)^{op}) \simeq \mathbf{pt}.$$

Note $B(d \downarrow \mathcal{C}^{op}) \simeq \mathbf{pt}$.

□

Corollary 5.25. *For any* $X : \mathcal{C} \to \mathcal{M}$ *and* $Y : \mathcal{C}^{op} \to \mathcal{M}$,

$$(s^{op})^* : \mathrm{hocolim}_{\mathcal{F}(\mathcal{C})^{op}}((s^{op})^* X) \xrightarrow{\sim} \mathrm{hocolim}_{\mathcal{C}}(X)$$

$$(t^{op})^* : \mathrm{hocolim}_{\mathcal{F}(\mathcal{C})^{op}}((t^{op})^* Y) \xrightarrow{\sim} \mathrm{hocolim}_{\mathcal{C}^{op}}(Y)$$

Definition 5.26. For any bifunctor $D : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{M}$, define homotopy coend

$$\int_{\mathbb{L}}^{c \in \mathcal{C}} D(c, c) \coloneqq \mathrm{hocolim}_{\mathcal{F}(\mathcal{C})^{op}}(\pi^* D)$$

where

$$\pi^{op} \coloneqq t^{op} \times s^{op} : \mathcal{F}(\mathcal{C})^{op} \longrightarrow \mathcal{C}^{op} \times \mathcal{C} \xrightarrow{D} \mathcal{M}$$

and $\pi^* \coloneqq (\pi^{op})^*$. Consider

$$\mathcal{C}^{op} \times \mathcal{C} \xrightarrow{\pi_1} \mathcal{C}^{op} \xrightarrow{Y} \mathcal{M}$$

$$\mathcal{C}^{op} \times \mathcal{C} \xrightarrow{\pi_2} \mathcal{C} \xrightarrow{X} \mathcal{M}$$

then

$$\mathrm{hocolim}_{\mathcal{C}}(X) \simeq \int_{\mathbb{L}}^{c \in \mathcal{C}} (\pi_2)^*(X)$$
$$\mathrm{hocolim}_{\mathcal{C}}(Y) \simeq \int_{\mathbb{L}}^{c \in \mathcal{C}} (\pi_1)^*(Y)$$

Recall for a bifunctor $X : I^{op} \times I \to \mathcal{C}$, the coend

$$\mathrm{coend}(X) := \int^{i \in I} X(i, i)$$

is the unversal (initial) cocone of X in $\mathcal{C}$.

Lemma 5.27. *coend*$(X) \cong \mathrm{colim}_{\mathcal{F}(\mathcal{C})^{op}}(\pi^* X)$.

Remark. The reason to work with a bigger category is because you may have more "free" categories connected to $\mathcal{F}(\mathcal{C})^{op}$, i.e. functor $\mathcal{F}(\mathcal{C})^{op} \to \mathcal{D}$ which is right homotopy cofinal, so you can transfer computation to $\mathcal{D}$-diagrams.

Properties of Homotopy coend:

1. (interchange) The flip functor $\tau : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}^{op}$ gives

$$\int_{\mathbb{L}}^{c \in \mathcal{C}} \tau^*(X)(c, c) \simeq \int_{bL}^{c \in \mathcal{C}^{op}} X(c, c)$$

2. (Fubini Theorem) For $X : \mathcal{C} \times \mathcal{D})^e \to \mathcal{M}$,

$$\int_{bL}^{(c,d) \in \mathcal{C} \times \mathcal{D}} X(c, d; c, d) \simeq \int_{\mathbb{L}}^{d \in \mathcal{D}} \int_{\mathbb{L}}^{c \in \mathcal{C}} X(c, d; c, d)$$

3. (Diagrams of diagrams) For $X : \mathcal{C} \times \mathcal{C}^{op} \to \mathcal{M}^{\mathcal{J}}$

$$(\int_{\mathbb{L}}^{c \in \mathcal{C}} X(c, c))_j = \int_{\mathbb{L}}^{c \in \mathcal{C}} X(c, c)_j.$$

4. (Commutativity with left Quillen derived functors) Given a left Quillen functor $F : \mathcal{M} \to \mathcal{N}$,

Proposition 5.28 (Shapiro Lemma for Model Categories)**.** *Let $\mathcal{M}$ be a simplicial model category and $\mathcal{C}$ be a small category*

6 Simplicial Bar Construction

6.1 Tensor and Cotensor

Notation/Terminology Let $\mathcal{M}$ be a simplicial (i.e. enriched over **sSet**) category. For any $X, Y \in \mathrm{Ob}(\mathcal{M})$, morphisms $\mathbf{Hom}(X, Y) \in \mathrm{Ob}(\mathbf{sSet})$ and the composition

$$\mathbf{Hom}_{\mathcal{M}}(Y, Z) \times \mathbf{Hom}_{\mathcal{M}}(X, Y) \xrightarrow{\circ} \mathbf{Hom}_{\mathcal{M}}(X, Z)$$

is a morphism in **sSet** which is natural in X, Y, Z.

Remark. The notation comes from the observation that $\mathbf{Hom}_{\mathcal{M}}(X, Y) = \mathrm{Hom}_{\mathcal{M}}(X, Y)$ and $\mathbf{Hom}_{\mathcal{M}}(X, Y)$ is the simplicial homotopies between maps.

Definition 6.1. $\mathcal{M}$ is **tensored** over **sSet** if there is a bifunctor

$$\boxtimes : \mathbf{sSet} \times \mathcal{M} \longrightarrow \mathcal{M}$$

$$(K, X) \longmapsto K \boxtimes X$$

such that there is a natural isomorphism

$$\mathbf{Hom}_{\mathcal{M}}(K \boxtimes X, Y) \cong \mathbf{Hom}_{\mathbf{sSet}}(K, \mathbf{Hom}_{\mathcal{M}}(X, Y)).$$

Definition 6.2. $\mathcal{M}$ is **cotensored** over **sSet** if there is a bifunctor

$$\{-,-\} : \mathbf{sSet}^{op} \times \mathcal{M} \longrightarrow \mathcal{M}$$

$$(K, X) \longmapsto \{K, X\}$$

such that there is a natural isomorphism

$$\mathbf{Hom}_{\mathcal{M}}(X, \{K, Y\}) \cong \mathbf{Hom}_{\mathbf{sSet}}(K, \mathbf{Hom}_{\mathcal{M}}(X, Y)).$$

Example 6.3. $\mathcal{M} = \mathbf{s}\mathcal{C}$ where $\mathcal{C}$ is bicomplete is a category tensored and cotensored over **sSet** with

- tensor $K \boxtimes X := \{\coprod_{K_n} X_n\}_{n \geq 0}$.
- $(K \times L) \boxtimes X = K \boxtimes (L \boxtimes X)$ and $\Delta[0]_* \boxtimes X \cong X$.
- $\mathbf{Hom}_{\mathbf{s}\mathcal{C}}(X, Y) := \{\mathrm{Hom}_{\mathcal{M}}(\Delta[n]_* \boxtimes X, Y)\}_{n \geq 0}$.
- cotensor is defined by

$$\mathbf{Hom}_{\mathcal{M}}(K \boxtimes X, Y) \cong \mathbf{Hom}_{\mathcal{M}}(X, \{K, Y\}).$$

Example 6.4. 1. For $\mathcal{C} = \mathbf{Set}$, $\mathcal{M} = \mathbf{sSet}$,

- $K \boxtimes X = K \times X = \{K_n \times X_n\}_{n \geq 0}$.
- $\{K, X\} = \mathbf{Hom}_{\mathbf{sSet}}(K, X) \coloneqq \{\mathrm{Hom}_{\mathbf{sSet}}(\Delta[n]_* \times K, X)$.

2. For $\mathcal{M} = \mathrm{Mod}_R$ where R is a rign and Mod_R is the category of left R-modules, $\mathcal{M} = \mathbf{s}\,\mathrm{Mod}_R$

- $K \boxtimes M \coloneqq \{R[K_n] \otimes_R M_n\}_{n \geq 0} \cong \{\bigoplus_{K_n} M_n\}_{n \geq 0}$.
- $\mathbf{Hom}_{\mathbf{s}\,\mathrm{Mod}_R}(M, N) = \{\mathrm{Hom}_{\mathbf{s}\,\mathrm{Mod}_R}(R[\Delta[n]_*] \otimes_R M, N)\}_{n \geq 0}$.
- $\{K, M\} = \mathbf{Hom}_{\mathbf{sSet}}(K, M)$ with pointwise simplicial R-module structure from M.

3. $\mathcal{C} = \mathbf{Comm}_k$ where k is a commutative ring, $\mathcal{M} = \mathbf{sComm}_k$.

- $K \boxtimes X = \{K_n \otimes X_n\}_{n \geq 0}$
- $\{K, X\} = \{\mathrm{Hom}_{\mathbf{sSet}}(\Delta[n]_*, X)$

4. $\mathcal{C} = \mathbf{Top} = \mathbf{CGWH}$.

- $K \boxtimes X = |K| \times X$.
- $\{K, X\} = \mathbf{Hom}_{\mathbf{Top}}(|K|, X)$.

6.2 Functor Tensor Products

Let $\mathcal{C}$ be a small category. Let $\mathcal{M}$ be a simplicial category tensored and cotensered over **sSet**.

Give $G : \mathcal{C}^{op} \to \mathbf{sSet}$ and $F : \mathcal{C} \to \mathcal{M}$, define

$$G \boxtimes_{\mathcal{C}} F \coloneqq \int^{c \in \mathrm{Ob}(\mathcal{C})} G(c) \boxtimes F(c) \coloneqq \mathrm{coeq}\ \{ \coprod_{c \xrightarrow{f} c'} G(c') \boxtimes F(c) \underset{1 \times f_*}{\overset{f^* \times 1}{\rightrightarrows}} \coprod_{c \in \mathrm{Ob}(\mathcal{C})} G(c) \boxtimes F(c)\}$$

Example 6.5. If $G : \mathcal{C}^{op} \to \mathbf{sSet}, c \mapsto \Delta[0]_* = \{*\}$, then

$$* \boxtimes_{\mathcal{C}} F = \operatorname*{colim}_{\mathcal{C}} F.$$

Intuition $G \boxtimes_{\mathcal{C}} F$ is the weighted colimit of F with respect to weights G.

Example 6.6 ((Left) Kan extension)**.** For $\mathcal{D}$ cocomplete

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "H"'] & \mathcal{D} \\ \mathcal{E} \arrow[ur, "\mathrm{Lan}_H(F)"'] & \end{tikzcd}$$

For any $e \in \mathrm{Ob}(\mathcal{E})$,

$$\mathrm{Lan}_H(F)(e) = \mathrm{Hom}_{\mathcal{E}}(H(-), e) \boxtimes_{\mathcal{C}} F.$$

Example 6.7 (Internal Geometric Realization)**.** For a simplicial category $\mathcal{M}$ tensored and cotensored over **sSet**, given any simplicial object $X : \Delta^{op} \to \mathcal{M}$, the internal geometric realization is defined as

$$|X_*| \coloneqq \Delta[\bullet]_* \boxtimes_{\Delta^{op}} X.$$

When $\mathcal{M} = \mathbf{Top}$, $\mathbf{s}\mathcal{M} = \mathbf{sTop}$, if $X : \Delta^{op} \to \mathbf{Set}$ is viewed as a discrete simplicial space,

$$|X| \simeq \mathbf{Hom_{sSet}}(\Delta[\bullet]_*, X) \boxtimes_\Delta \Delta^* \cong X \boxtimes_\Delta \Delta^*.$$

Exercise 6.8. For $\mathcal{M} = \mathbf{sSet}$, $X : \Delta^{op} \to \mathbf{sSet}$ a bisimplicial set,

$$|X| \cong \mathrm{diag}(X) = \{X_{nn}\}_{n \geq 0}.$$

6.3 Two-sided Simplicial Bar Construction

Definition 6.9. Given $G : \mathcal{C}^{op} \to \mathbf{sSet}$ and $F : \mathcal{C} \to \mathcal{M}$, define

$$\begin{aligned} B_*(G, \mathcal{C}, F) : \Delta^{op} &\longrightarrow \mathcal{M} \\ []n] &\longmapsto B_n(G, \mathcal{C}, F) \end{aligned}$$

where

$$B_n(G, \mathcal{C}, F) = \coprod_{\left(c_n \xleftarrow{f_{n-1}} c_{n-1} \leftarrow \cdots \xleftarrow{f_0} c_0\right) \in \mathcal{N}_n\mathcal{C}} G(c_n) \boxtimes F(c_0)$$

with face maps

$$\begin{aligned} d_i : B_n(G, \mathcal{C}, F) &\longrightarrow B_{n-1}(G, \mathcal{C}, F) \\ (y|f_{n-1}, \cdots, f_0|x) &\mapsto d_i(y|f_{n-1}, \cdots, f_0|x) \end{aligned}$$

defined by

$$d_i(y|f_{n-1}, \cdots, f_0|x) = \begin{cases} (y|f_{n-1}, \cdots, f_1|f_0 \cdot x), & i = 0, \\ (y|f_{n-1}, \cdots, f_i \circ f_{i-1}, \cdots, f_0|x), & 1 \leq i \leq n-1, \\ (y \cdot f_{n-1}|f_{n-2}, \cdots, f_0|x), & i = n. \end{cases}$$

and degeneracy maps

$$\begin{aligned} s_j : B_n(G, \mathcal{C}, F) &\longrightarrow B_{n+1}(G, \mathcal{C}, F) \\ (y|f_{n-1}, \cdots, f_0|x) &\mapsto (y|f_{n-1}, \cdots, \mathrm{Id}_{c_j}, \cdots, f_0|x) \end{aligned}$$

Note

$$\operatorname*{colim}_{\Delta^{op}} (B_*(G, \mathcal{C}, F)) \cong G \boxtimes_{\mathcal{C}} F.$$

Therefore $B_*(G, \mathcal{C}, F)$ is a fattened (thickened) tensor product. There is a canonical map

$$B_*(G, \mathcal{C}, F) \to G \boxtimes_{\mathcal{C}} F.$$

6.3.1 Examples

1. $\mathcal{M} = \mathbf{Set}$. $G : \mathcal{C}^{op} \to \mathbf{Set} \hookrightarrow \mathbf{sSet}, c \mapsto *$ and $F : \mathcal{C} \to \mathbf{Set}, c \mapsto *$ then

$$B_*(*, \mathcal{C}, *) = \mathcal{N}_*\mathcal{C}.$$

2. $\mathcal{M} = \mathbf{Set}$ or Top with $G = *$ and $F : \mathcal{C} \to \mathbf{Set}$, then

$$B_*(*, \mathcal{C}, F) = \mathcal{N}_*(\mathcal{C}_F).$$

3. $\mathcal{M} = \mathbf{Set}$ and for $c \in \mathrm{Ob}(\mathcal{C})$, we have

$$\begin{aligned} h^c : \mathcal{C} &\longrightarrow \mathbf{Set} & \qquad h_c : \mathcal{C}^{op} &\longrightarrow \mathbf{Set} \\ d &\longmapsto \mathrm{Hom}_{\mathcal{C}}(c, d) & b &\longmapsto \mathrm{Hom}_{\mathcal{C}}(b, c) \end{aligned}$$

then

$$B_*(*, \mathcal{C}, h^c) \cong \mathcal{N}(c \downarrow \mathcal{C}), \quad B(h_c, \mathcal{C}, *) \cong \mathcal{N}_*(\mathcal{C} \downarrow c).$$

Varying $c \in \mathrm{Ob}(\mathcal{C})$ in the last two formulas gives us two functors

$$\begin{aligned} B_*(*, \mathcal{C}, \mathcal{C}) : \mathcal{C}^{op} &\longrightarrow \mathbf{sSet} & \qquad B_*(\mathcal{C}, \mathcal{C}, *) : \mathcal{C} &\longrightarrow \mathbf{sSet} \\ c &\longmapsto B_*(*, \mathcal{C}, h^c) & c &\longmapsto B(h_c, \mathcal{C}, *) \end{aligned}$$

and we have

$$B_*(*, \mathcal{C}, \mathcal{C}) \cong \mathcal{N}((-) \downarrow \mathcal{C}), \quad B(\mathcal{C}, \mathcal{C}, *) \cong \mathcal{N}_*(\mathcal{C} \downarrow (-)).$$

4. (Exercise) For $F : \mathcal{C} \to \mathcal{D}$,

$$B_*(*, \mathcal{C}, \mathcal{D}) \cong \mathcal{N}((-) \downarrow F), \quad B(\mathcal{D}, \mathcal{C}, *) \cong \mathcal{N}_*(F \downarrow (-)).$$

which is a generalization of the previous one.

5. Let G be a discrete group, $\mathcal{C} = \underline{G}$ where $\mathrm{Ob}(\underline{G}) = \{*_G\}$ then $\mathrm{Hom}_{\underline{G}}(*, *) = G$.

$$B_*(*, \underline{G}, \underline{G}) \cong E_*(G), \quad B(\underline{G}, \underline{G}, *) \cong E_*G.$$

where the first is as right G-action and the second is as left G-action.

Definition 6.10. The **bar construction** is

$$B(G,\mathcal{C},F) \coloneqq |B_*(G,\mathcal{C},F)|_{\mathcal{M}} = \Delta[\bullet]_* \boxtimes_{\Delta^{op}} B_*(G,\mathcal{C},F).$$

The canonical functor

$$\Delta[\bullet]_* \longrightarrow \Delta[0]_*$$

induces a functor

$$\varepsilon : B(G,\mathcal{C},F) \longrightarrow G \boxtimes_{\mathcal{C}} F$$

in $\mathcal{M}$.

Lemma 6.11. *For any* $F : \mathcal{C} \to \mathcal{M}$*,*

$$B_*(*,\mathcal{C},F) \cong B_*(*,\mathcal{C},\mathcal{C}) \boxtimes_{\mathcal{C}} F.$$

Thus one-sided bar construction is a weight colimit.

Proof. Tensor products are all coends, and the following commutes.

$$\begin{aligned} B(*,\mathcal{C},F) &\coloneqq \Delta[\bullet]_* \boxtimes_{\Delta^{op}} B_*(*,\mathcal{C},F) \\ &\cong \Delta[\bullet]_* \boxtimes_{\Delta^{op}} B_*(*,\mathcal{C},\mathcal{C} \boxtimes_{\mathcal{C}} F) \\ &\cong (\Delta[\bullet]_* \boxtimes_{\Delta^{op}} B_*(*,\mathcal{C},\mathcal{C})) \boxtimes_{\mathcal{C}} F \\ &= B_*(*,\mathcal{C},\mathcal{C}) \boxtimes_{\mathcal{C}} F. \end{aligned}$$

□

Definition 6.12 (BK formula for homotopy colimit)**.** $\operatorname{hocolim}_{\mathcal{C}}(F) \coloneqq B(*,\mathcal{C},F) \cong \mathcal{N}_*((-) \downarrow \mathcal{C}) \boxtimes_{\mathcal{C}} F$.

Remark. The concrete formula is

$$\mathbb{L} \operatorname*{colim}_{\mathcal{C}}(F) \cong B(*,\mathcal{C},QF)$$

where $Q : \mathcal{M} \to \mathcal{M}$ is a cofibrant replacement functor on $\mathcal{M}$.

6.4 Cyclic Bar Construction of a Discrete Group

Notation. We will use $\Gamma_* G = B_*^{cyc}(G)$ to denote the cyclic bar construction.

Definition 6.13. The cyclic bar construction of G is given by

$$\begin{aligned} \Gamma_*(G) : \Delta C^{op} &\to \mathbf{Top} \\ [n] &\longmapsto G^{n+1} \end{aligned}$$

with
$$d_i : \Gamma_n G \longrightarrow \Gamma_{n-1} G$$
$$(g_0, \cdots, g_n) \mapsto d_i(g_0, \cdots, g_n)$$
where
$$d_i(g_0, \cdots, g_n) = \begin{cases} (g_0, \cdots, g_i g_{i+1}, \cdots, g_n), & 0 \le i < n, \\ (g_n g_0, g_1, \cdots, g_{n-1}), i = n. \end{cases}$$
and
$$s_j : \Gamma_n G \longrightarrow \Gamma_{n+1} G$$
$$(g_0, \cdots, g_n) \mapsto (g_0, \cdots, g_{j-1}, 1, g_j, \cdots, g_n)$$
and
$$t : \Gamma_n G \longrightarrow \Gamma_n G$$
$$(g_0, \cdots, g_n) \mapsto (g_n, g_0, g_1, \cdots, g_{n-1})$$

Remark. If G is a discrete group, and let $A = k[G]$ be the group algebra, then
$$C_*^{HH}(A) \cong k[\Gamma_* G].$$

Remark. Fix $z \in \mathcal{Z}(G)$ a central element, can define a twisted cyclic map
$$t_n(z) : B_n G \longrightarrow B_n G$$
$$(g_1, \cdots, g_n) \mapsto (z(g_1 \cdots g_n)^{-1}, g_1, \cdots, g_{n-1})$$
then $(t_n(z))^{n+1} = \mathrm{Id}_{B_n G}$.

Recall $BG = |B_* G|$.

Theorem 6.14. *For any discrete simplicial/topological group G, there is a canonical S^1 equivariant homotopy equivalence*
$$\gamma : |\Gamma_* G| \to \mathscr{L}(BG) = \mathrm{Map}(S^1, BG)$$
where the free loop space $\mathscr{L}(BG)$ carries a natural S^1-action.

Lemma 6.15. *For any discrete group G,*
$$F : \mathrm{hocolim}_{\underline{G}}(\mathrm{Ad}) \cong \gamma_* G,$$
where
$$\mathrm{Ad} : \underline{G} \longrightarrow \mathbf{Set}$$
$$* \longmapsto G$$
$$g \longmapsto (\mathrm{Ad}_g : G \to G, \gamma \mapsto g \gamma g^{-1}).$$

Proof. By definiton,

$$\begin{aligned}\mathrm{hocolim}_{\underline{G}}(\mathrm{Ad}) &= |B_*(*,\underline{G},\underline{G}) \boxtimes_{\underline{G}} \mathrm{Ad}_G| \\ &\cong |E_*G \boxtimes_{\underline{G}} \mathrm{Ad}_G| \\ &\cong \mathcal{N}_*(\underline{EG}) \boxtimes_{\underline{G}} \mathrm{Ad}_G\end{aligned}$$

where $E_*G = \mathcal{N}_*(* \downarrow \underline{G}) = \mathcal{N}_*(\underline{EG})$. [1]

Explicitly, $\underline{EG} = * \downarrow \underline{G}$ is the canonical groupoid attached to G, with

- objects are $g \in G$, and
- $\mathrm{Hom}_{\underline{EG}}(g_1, g_2) = \{h \in G | g_2 = hg_1\} = \{g_2 g_1^{-1}\}$.

The nerve of $\underline{EG}$ is a simplicial set

$$E_*G : \Delta^{op} \longrightarrow \mathbf{Set}$$

$$[n] \longmapsto G^{n+1}$$

with face and degeneracy maps

$$d_i : G^{n+1} \longrightarrow G^n \qquad\qquad s_j : G^{n+1} \longrightarrow G^{n+2}$$

$$(g_0, \dots, g_n) \mapsto (g_0, \cdots, g_{i-1}, g_{i+1}, \cdots, g_n) \qquad (g_0, \dots, g_n) \mapsto (g_0, \cdots, g_{j-1}, g_j, g_j, g_{i+1}, \cdots, g_n)$$

Note we have the identification

$$\mathcal{N}_n(\underline{EG}) \xrightarrow{\cong} E_nG$$

$$(g_0 \xrightarrow{g_1 g_0^{-1}} g_1 \to \cdots \to g_n) \to (g_0, \cdot, g_n)$$

and the face map d_i^{Nerve} which skips g_i in an n-composable morphisms in $\mathcal{N}_n(\underline{EG})$ by composing morphisms can be identified with $d_i^{E_*G}$ which skips g_i in the $n+1$-tuple in E_nG.

Note E_*G is equipped with a right G-action given by

$$E_nG \times G \longrightarrow E_nG$$

$$((g_0, \cdots, g_n), z) \mapsto (g_0 z, \cdots, g_n z)$$

hence we can take the G-orbit of $E_*G \times \mathrm{Ad}_G$ levelwise by

$$E_nG \times_G G \cong (G^{n+1} \times G)/G$$

[1]In general, we have $B(*, \mathcal{C}, \mathcal{C}) \cong \mathcal{N}_*((-) \downarrow \mathcal{C}) : \mathcal{C}^{op} \to \mathbf{Set}, c \mapsto \mathcal{N}_*(c \downarrow \mathcal{C})$.

where
$$z \cdot (g_0, \cdots, g_n; g) = (g_0 z^{-1}, \cdots, g_n z^{-1}, zgz^{-1}).$$
So we have

$$\begin{aligned} F_* : E_*G \times_G \mathrm{Ad}_G &\longrightarrow \Gamma_* G \\ [g_0, \cdots, g_n; g] &\longmapsto (g_n g g_0^{-1}, g_0 g_1^{-1}, \cdots, g_{n-1} g_0^{-1}) \end{aligned}$$

which is compatible with face and degeneracy maps, and it has an inverse

$$\begin{aligned} F_*^{-1} : \Gamma_* G &\longrightarrow E_*G \times_G \mathrm{Ad}_G \\ (\gamma_0, \cdots, \gamma_n) &\mapsto [\gamma_1 \cdots \gamma_n, \gamma_2 \cdots \gamma_n, \cdots, \gamma_n, 1, \gamma_0\gamma_1 \cdots \gamma_n] \end{aligned}$$

□

Corollary 6.16. *There is a canonical decomposition of cyclic sets* $\Gamma_* G = \coprod_{\langle z \rangle \in \langle G \rangle} \Gamma_*(G, z)$ *where* $\langle G \rangle$ *is the set of conjugacy classes of elements of* G *and*

$$\Gamma_*(G, z) = \left\{\Gamma_n(G, z) = \{(\gamma_0, \cdots, \gamma_n) \in \Gamma_n G : \gamma_0\gamma_1 \cdots \gamma_n \in \langle z \rangle\}\right\}$$

Proof. Note

$$\begin{tikzcd} \Gamma_* G \cong \mathrm{hocolim}_{\underline{G}} \mathrm{Ad} \arrow[r, "\cong"] \arrow[d] & E_*G \times_G \mathrm{Ad} \arrow[d] \\ \mathrm{colim}_{\underline{G}} \mathrm{Ad} \arrow[r, equal] & \{*\} \times_G \mathrm{Ad} = \langle G \rangle \end{tikzcd}$$

Simplicially, in degree n, we have

$$\begin{aligned} \pi_0 : E_n G \times_G \mathrm{Ad} &\to \langle G \rangle \\ [g_0, \cdots, g_n; z] &\longmapsto \langle z \rangle \end{aligned}$$

□

6.4.1 Simplicial Models of Circles

There are two simplicial models of S^1.

1. $B_* C = \{\mathbb{Z}/(n+1)\}_{n \geq 0}$, and
2. $B_* \mathbb{Z} = \{\mathbb{Z}^n\}_{n \geq 0}$.

There is a natural homotopy equivalence $|C_*| \simeq |B_*\mathbb{Z}|$ induced by the following map of simplicial sets (CHECK)

$$\begin{aligned} f_n : C_n &\longrightarrow \mathbb{Z}^n \\ t_n^0 &\longmapsto (0, \cdots, 0) \\ t_n^i &\longmapsto (0, \cdots, 1, \cdots, 0) \end{aligned}$$

Furthermore, it is a map of cyclic sets with

$$\begin{aligned} t_n : \mathbb{Z}^n &\longrightarrow \mathbb{Z}^n \\ (m_1, \cdots, m_n) &\mapsto (1 - (m_1 + \cdots + m_n), m_1, \cdots, m_{n-1}). \end{aligned}$$

6.5 Nerves

6.5.1 Twisted Nerve of a Group

Let G be a discrete group G, fix $z \in \mathcal{Z}(G)$ an element in the center of G. We can define $B_*(G, z)$ by add a cyclic map to the nerve of G as follows

$$\begin{aligned} t_n : B_n G = G^n &\longrightarrow B_n G = G^n \\ (g_1, \cdots, g_n) &\longmapsto \left(z \cdot (g_1 \cdots g_n)^{-1}, g_1, \cdots, g_{n-1}\right). \end{aligned}$$

then one can check

$$t_n^{n+1}(g_1, \cdots, g_n) = (zg_1z^{-1}, \cdots, zg_nz^{-1}) = (g_1, \cdots, g_n).$$

Notation. When $z = 1$, we write $B_*G = B_*(G, 1)$.

Proposition 6.17. *Given* G *and* $z \in \mathcal{Z}(G)$ *as above. We can define a group homomorphism*

$$\begin{aligned} \gamma_z : \mathbb{Z} \times G &\longrightarrow G \\ (n, g) &\longmapsto z^n g = g z^n. \end{aligned}$$

Up to homotopy, the S^1-action on $BG = |B_*(G, z)|$ is induced by the above cyclic structure, which is given by following map

$$S^1 \times BG \cong |B_*\mathbb{Z}| \times |B_*G| \cong |B_*(\mathbb{Z} \times G)| \xrightarrow{B\gamma_z} BG.$$

Recall the following adjunction

$$F : \mathbf{Set}^{\Delta^{op}} \rightleftarrows \mathbf{Set}^{\Delta C^{op}} : U$$

Proof. It suffices to show on π_1:

$$\begin{tikzcd} {|F(B_*G)|} \arrow[r, "|ev|"] \arrow[d, "(p_1,p_2)"'] & {|B_*G|} \arrow[d, equal] \\ {|C_*| \times |B_*G|} \arrow[r] & {|B_*G|} \end{tikzcd}$$

Check $t_1^* = \tau_1$. ☐

Definition 6.18. $X(G,z) := ES^1 \times_{S^1} |B_*(G,z)| =: |B_*(G,z)|_{hS^1}$ is the S^1-equivariant Borel homotopy quotient of $|BG|$ by S^1.

For any S^1-space X, we have the following fibration

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d, equal] & ES^1 \times X \arrow[r] \arrow[d, "\simeq"] & ES^1 \times_{S^1} X \arrow[d, equal] \\ {|B_*\mathbb{Z}|} \arrow[r, "\mu_*"] & BG \arrow[r] & X(G,z) \end{tikzcd}$$

where

$$\mu : \mathbb{Z} \to G$$

$$1 \longmapsto z.$$

Claim 6.19. *$X(G,z)$ is equivalent to the classifying space (nerve) of the crossed module defined by the map $\mu : \mathbb{Z} \to G$ together with the trivial action of G on $\mathbb{Z}$.*

6.5.2 Crossed Modules

Digression (on crossed modules) Let G be a discrete group and A an abelian group.

Fact 6.20. $H^2(G,A) \cong \{\text{group extensions } 1 \to A \to E \to G \to 1\}/\sim$, which has 1-1 correspondence with maps

$$G \times G \longrightarrow A$$

$$(g_1, g_2) \mapsto s(g_1)s(g_2)s(g_1g_2)^{-1}.$$

Question 6.21. *Is there a similar interpretation for $H^3(G,A)$?*

Definition 6.22. A crossed module is a group homomorphism $\mu : M \to N$ together with

$$\rho : N \longrightarrow \mathrm{Aut}(M)$$

$$n \longmapsto (\rho(n) : m \mapsto^n m)$$

satisfying

1. $\mu({}^{n}m) = n\mu(m)n^{-1}$ for any $m \in M, n \in N$.
2. ${}^{\mu(m)}(m') = m \cdot m' \cdot m^{-1}$ for any $m \in M, n \in N$.

Equivalently, given any $\mu : M \to N$, there is a commutative diagram

$$\begin{tikzcd} M \arrow[r, "Ad"] \arrow[d, "\mu"'] & Aut(M) \arrow[d, "act"] \\ N \arrow[r, "Ad_\mu"'] \arrow[ur, dashed, "\exists\rho"] & Hom(M,N) \end{tikzcd}$$

where
$$Ad(m)(m') = mm'm^{-1}$$
and
$$Ad_\mu(n) : M \longrightarrow N$$
$$m' \longmapsto n\mu(m')n^{-1}.$$
and
$$act : Aut(M) \longrightarrow Hom(M,N)$$
$$\sigma \longmapsto (\mu \circ \sigma : M \xrightarrow{\sigma} M \xrightarrow{\mu} N).$$

Then a crossed module structure on μ is equivalent to a lifting $\rho : N \to Aut(M)$.

Example 6.23. 1. Given $\mu : M \hookrightarrow N$, the normal morphisms

$$\rho : N \longrightarrow Aut(M)$$
$$n \longmapsto M \to M, m \mapsto nmn^{-1}.$$

2. (Whitehead, 1949) Given a Serre fibration $F \to E \to B$ of pointed spaces, $\mu : \pi_1(F) \to \pi_1(E)$ is a crossed module.

3. (Loday) What are group objects in **Cat**? Suppose $\underline{G}$ is a group object in **Cat**, then $\mathcal{N}_*\underline{G}$ is a group object in **sSet**, i.e. a simplicial group.
Recall for any simplicial group G_*, its Moore complex is defined by $K_n = \bigcap_{i=1}^{n} \ker(d_i^{(n)}) \subseteq G_n$ with $\partial_n = d_0^{(n)}$. If we take $G_* = \mathcal{N}_*(\underline{G})$, then its Moore complex is a crossed module.

Lemma 6.24. *Given a crossed module* $(\mu : M \to N, \rho)$*, consider* $A = \ker(\mu)$ *and* $G = \mathrm{coker}(\mu)$*, then*

$$1 \longrightarrow A \longrightarrow M \xrightarrow{\mu} N \longrightarrow G \longrightarrow 1$$

1. G *is a group and* A *is a central subgroup of* M.
2. *the action map* $N \times M \to M, (n, m) \mapsto^{n} m$ *induces a well-defined* G*-structure on* A.

Proof. Exercise. $\square$

Fix G and A, define an equivalence relation on the set of all crossed modules $\mu : M \to N, \rho)$ such that $A \cong \ker(\mu)$ and $G \cong \operatorname{coker}(\mu)$. The equivalence relation is generated by a map of crossed modules (α, β) as follows.

$$\begin{tikzcd}
1 \arrow[r] & A \arrow[r] \arrow[d, equal] & M \arrow[r, "\mu"] \arrow[d, "\alpha"] & N \arrow[r] \arrow[d, "\beta"] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & A \arrow[r] & M' \arrow[r, "\mu'"] & N' \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

Proposition 6.25. *There is a canonical isomorphism*

$$\mathbf{CrMod}(G, A)/\sim \;\to\; H^3(G, A)$$

$$\Big[(M \xrightarrow{\mu} N, \rho)\Big]_{\sim} \;\mapsto\; k(M, N)$$

where $k(M, N)$ *is called the* **MacLane invariant of crossed modules**.

{thm:moore}

Theorem 6.26 (Loday)**.** *The following are equivalent:*

1. *a crossed module of groups* $(\mu : M \to N, \rho : N \to \mathrm{Aut}(M))$.
2. *a simplicial group* $G_* \in \mathbf{sGr}$ *whose Moore (nonabelian chain) complex* K_*G *has length 1.*
3. *a group object in* **Cat**.
4. *a "categorical group", which is a pair* (N, G) *where* $N \subseteq G$ *together with two group homomorphisms* $s, t : G \to N$ *such that*
   (a) $s|_N = t|_N = \mathrm{Id}_N$, *and*
   (b) $[\ker(s), \ker(t)] = 1$.

Moore Complex For a simplicial abelian group A_*, there are two chain complexes associated to it.

1. $C_*(A, \mathbb{Z})$ where $C_n(A, \mathbb{Z}) = A$ with differential $\partial_n = \displaystyle\sum_{i=0}^{n} (-1)^i d_i$.

2. $N_*(A)$ where $N_n(A) = \bigcap_{i=1}^{n} d_i^{(n)} \subseteq A_n$ with differential $\partial_n = d_0|_{N_n(A)}$.

For a simplicial group, its Moore complex is the nonabelian version of Dold-Kan normalization.

Definition 6.27. The homotopy groups of a simplicial group are defined by $\pi_n(G_*) := H_n(K_*(G), \partial) \cong \pi_n(|G_*|)$ for all $n \geq 0$.

Lemma 6.28. $\pi_n(B|G_*|) = \pi_{n-1}(|G_*|) = H_{n-1}(K_*(G), \partial)$.

Classifying Space of Crossed Modules Given a crossed module $(M \xrightarrow{\mu} N, \rho)$, define $\mathcal{C}$ by

- $\mathrm{Ob}(\mathcal{C}) = N$, and
- $\mathrm{Mor}(C) = M \rtimes N$ with

$$\begin{aligned} s : \mathrm{Mor}(\mathcal{C}) &\to \mathrm{Ob}(\mathcal{C}) \\ (m, n) &\longmapsto n \end{aligned}$$

  and

$$\begin{aligned} s : \mathrm{Mor}(\mathcal{C}) &\to \mathrm{Ob}(\mathcal{C}) \\ (m, n) &\longmapsto \mu(n) \cdot n \end{aligned}$$

Therefore if we take the nerve of $\mathcal{C}$ we will get $B_0\mathcal{C} = N$, $B_1\mathcal{C} = M \rtimes N$ and $B_2\mathcal{C} = M \rtimes (M \rtimes N)$, and $B_*\mathcal{C}$ is a simplicial group.

Recall there is a Kan loop group adjunction

$$\mathbb{G} : \mathbf{sSet}_0 \rightleftarrows \mathbf{sGr} : \overline{W} :$$

which is a Quillen adjunction. Here $\mathbf{sSet}_0$ is the category of reduced simplicial sets X_*, i.e. $X_0 = \{\mathbf{pt}\}$. In particular $|\overline{W}G| \simeq \Omega X$.

Now we take $BB\mathcal{C} = |\overline{W}B_*\mathcal{C}|$ and

$$\pi_i BB\mathcal{C}) = \begin{cases} \operatorname{coker} \mu, & i = 1, \\ \ker \mu, & i = 2, \\ 0, & i \geq 3. \end{cases}$$

and $k \in H^3(B\pi_1, \pi_2) \cong H^3(BG, A)$.

Proof of Theorem 6.26. 1. $1 \implies 2$: take $B_*C(M, N)$ where $s_0 : N \hookrightarrow M \rtimes N$ is the natural inclusion.

2. 3 $\implies$ 2: take $K_*(B_*C) \cong \{N \xleftarrow{\partial_0 = d_0} \ker d_1 \leftarrow 1\}$, where $d_0 s_0 = d_1 s_0 = \mathrm{Id} \iff s|_N = t|_N = \mathrm{Id}_N$.

3. 4 $\implies$ 1 comes from the observation that $[\ker(d_0, \ker(d_1)] = 1 \iff$ axiom (b) in the definition of crossed modules.

□

Question 6.29. *What is a correct definition of normal maps of groups? What is the homotopy invariant version of normality?*

Question 6.30. *For a noncommutative ring spectrum, what does it mean to be simple?*

Homotopy Normal Maps Let $\mu : M \to N$ be a map between (simplicial) groups in **sGr**.

In **Gr**, μ is normal if and only if

- μ is injective,
- $\mathrm{Im}(\mu) \trianglelefteq N$.

This gives an exact sequence

$$M \longrightarrow N \longrightarrow N/M$$

which gives a homotopy fibration sequence

$$BM \longrightarrow BN \longrightarrow B(N/M)$$

Definition 6.31. A map of groups $\mu : M \to N$ is **homotopy normal** if there exists a pointed connected space with $\alpha : BN \to W$ such that

$$BM \longrightarrow BN \longrightarrow W$$

is a homotopy fibration sequence. Or equivalently, $B\mu \simeq \mathtt{hofib}_*(\alpha)$.

Remark. For any pointed map of connected spaces, $f : X \to Y$, there is a Puppe sequence

$$\cdots \longrightarrow \Omega X \longrightarrow \Omega Y \longrightarrow \Omega X//\Omega Y \longrightarrow X \longrightarrow Y \xrightarrow{\alpha} W$$

Theorem 6.32 ([FH], [P]). *(Higher normality) A map $\mu : M \to N$ is homotopy normal (in* **sGr***) if there exists a simplicial group Γ_* with an isomorphism $\Gamma_0 \cong N$ which extends to a N-equivariant isomorphism of simplicial sets*

$$\mathrm{hocolim}_M(N) = N \times_M EM \xrightarrow{\cong} \Gamma_*$$

Theorem 6.33. *To give such an extension is equivalent to extend μ to a crossed module $(\mu : M \to N, \rho)$.*

Recall for a crossed module $(M \xrightarrow{\mu} N, \rho)$ we associate it with a simplicial group $B_*C(M, N)$, which gives us $BB\mathcal{C} = B|B_*\mathcal{C}|$ the classifying space of crossed modules.

$$\begin{aligned} &\pi_1(BB\mathcal{C}) \cong \mathrm{coker}(\mu), \\ &\pi_2(BB\mathcal{C}) \cong \ker(\mu), \\ &\pi_i(BB\mathcal{C}) = 0, \qquad i \geq 3. \end{aligned}$$

Example 6.34. For a discrete group G together with $z \in \mathcal{Z}(G)$, we have $B_*(G, z)$ the twisted nerve of G and $X(G, z) := ES^1 \times_{S^1} |B_*(G, z)|$. To describe the homotopy type of this space depends on (G, z).

Observe there is a sphere fibration

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d, equal] & ES^1 \times B(G,z) \arrow[r, two heads] \arrow[d, "\simeq"] & ES^1 \times_{S^1} B(G,z) \arrow[d, equal] \\ S^1 \arrow[r] & \{*\} \times BG \arrow[r] & X(G,z) \end{tikzcd}$$

Consider $\mu : \mathbb{Z} \to G, n \mapsto z^n$ together with the trivial G-action $\rho : G \to \mathrm{Aut}(\mathbb{Z})$, this crossed module (μ, ρ) gives us a group homomorphism

$$\begin{aligned} \gamma_z : \mathbb{Z} \times G &\longrightarrow G \\ (n, g) &\longmapsto z^n g = g z^n \end{aligned}$$

which induces

$$S^1 \times BG \cong B(\mathbb{Z} \times G) \xrightarrow{B\gamma_z} BG$$

or equivalently, a fibration

$$B\mathbb{Z} \xrightarrow{\mu} BG \xrightarrow{\alpha} X(G, z).$$

By Proposition 1, $X(G, z)$ is the classifying space of $BB\mathcal{C}(\mathbb{Z}, G, \rho)$, and

$$\begin{aligned} &\pi_1(X(G, z)) \cong \mathrm{coker}(\mu) = G/\mathrm{Im}(\mu), \\ &\pi_2(X(G, z)) \cong \ker(\mu), \\ &\pi_i(X(G, z)) = 0, \qquad i \geq 3. \end{aligned}$$

Proposition 6.35.

1. *If z has infinite order, then $X(G, z) \cong B(G/\langle z \rangle)$.*
2. *If z has finite order, then*

$$\begin{aligned} &\pi_1(X(G, z)) \cong G/\langle z \rangle, \\ &\pi_2(X(G, z)) \cong \mathbb{Z}, \\ &\pi_i(X(G, z)) = 0, \qquad i \geq 3. \end{aligned}$$

In the second case, π_1 acts trivially in π_2,

$$k(G, z) \in H^3(B\pi_1, \pi_2) = H^3(G/\langle z\rangle, \mathbb{Z}).$$

When $k(G, z) = 0$, by Postnikov decomposition,

$$X(G, z) = K(G/\langle z\rangle, 1) \times K(\mathbb{Z}, 1) = B(G/\langle z\rangle) \times BS^1.$$

This happens in two examples.

1. When G is any discrete group and $z = 1$, $B(G, 1) = BG$ and $X(G, 1) = BG \times BS^1$.
2. When G is abelian and $z \in G$ has finite order, $X(G, z) = B(G/\langle z\rangle) \times BS^1$. Note in this case, G and $\mathbb{Z}$ are abelian groups and G acts on $\mathbb{Z}$ trivially, so $B_n\mathcal{C} = \mathbb{Z} \times G^n$ and $B_*\mathcal{C}$ is an abelian simplicial group. By a theorem in [M], any abelian simplicial group is the product of Eilenberg-Maclane spaces
$$|\Gamma_*| \simeq \prod_{i \geq 1} K(\pi_i(\Gamma_*), i)$$
or equivalently, Postnikov's invariants of $|\Gamma_*|$ are trivial.

Theorem 6.36. *For any group and any finite order $z \in \mathcal{Z}(G)$, there is a rational equivalence*

$$X(G, z) \simeq_{\mathbb{Q}} B(G/\langle z\rangle)_{\mathbb{Q}} \times BS^1_{\mathbb{Q}}.$$

6.6 Cyclic Nerve

Kapranov-Dyckenhoff 2015 [DK]

Definition 6.37. Let

$$\begin{aligned} \Psi : \Delta C &\longrightarrow \textbf{Cat} \\ [n] &\longmapsto \Psi([n]) \end{aligned}$$

where $\Psi([n])$ is the path (free) category of the cyclic quiver

$$\begin{tikzcd} 0 \arrow[rrr, bend right, "x_0"'] & 1 \arrow[l, "x_1"'] & \cdots \arrow[l] & n \arrow[l, "x_n"'] \end{tikzcd}$$

Then the cyclic nerve is

$$\begin{aligned} \mathcal{N}^{cyc}_* : \textbf{Cat} &\to \textbf{Set}^{\Delta C^{op}} \\ C &\longmapsto \mathcal{N}^{cyc}_*\mathcal{C} \end{aligned}$$

where
$$\mathcal{N}^{cyc}_n\mathcal{C} = \{c_0 \xleftarrow{x_1} c_1 \xleftarrow{x_2} c_2 \leftarrow \cdots \xleftarrow{x_n} c_n \xleftarrow{x_0} c_0\}$$
The structure maps are given as follows. For any $f : [m] \to [n]$, there is an unique factorization $f = \tau \circ \alpha$,

- For $\alpha \in \mathrm{Hom}_\Delta([m],[n])$,
$$\mathcal{N}^{cyc}_\alpha(\mathcal{C}) : (c_0 \leftarrow c_1 \leftarrow \cdots \leftarrow c_n \leftarrow c_0) = (c_{\alpha(0)} \leftarrow c_{\alpha(1)} \leftarrow \cdots \leftarrow c_{\alpha(n)} \leftarrow c_{\alpha(0)}).$$
- For $\tau \in \mathrm{Hom}_{\Delta C}([n],[n])$,
$$\mathcal{N}^{cyc}_\alpha(\mathcal{C}) : (c_0 \leftarrow c_1 \leftarrow \cdots \leftarrow c_n \leftarrow c_0) = (c_n \leftarrow c_0 \leftarrow \cdots \leftarrow c_{n-1} \leftarrow c_n).$$

Lemma 6.38. *1. If G is a group, $\Gamma_*(G) = \mathcal{N}^{cyc}_*(\underline{G})$.*

*2. $\mathcal{N}^{cyc}_*$ is the restriction functor for certain map of small categories $b^* : \Delta C \to \mathscr{G} \subset$ **Gr**, i.e. $\mathcal{N}^{cyc}_* = (b^{cyc})^*$.*

Proof. Consider the following adjunctions
$$U : \mathbf{Cat} \rightleftarrows \mathbf{Grpd} \rightleftarrows \mathbf{Gr} : (-)$$
Define
$$b^{cyc} = \Psi : \Delta C \longrightarrow \mathbf{Cat}$$
$$[n] \longmapsto \mathbb{F}^{n+1} = \mathbb{F}\langle x_0, \cdots, x_n\rangle$$
Check
$$\Gamma_n(G) = G^{n+1} \cong \mathrm{Hom}_{\mathbf{Gr}}(b^{cyc}([n]), G) \cong \mathrm{Hom}_{\mathbf{Gr}}(U\Psi([n]), G) \cong \mathrm{Hom}_{\mathbf{Cat}}(\Psi([n]), \underline{G}) = \mathcal{N}_n(\underline{G})$$
on morphisms. □

Theorem 6.39. *For any (simplicial topological) group G, there is a canonical homotopy equivalence*
$$\gamma : |\Gamma_* G| \xrightarrow{\sim} \mathscr{L}(BG) = \mathrm{Map}(S^1, BG)$$
which is S^1-equivariant.

The left S^1-action comes from cyclic structure and the right S^1-action comes from the action of S^1 on itself.

Corollary 6.40. *There are canonical equivalences*

1. $HH_*(k[G]) \cong H_*(\mathscr{L}(BG;k)$.
2. $HC_*(k[G]) \cong H^{S^1}_*(\mathscr{L}(BG);k)$.

6.6.1 Burghelea Decomposition

Free Loop Space Let X be a pointed connected space. $\mathscr{L}X = \mathrm{Map}(S^1, X)$.

The S^1-action on $\mathscr{L}X$ is given by

$$S^1 \times \mathscr{L}X \longrightarrow \mathscr{L}X$$

$$(z, \theta) \longmapsto (z, \theta) : z' \mapsto \theta(z \cdot z')$$

Theorem 6.41 (Goodwillie, Loday 2016). *For any topological group G, there is a canonical S^1-equivariant homotopy equivalence*

$$|\Gamma_* G| \xrightarrow{\sim} \mathscr{L}BG$$

Proof. (Detailed sketch) Let G be a discrete group, there is a simplicial map

$$\mathrm{pr}_* : \Gamma_* G \longrightarrow B_* G = B_*(G, 1)$$

$$(g_0, \cdots, g_n) \longmapsto (g_1, \cdots, g_n)$$

which can be extended to a morphism of cyclic sets.

CHECK

Define

$$\gamma^\# : S^1 \times |\Gamma_* G| \xrightarrow{act} |\Gamma_* G| \xrightarrow{|\mathrm{pr}_*|} |B_* G|$$

Note $\gamma^\#$ is S^1-equivariant (because both are). γ is defined to be the adjoint map

$$\gamma : |\Gamma_* G| \longrightarrow \mathrm{Map}(S^1, BG)$$

and γ is S^1-equivariant because $\gamma^\#$ is. Recall $|B_* G| = |B_*(G, 1)|$ comes with trivial S^1-action, so S^1-action on $\mathscr{L}BG$ is the standard one.

Need to check γ is a homotopy equivalence.

If X is pointed, there is a natural fibration sequence

$$\Omega X \longrightarrow \mathscr{L}X \xrightarrow{ev} X$$

For $X = |B_* G|$, we have

$$\begin{tikzcd}
G \arrow[r, "in"] \arrow[d, "\sim"', "\overline{\gamma}"] & {|\Gamma_* G|} \arrow[r, "{|pr|}"] \arrow[d, "\gamma"] & {|B_* G|} \\
\Omega BG \arrow[r] & \mathscr{L}BG \arrow[r, "ev"] & BG
\end{tikzcd}$$

where

$$G \longrightarrow \Gamma_* G$$

$$g \mapsto (g, 1, \cdots, 1)$$

Lemma 6.42. $\overline{\gamma} : G \to \Omega BG$ *is the classical homotopy equivalence.*

Remark. $\overline{\gamma}$ is called a **topological group completion**. Quillen shows that $\overline{\gamma}$ induces

$$\begin{tikzcd}
H_*(M,k) \arrow[rr, "\overline{\gamma}"] \arrow[dr] & & H_*(\Omega BM, k) \\
& H_*(M,k)[\pi_0(M)^{-1}] \arrow[ur, "\cong"] &
\end{tikzcd}$$

where $\pi_0(M)^{-1}$ is a 2-sided Ore set.

For M a topological monoid, $\overline{\gamma} : M \to \Omega BM$ is homotopy equivalence if and only if M is group like, $\pi_0(M)$ is a monoid group.

Proof. We will describe the classical map $\overline{\gamma}'$ simplicially and show that $\overline{\gamma}' = \overline{\gamma}$.

Recall that $\overline{\gamma}'$ is defined by looking at its adjoint

$$\begin{aligned}
(\overline{\gamma}')^{\#} : S^1 \times G &\longrightarrow BG \\
C_1 \times G &\longrightarrow B_1 G = G \\
(t, g) &\longmapsto g
\end{aligned}$$

Check there is a unique simplicial map that extends to a map $C_* \times G \to B_* G$.

On the other hand, by definition of S^1-action on $|\Gamma_* G|$

$$\begin{tikzcd}
S^1 \times |\Gamma_* G \arrow[r, "(p_1,p_2)^{-1}"] & {|F(\Gamma_* G)|} \arrow[r, "|ev|"] & {|\Gamma G|} \\
S^1 \times G = |C_* \times G| \arrow[u, hook, "|in|"] & &
\end{tikzcd}$$

and $\overline{\gamma}'$ is the realizationof

$$\begin{tikzcd}
C_* \times G \arrow[r, "in"] & C_* \times \Gamma_* G \arrow[r, "ev"] & \Gamma_* G \arrow[r, "pr"] & B_* G \\
(t, g) \arrow[r, mapsto] & (t, (g, 1)) \arrow[r, mapsto] & t_*(g, 1) = (1, g) \arrow[r, mapsto] & g
\end{tikzcd}$$

which can be identified with $\overline{\gamma}$. ☐

☐

Corollary 6.43. *For any discrete group* G *and any commutative ring* k*,*

$$\begin{aligned}
HH_*(k[G]) &\cong H_*^{simp}(\mathscr{L}(BG), k) \cong H_*^{sing}(\mathscr{L}(BG), k) \\
HC_*(k[G]) &\cong H_*^{S^1}(\mathscr{L}BG, k).
\end{aligned}$$

Proof. Recall $k[\Gamma_* G] \cong C_*^{Hoch}(k[G])$ (by inspection), so

$$HH_*(k[G]) \coloneqq H_*(C_*^{Hoch}(k[G]) \cong H_*(k[\Gamma_* G]) \cong H_*^{simp}(\mathscr{L}BG).$$

□

MacLane Isomorphism Let $A = k[G]$. Let M be a bimodule over A

$$HH_*(k[G], M) \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} H_*^{EM}(G, \tilde{M})$$

where $\tilde{M}$ is the (right) G-module defined by $\tilde{M} = M$ as k-module.

$$\tilde{M} \times G \longrightarrow \tilde{M}$$

$$(m, g)n \mapsto m^g = g^{-1} m g$$

Proof. Let $C_n^{Hoch}(k[G], M) \coloneqq (M \bigotimes_k k[G]^{\otimes n}, d^{Hoch})$ and $C_n^{EM}(G, \tilde{M}) \coloneqq (\tilde{M} \times k[G^n], d^{EM})$ where

$$d_n^{EM} : C_n^{EM} \longrightarrow C_{n-1}^{EM}$$

$$(m, g_1, \cdots, g_n) \mapsto (m^{g_1}, g_2, \cdots, g_n) + \textstyle\sum_{i=1}^{n-1} (-1)^i (m, g_1, \cdots, g_i g_{i+1}, \cdots, g_n) + (-1)^n (m, g_1, \cdots, g_{n-1}).$$

For

$$\Phi_n : C_n^{Hoch}(k[G], M) \rightleftarrows C_n^{EM}(k[G], \tilde{M}) : \Phi_n^{-1}$$

$$(m, g_1, \cdots, g_n) \longmapsto (g_1 \cdots g_n \cdot m, g_1, \cdots, g_n)$$

$$(g_n^{-1} \cdots g_1^{-1} \tilde{m}, g_1, \cdots, g_n) \longleftarrow (\tilde{m}, g_1, \cdots, g_n)$$

check $\phi_{n+1} \circ d^{Hoch} \cong d^{EM} \circ \Phi_n$, which induces isomorphism on cohomology.

□

Burghelea decomposition Ref: Quillen diaries 1992 April-October, Clay Institute, Quillen notebooks, "cyclic torsors" over Segal groupoids.

Burghelea, Goodwillie-Fiedorowicz

For any $z \in G$, $G_z = C_G(z) = \{g \in G : gz = zg\}$. Note $z \in G_z$, so we can define $B_*(G_z, z)$ and $X(G_z, z) = ES^1 \times_{S^1} |B(G_z, z)|$. And $\mathbb{Z} \to G_z, n \mapsto z^n$ defines a crossed module.

Note G_z and $X(G_z, z)$ depends only on the conjugacy class of z.

The set of conjugacy classes of all elements in G

$$\langle G \rangle = \langle G \rangle^{Fin} \coprod \langle G \rangle^{\infty}$$

consisting of finite order ones and infinite order ones.

Recall that $\Gamma_*(G,z)$ contains cycic subsets $\Gamma_*(G,z)$ and we have a cyclic isomorphism $\Gamma_* G \cong \coprod_{\langle z\rangle} \Gamma(G,z)$, which comes from a homotopy decomposition, where

$$\Gamma_* G \cong \mathrm{hocolim}_G(\mathrm{Ad}) \xrightarrow{\pi_0} \operatorname*{colim}_G(\mathrm{Ad}) = \langle G\rangle$$

and $\Gamma_*(G,z) = \mathtt{fiber}_{\langle z\rangle}(\pi_0)$, which implies

Lemma 6.44. $k[\Gamma_* G] \cong \bigoplus_{\langle z\rangle} k[\Gamma_*(G,z)]$.

Proof. Apply $k[-]$. $\square$

{lem:classifying-cyclic}

Lemma 6.45 (Loday, Fiedorowicz). *For every $z \in G$, there is a natural inclusion*

$$\begin{aligned} i : B_*(G_z, z) &\longrightarrow \Gamma_*(G,z) \\ (g_1, \cdots, g_n) &\mapsto ((g_1 \cdots g_n)^{-1} z, g_1, \cdots, g_n) \end{aligned}$$

with properties

1. i *is a map of cyclic sets.*
2. $i_* : k[B_*(G_z,z)] \to k[\Gamma_*(G,z)]$ *induces an isomorphism on cyclic homology.*

Proof. 1. Check

$$\begin{tikzcd} B_n(G_z,z) \arrow[r, "i_n"] \arrow[d, "t_n^B"] & \Gamma_n(G,z) \arrow[d, "t_n^\Gamma"] \\ B_n(G_z,z) \arrow[r, "i_n"] & \Gamma_n(G,z) \end{tikzcd}$$

where

$$\begin{aligned} t_n^B(g_1, \cdots, g_n) &= ((g_1 \cdots g_n)^{-1} z, g_2, \cdots, g_n) \\ t_n^\Gamma(g_0, \cdots, g_n) &= (g_n, \cdots, g_0, \cdots, g_{n-1}) \end{aligned}$$

2. By the next lemma, it suffices to show that

$$i : k\,[B_*(G_z,z)] \hookrightarrow k\,[\Gamma_*(G_z,z)]$$

is Hochschild equivalence. Notice i factors as

$$\begin{tikzcd} k\,[B(G_z,z)] = k\,[B_*(G_z)] \cong C_*^{\mathrm{Hoch}}(k\,[G_z] \arrow[r, "\alpha"] & C_*^{EM}(G, k\,[\langle z\rangle]) \arrow[d, "\cong", "\Phi^{-1}"'] \\ & k\,[\Gamma_*(G,z)] \arrow[d, hook] \\ & k\,[\Gamma_* G] \end{tikzcd}$$

where α is an isomorphism by Shapiro lemma: for $G_z \subseteq G$,

$$H_*(G_z, k) \cong H_*(G, \mathrm{Ind}_{G_z}^{G}(k)),$$

and $\mathrm{Ind}_{G_z}^{G}(k) = k[\langle\rangle]$.

□

Lemma 6.46. *A map of cyclic* k*-modules* $f : E \to E'$ *is a cyclic equivalence*

$$f_* : \mathrm{Tor}_*^{\Delta C^{op}}(k, E) \cong \mathrm{Tor}_*^{\Delta C^{op}}(k, E')$$

if and only if

$$f_* : \mathrm{Tor}_*^{\Delta^{op}}(k, E) \cong \mathrm{Tor}_*^{\Delta^{op}}(k, E')$$

Proof. If follows from Connes' Periodicity sequence

$$\cdots \longrightarrow HH_n(E) \longrightarrow HC_n(E) \longrightarrow HC_{n-2}(E) \longrightarrow HH_{n-1}(E) \longrightarrow \cdots$$

which is the Gysin sequence of the following Borel fibraion

$$\cdots \longrightarrow S^1 \longrightarrow X \longrightarrow ES^1 \times_{S^1} X \longrightarrow BS^1$$

□

6.6.2 Cyclic Homology of Group Algebras

Theorem 6.47. *Let* G *be a discrete group and* k *be a commutative ring.*

$$\begin{aligned} HH_*(k[G]) &\cong \bigoplus_{\langle z\rangle \in \langle G\rangle} H_*(k[B_*(G_z, z)]) \\ HC_*(k[G]) &\cong \bigoplus_{\langle z\rangle \in \langle G\rangle} HC_*(k[B_*(G_z, z)]) \\ HC_*(k[G]) &\cong \bigoplus_{\langle z\rangle \in \langle G\rangle} H_*^{top}(X_*(G_z, z), k) \end{aligned}$$

Notation For any $z \in G$, $G_z = C_G(z)$ and $z \in \mathscr{Z}(G_z)$. The twisted nerve of (G_z, z) is $B(G_z, z) \in \mathbf{Set}^{\Delta C^{op}}$.

Corollary 6.48. *If* G *is torsion-freee,*

$$\begin{aligned} HC_*(k\,[G]) &\cong \bigoplus_{\langle z\rangle \in \langle G\rangle, z\neq 1} H_*(G_z/\langle z\rangle, k) \bigoplus H_*(BS^1 \times BG, k) \\ &\cong \bigoplus_{\langle z\rangle \in \langle G\rangle, z\neq 1} H_*(G_z/\langle z\rangle, k) \bigoplus \bigoplus_{i\in\mathbb{N}} H_{*-2i}(G, k) \end{aligned}$$

Corollary 6.49. *For abelian group* G*,*

6.7 Homotopy Simplicial Groups

Example of homotopy simplicial $\mathcal{T}$-algebras over a (simplicial) algebraic theory $\mathcal{T}$ of groups.

Definition 6.50. Let $\mathcal{G}$ be the (skeleton) of the full subcategory of **Gr** consisting of finitely generated free groups with

- objects $\{\langle n\rangle = \mathbb{F}\langle x_1, \cdots, x_n\rangle\}_{n\geq 0}$, and
- morphisms $\mathrm{Hom}_{\mathcal{G}}(\langle n\rangle, \langle m\rangle) = \mathrm{Hom}_{\mathbf{Gr}}(\langle n\rangle, \langle m\rangle)$.

There is a strict monoidal structure on $\mathcal{G}$ with $\langle n\rangle \coprod \langle m\rangle = \langle n\rangle * \langle m\rangle = \langle n+m\rangle$. $\mathcal{G}^{op}$ has an opposite monoidal structure $\prod$.

Starting Point. Consider the following composite

$$\mathbf{Gr} \hookrightarrow \mathrm{Fun}(\mathbf{Gr}^{op}, \mathbf{Set}) \xrightarrow{Res} \mathrm{Fun}(\mathcal{G}^{op}, \mathbf{Set})$$

which gives

$$\mathbf{Gr} \xrightarrow{\quad\cong\quad} \mathrm{Fun}^{\otimes}(\mathcal{G}^{op}, \mathbf{Set})$$

$$\Gamma \longmapsto \underline{\Gamma} : \langle n\rangle \mapsto \mathrm{Hom}_{\mathbf{Gr}}(\langle n\rangle, \Gamma) = \Gamma^n$$

Note for each $n \geq 0$ and $1 \leq k \leq n$, there are maps

$$i_{n,k} : \langle 1\rangle \longrightarrow \langle n\rangle = \textstyle\coprod_k \langle 1\rangle$$

$$x \longmapsto x_k$$

which corresponds to opposite maps $p_{n,k}$ in $\mathcal{G}^{op}$.

If we replace **Gr** by **sGr** and **Set** by **sSet**, we have similar argument for simplicial groups and given any functor $A : \mathcal{G}^{op} \to \mathbf{sSet}$, we can patch maps $p_{n,k}$ to get

$$A(p_n) \coloneqq \textstyle\prod_{k=1}^{n} A(P_{n,k}) : A(\langle n\rangle) \longrightarrow \prod_{i=1}^{k} A(\langle 1\rangle) = A(\langle n\rangle)^n$$

Definition 6.51. $A : \mathcal{G}^{op} \to \mathbf{sSet}$ is **(strictly) product-preserving** if $A(p_n)$ are isomorphisms in **sSet** for any $n \geq 1$. Then

$$\mathbf{sGr} \cong \mathrm{Fun}^{\otimes}(\mathcal{G}^{op}, \mathbf{sSet})$$

and $A(p_o) = \mathbf{pt}$.

Definition 6.52 (B, Badzioch [B])**.** A **homotopy simplicial group** is a functor $A : \mathcal{G}^{op} \to \mathbf{sSet}$ such that $A(p_n)$ are weak homotopy equivalences in **sSet**.

Remark. The full subcategory of all weakly product-preserving functors cannot carry a model structure because it is not closed under colimits. (Examples?)

Badzioch Model Category is a model structure on $\mathbf{sSet}^{\mathcal{G}^{op}}$ in which homotopy simplicial groups are fibrant objects.

Definition 6.53. Let $\mathbf{sGr}^h \coloneqq L_{\mathcal{S}}(\mathbf{sSet}^{\mathcal{G}^{op}})$ be the left Bousfield localization of $\mathbf{sSet}^{\mathcal{G}^{op}}$ with BK projective model structure with respect to

$$\mathcal{S} = \{j_n : \textstyle\coprod_{k=1}^n \mathrm{Hom}_{\mathcal{G}}(-,\langle 1\rangle) \xrightarrow{\coprod_{k=1}^n (i_{n,k})_*} \mathrm{Hom}_{\mathcal{G}}(-,\langle n\rangle)\}_{n\geq 1}$$

The localization induces a Quillen pair

$$j = \mathrm{Id} : \mathbf{sSet}^{\mathcal{G}^{\mathcal{G}^{op}}} \rightleftarrows \mathbf{sGr}^h : \mathrm{Id} = q$$

which induces an adjunction on homotopy category

$$\overline{\mathrm{Id}} : \mathrm{Ho}(\mathbf{sSet}^{op}) \rightleftarrows \mathrm{Ho}(\mathbf{sGr}^h) : \mathbb{R}\mathrm{Id}$$

Definition 6.54. The localization with respect to $\mathcal{S}$ is the adjunction unit $\{L_{\mathcal{S}}(X) : X \to \mathbb{R}q \circ \bar{j}(X)\}_{X\in Ob(\mathrm{Ho}(\mathbf{sSet}^{\mathcal{G}^{op}}))}$ of this pair

$$\bar{j} : \mathrm{Ho}(\mathbf{sSet}^{\mathcal{G}^{op}}) \rightleftarrows \mathrm{Ho}(\mathbf{sGr}^h) : \mathbb{R}q$$

Proposition 6.55 ([B])**.** *Homotopy simplicial groups are precisely the fibrant objects in* $\mathbf{sGr}^h$ *(which are objectwise fibrant as simplicial set.*

The natural inclusion $J : \mathbf{sGr} \hookrightarrow \mathbf{sSet}^{\mathcal{G}^{op}}$ has a left adjoint

$$K : \mathbf{sSet}^{\mathcal{G}^{op}} \longrightarrow \mathbf{sGr}$$

where $J(\Gamma) = \underline{\Gamma}$ and

$$K(\Gamma) \coloneqq \int^{n\in Ob(\mathcal{G})} \underline{\Gamma}\langle n\rangle \boxtimes \mathbb{F}\langle n\rangle.$$

Theorem 6.56 (Rigidification Theorem)**.** *1.* $\mathbb{L}K$ *maps* $\overline{S} \in \mathrm{Ho}(\mathbf{sSet}^{\mathcal{G}^{op}})$ *to isomorphisms in* $\mathrm{Ho}(\mathbf{sGr})$*. Hence* $K^h : \mathbf{sGr}^h \rightleftarrows \mathbf{sGr} : J^h$ *is a Quillen pair.*

2. (K^h, J^h) *is a Quillen equivalence,* $\mathrm{Ho}(\mathbf{sGr}^h) \cong \mathrm{Ho}(\mathbf{sGr})$.

Remark. 1. The above theorem generalizes to (multi-sorted) algebraic theories.

Remark. If we consider

$$\mathcal{G} \hookrightarrow \mathbf{Gr} \hookrightarrow \mathrm{Ho}(\mathbf{Top}_{0,*})$$

$$\langle n\rangle \longmapsto B\langle n\rangle$$

in $\mathrm{Ho}(\mathbf{Top}_{0,*})$, notice $\mathcal{G} \cong \mathcal{B}$ where $Ob(\mathcal{B}) = \{\bigvee_{i=1}^n S^1\}_{n\geq 0}$.

6.7.1 Cyclic Homology

Let Γ be a discrete group, we can define

$$\begin{aligned} B^{cyc}\Gamma : \Delta C^{op} &\to \mathbf{Set} \\ [n] &\longmapsto \Gamma^{n+1} \end{aligned}$$

and extend to a functor

$$\begin{aligned} k\,[-] : \mathbf{Set}^{\Delta C^{op}} &\longrightarrow \mathrm{Mod}_k^{\Delta C^{op}} \\ (\Delta C^{op} \to \mathbf{Set}) &\mapsto (\Delta C^{op} \to \mathbf{Set} \xrightarrow{k[-]\,\mathrm{Mod}_k}) \end{aligned}$$

and we have $k[B^{cyc}\Gamma] \cong C_*^{Hoch}(k[\Gamma])$.

Key Observation There is a functor

$$\begin{aligned} \Psi_{cyc} : \Delta C &\longrightarrow \mathcal{G} \\ [n] &\longmapsto \langle n+1 \rangle \end{aligned}$$

which induces

$$\Psi_{cyc}^*(\underline{\Gamma}) : \Delta C^{op} \longrightarrow \mathcal{G}^{op} \xrightarrow{\underline{\Gamma}} \mathbf{Set}$$

such that $B^{cyc}\Gamma \cong \Psi_{cyc}^*(\underline{\Gamma})$.

Define for any $\Gamma : \mathcal{G}^{op} \to \mathbf{sSet}$, we can define

$$B^{cyc}(\Gamma) \coloneqq \Psi_{cyc}^*(\Gamma) : \Delta C^{op} \longrightarrow \mathbf{sSet}$$

and

$$k\,[B^{cyc}(\Gamma)] = k\left[\Psi_{cyc}^*(\Gamma)\right] : \Delta C^{op} \longrightarrow \mathbf{s}\,\mathrm{Mod}_k \xrightarrow{N} \mathrm{Ch}_k$$

Lemma 6.57. *If Γ is a homotopoy simplicial group, then $HC_*(k[\Gamma])$ depends only on $\mathcal{S}$-local weak equivalence class of Γ, i.e.*

$$HC_*(k\,[-]) : (\mathbf{sGr}^h)^{fib} \longrightarrow \mathrm{Mod}_k$$

Corollary 6.58. $HC_*(k[\Gamma])$ *depends only on the homotopy type of the classifying space of Γ.*

Question 6.59. *What is the center/centralizer of homotopy simplicial group? What is the twisted nerve?*

7 Realizable (Voedvodsky) Homotopy Colimits

Voedvodsky (2010): from BK construction to simplicial descent category, proposed axiomatics for simplicial descent categories.

Barwick-Kan (2012): the fifth model for $(\infty, 1)$-categories.

B. Rodzigez-Gonzalez [G] (2016) used Barwick-Kan model **RelCat** (the fifth model for ∞-category) to make it concrete.

Motivation For $f : \mathcal{I} \to \mathcal{J}$ in **Cat**. Let $\mathcal{M}$ be a (nice) model category, there is a Quillen pair

$$f_! : \mathcal{M}^{\mathcal{I}} \longrightarrow \mathcal{M}^{\mathcal{J}} : f^*$$

where the left adjoint is defined by left Kan extension. Thus we have a (left) derived functor

$$\mathbb{L}f_! : \mathrm{Ho}(\mathcal{M}^{\mathcal{I}}) \longrightarrow \mathcal{M}^{\mathcal{J}}.$$

When $\mathcal{J} = \{*\}$, we have $p : \mathcal{I} \to \{*\}$ and $\mathbb{L}p_! : (X) = \mathrm{hocolim}_{\mathcal{J}} X$.

Note $\mathbb{L}f_!$ is defined on $\mathrm{Ho}(\mathcal{M}^{\mathcal{I}})$ but not $\mathrm{Ho}(\mathcal{M})^{\mathcal{I}}$, so we want to lift diagrams from $\mathrm{Ho}(\mathcal{M}^{\mathcal{I}})$ to $\mathcal{M}^{I}$.

Want notion of "hocolim" at the level of the diagram categories.

Example 7.1 (Bousfield-Kan)**.** $\mathrm{hocolim}_{\mathcal{I}}^{BK} X \coloneqq |B_*(*, \mathcal{I}, X)|_{\mathcal{M}}$.

Question 7.2. *Replace B_* by "smaller resolutions".*

Theorem 7.3. *Homotopy colimits are realizable if and only if they satisfy Voedvodsky's axiomatics.*

7.1 Relative Categories

Definition 7.4. A **relative category** is a pair $(\mathcal{C}, \mathcal{W})$ where

- $\mathcal{C}$ is the underlying category, and
- $\mathcal{W} \subset \mathcal{C}$ is a (wide) subcategory (of weal equivalences in $\mathcal{C}$).

A **relative functor** $F : (\mathcal{C}, \mathcal{W}_{\mathcal{C}}) \to (\mathcal{D}, \mathcal{W}_{\mathcal{D}})$ is a functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(\mathcal{W}_{\mathcal{C}}) \subseteq \mathcal{W}_{\mathcal{D}}$.

Denote by **RelCat** the category of all (small) relative categories and relative functors.

Theorem 7.5 ([BK])**.** *There is a cofibrantly generated left proper model structure on* **RelCat** *that is Quillen equivalent to $\mathcal{CSS}$ and* **QCat***.*

Sketch of Construction

Definition 7.6. A relative category $(\mathcal{C}, \mathcal{W})$ is called

- **maximal** if $\mathcal{W} = \mathcal{C}$.
- **minimal** if $\mathcal{W} = \mathrm{Ob}(\mathcal{C})$ has no identity morphisms.
- a relative poset if $\mathcal{C} = \mathcal{P}$ is a poset category.

RelPos is the full subcategory of **RelCat** with objects being relative posets.

Notation For a category $\mathcal{C}$, we denote by $\mathcal{C}_{\max}$ and $\mathcal{C}_{\min}$ the maximal (resp. minimal) relative model structure on $\mathcal{C}$.

Example 7.7. Let $\mathcal{C} = [n] = \{0 \to 1 \to \cdots \to n\}$ be a poset category, $[n]_{\min}$ and $[n]_{\max}$ are distinguished relative posets.

Definition 7.8. Let $\mathcal{P} \in$ **RelPos**

- the **terminal subdivision** of $\mathcal{P}$ is the relative poset $\xi_t\mathcal{P}$ with
  - objects are monomorphisms in **RelPos** $x : [n]_{\min} \to \mathcal{P}, \forall n \geq 0$.
  - morphisms are commutative diagrams of the form

$$\begin{tikzcd}
{[n]_{\min}} \arrow[rr] \arrow[dr, "x_n"] & & {[m]_{\min}} \arrow[dl, "x_m"] \\
& \mathcal{P} &
\end{tikzcd}$$

  - weak equivalences are commutative diagrams as above such that $x_n([n]) \xrightarrow{\sim} x_m([m]) \in \mathcal{W}(\mathcal{P})$ are weak equivalences in $\mathcal{P}$.
- the **initial subdivision** in **RelPos** of $\mathcal{P}$ is given by the relative poset $\xi_i(\mathcal{P})$ such that $\xi_i\mathcal{P} := \xi_t(\mathcal{P}^{op})^{op}$. More explicitly, $\xi_i(\mathcal{P})$ is described by
  - objects are monomorphisms $x : [n]_{\min} \to \mathcal{P}^{op}, \forall n \geq 0$.
  - morphisms are commutative diagrams of the form

$$\begin{tikzcd}
{[m]_{\min}} \arrow[rr] \arrow[dr, "x_m"] & & {[n]_{\min}} \arrow[dl, "x_n"] \\
& \mathcal{P}^{op} &
\end{tikzcd}$$

  - weak equivalences are commutative diagrams as above such that $x_n([n]) \xrightarrow{\sim} x_m([m]) \in \mathcal{W}(\mathcal{P})$ are weak equivalences in $\mathcal{P}$.

Note $\xi_t(\mathcal{P})$ and $\xi_i(\mathcal{P})$ come with two projection functors

$$\begin{aligned} \pi_t : \xi_t(\mathcal{P}) &\longrightarrow \mathcal{P} \\ (x_n \to x_m) &\longmapsto x_n([n]) \to x_m([m]) \\ \pi_i : \xi_t(\mathcal{P}) &\longrightarrow \mathcal{P} \\ (x_n \to x_m) &\longmapsto x_n([n]) \to x_m([m]) \end{aligned}$$

There are canonical extensions of ξ_t and ξ_i to endofunctors on **RelPos**.

$$\xi_t : \mathbf{RelPos} \longrightarrow \mathbf{RelPos}$$

- on objects $\xi_t(\mathcal{P})$
- on morphisms, for each $[n]_{\min} \to \mathcal{P}$ and $[m]_{\min} \to \mathcal{Q}$ there is a unique epi $[n]_{\min} \twoheadrightarrow [m]_{\min}$ such that

$$\begin{tikzcd} {[n]_{\min}} \arrow[r, two heads] \arrow[d, hook] & {[m]_{\min}} \arrow[d, hook] \\ \mathcal{P} \arrow[r, "f"] & \mathcal{Q} \end{tikzcd}$$

commutes.

The extension $\xi_i : \mathbf{RelPos} \to \mathbf{RelPos}$ is defined similarly.

Definition 7.9. The two-fold subdivision of $\mathcal{P}$ is defined by $\xi\mathcal{P} \coloneqq \xi_i\xi_t\mathcal{P}$.

Remark. $\xi\mathcal{P}^{\mathrm{conj}} = \xi_t\xi_i\mathcal{P} \neq \xi\mathcal{P}$, what would happen to the main theorem in this case?

Example 7.10. Let $\mathcal{P} = [2]_{\min} = \{0 \to 1 \to 2\}$. Then

$$\xi_t([2]_{\min} = \left\{ \begin{tikzcd} & & 1 \arrow[dl] \arrow[d] \arrow[dr] & & \\ & 01 \arrow[r] & 012 & 12 \arrow[l] & \\ 0 \arrow[ur] \arrow[urr] \arrow[rr] & & 02 \arrow[u] & & 2 \arrow[ll] \arrow[ull] \arrow[ul] \end{tikzcd} \right\}$$

and $\xi_i[2]_{\min}$ is given by the opposite diagram.

Definition 7.11. The **classification diagram** (Rezk) of a relative category $\mathcal{C}, \mathcal{W})$ is the simplicial space $\mathcal{N}\mathcal{C}_* \in \mathbf{sSet}^{\Delta^{op}}$ with

$$\begin{aligned} \mathcal{N}\mathcal{C}_* : \Delta^{op} &\longrightarrow \mathbf{sSet} \\ [n] &\longmapsto \mathcal{N}\mathcal{C}_n = \mathcal{N}_*(W(\mathcal{C}^{[n]})) \end{aligned}$$

where $\mathcal{N}_*$ is the usual simplicial nerve and $\mathcal{C}^{[n]} = \mathrm{Fun}([n], \mathcal{C})$ and $\mathcal{W}(\mathcal{C}^{[n]}) := We(\mathcal{C}^{[n]})$ the natural transformations with objectwise weak equivalences.

Exercise 7.12. Chcek (by decoding notations) $\mathcal{N}\mathcal{C}_n([m]) = \{[n]_{\min} \times [m]_{\max} \longrightarrow \mathcal{C}\}$.

Lemma 7.13. *The classification diagram functor $\mathcal{N}$ has a left adjoint*

$$\mathsf{K} : \mathbf{sSet}^{\Delta^{op}} \rightleftarrows \mathbf{RelCat} : \mathcal{N}$$

such that $\mathsf{K}(\Delta[n]^t_* \times \Delta[m]_*) = [n]_{\min} \times [m]_{\max}$*, where* $(-)^t$ *is the transpose embedding.*

In particular K takes values in **RelPos** $\subset$ **RelCat**. Hence we can define the two-subdivided functor

$$\mathsf{K}_\xi : \mathbf{sSet}^{\Delta^{op}} \xrightarrow{\mathsf{K}} \mathbf{RelPos} \xrightarrow{\xi = \xi_i \xi_t} \mathbf{RelPos} \hookrightarrow \mathbf{RelCat}$$

Proposition 7.14 ([BK], Section 3)**.** K_ξ *has a right adjoint denoted by*

$$\mathcal{N}_\xi : \mathbf{RelCat} \rightleftarrows \mathbf{sSet}^{\Delta^{op}} : \mathsf{K}_\xi \tag{7.1}$$ {{eq:rel-adj}}

Note there is a natural transformation $\pi : \mathcal{N} \Rightarrow \mathcal{N}_\xi$ induced by $\pi = \pi_t \pi_i : \xi \Rightarrow \mathrm{Id}$.

Recall $\mathbf{sSet}^{\Delta^{op}}$ carry the Reedy model structure with weak equivalences being levelwise weak equivalences of simplicial sets.

Theorem 7.15 (Main Theorem, [BK], Theorem 6.1)**.** *The adjunction 7.1 induces the BK model structure on* **RelCat** *with*

- *weak equivalences are maps* $f : \mathcal{C} \to \mathcal{D}$ *such that* $\mathcal{N}_\xi(f)$ *are Reedy equivalences in* $\mathbf{sSet}^{\Delta^{op}}$.
- *Fibrations are maps* $f : \mathcal{C} \to \mathcal{D}$ *such that* $\mathcal{N}_\xi(f)$ *are Reedy fibrations in* $\mathbf{sSet}^{\Delta^{op}}$.
- *cofibrant objects are relative posets.*

7.2 2-Category

Assumptions (on relative categories): small

- $(\mathcal{C}, \mathcal{W})$ is saturated: in the formal localization $\gamma : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$, $\gamma^{-1}(\text{Iso}) = \mathcal{W}$. For instance, if $(\mathcal{C}, \mathcal{W})$ comes from a model category, then $\mathcal{W}$ is saturated.
- $(\mathcal{C}, \mathcal{W})$ is closed under finite coproducts. That is, $\mathcal{C}$ has an initial object $\emptyset$ and both $\mathcal{C}$ and $\mathcal{W}$ are closed under finite coproducts.

Notation If $\mathcal{C}$ is a category and $(\mathcal{D}, \mathcal{W})$ is a relative category, then $\mathcal{D}^{\mathcal{C}} = \text{Fun}(\mathcal{C}, \mathcal{D})$ has obvious relative structure with $\mathcal{W}^{\mathcal{C}} \subset \mathcal{D}^{\mathcal{C}}$ defined pointwise.

Definition 7.16. A **relative natural transformation** $\tau : F \Rightarrow G$ between two functors $F, G : (\mathcal{C}, \mathcal{W}_{\mathcal{C}}) \to (\mathcal{D}, \mathcal{W}_{\mathcal{D}})$ is a morphism of in $\mathcal{D}^{\mathcal{C}}[(\mathcal{W}^{\mathcal{D}})^{-1}] := \text{Fun}(\mathcal{C}, \mathcal{D})[(\mathcal{W}^{\mathcal{D}})^{-1}]$. More explicitly, τ is represented by zigzags of natural transformations with reversed arrows being weak equivalences.

Proposition 7.17. **RelCat** *with relative functors and relative natural transformations is a 2-category.*

Definition 7.18. A bicategory is a tuple $(\mathcal{B}, 2, c, a, l, r)$ consisting of the following data:

- Objects (0-cells) are $B_0 = \text{Ob}(\mathcal{B})$
- Hom categories: for any $X, Y \in \text{Ob}(\mathcal{B})$, $B(X, Y)$ is a category with
  - objects (1-cells) $B_1 = \text{Ob}(B(X, Y))$, and
  - morphisms (2-cells) $B_2 = \text{Ob}(B(X, Y))$
  - $\circ$ and Id in $B(X, Y)$ are **vertical compositions** and identity 2-cells.
  - isomorphisms are invertible 2-cells.
- Identity 1-cells: for any $X \in B_0$, $1_X : * \to B(X, X)$ is a functor called the identity cell of X.
- Horizontal compositions: for any $X, Y, Z \in B_0$, $c_{X,Y,Z} : B(Y, Z) \times B(X, Y) \to B(X, Z)$.
- Associations: for any $W, X, Y, Z \in B_0$,

  $$a_{W,X,Y,Z} : c_{W,Y,Z}(c_{X,Y,Z} \times \text{Id}_{B(W,X)}) \xrightarrow{\sim} c_{W,Y,Z}(\text{Id}_{B(Y,Z)} \times c_{W,X,Y}).$$

  a natural isomorphism of functors $B(Y, Z) \times B(X, Y) \times B(W, X) \to B(W, Z)$.

Example 7.19.
1. usual categories are discrete bicategories.
2. monoidal categories are one object bicategories.
3. bimodules: B_0 is the set of rings and B_1 are $R-S$-bimodules and B_2 are bimodule morphisms.

Definition 7.20. A 2-category is a bicategory $(B, 1, c, a, l, r)$ where a, l, r are identity natural transformations.

$$\begin{tikzcd} \mathbf{BiCat} \arrow[r, shift left, "\text{strict}"] & 2-\mathbf{Cat} \arrow[l, hook, shift left] \\ \mathbf{MonCat} \arrow[u, hook, "\mathtt{one\backslash object}"] \arrow[r, shift left, "\text{strict}"] & \mathbf{StrictMonCat} \arrow[u, hook] \arrow[l, hook, shift left] \end{tikzcd}$$

In **RelCat**m horizontal compositions are conpositions of relative functors.

Definition 7.21.
- A **relative isomorphism** in **RelCat** is an invertible 2-cell, i.e. a relative natural transformation $\tau : F \dashrightarrow G$ which is an isomorphism in $\mathrm{Fun}(F, G)[W^{-1}]$.
- An equivalence of relative categories is a relative functor $F : (\mathcal{C}, \mathcal{W}_{\mathcal{C}}) \Rightarrow (\mathcal{D}, \mathcal{W}_{\mathcal{D}})$ such that there is a relative functor $G : (\mathcal{D}, \mathcal{W}) \to (\mathcal{C}, \mathcal{W})$ with relative isomorphisms

$$\tau : F \circ G \xrightarrow{\sim} \mathrm{Id}_{\mathcal{D}}$$

$$\rho : G \circ F \xrightarrow{\sim} \mathrm{Id}_{\mathcal{C}}.$$

Remark. Barwick-Kan (Chapter 33) define relative homotopy equivalence which is a relative equivalence, where ρ and τ are represented by zigzags with all arrows being natural isomorphisms in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})[\mathcal{W}^{-1}]$.

Lemma 7.22. *The localization is a 2-functor*

$$\mathrm{loc} : \mathbf{RelCat} \longrightarrow \mathbf{Cat}$$

$$(\mathcal{C}, \mathcal{W}) \longmapsto \mathcal{C}[\mathcal{W}^{-1}]$$

Lemma 7.23. *For any small* $I \in \mathbf{Cat}$*, the exponentiation functor is a 2-functor:*

$$(-)^I : \mathbf{RelCat} \to \mathbf{RelCat}$$

$$(\mathcal{C}, \mathcal{W}) \longmapsto (\mathcal{C}^I, \mathcal{W}^I)$$

Definition 7.24. A **relative adjunction** consists of

- relative functors $F : (\mathcal{C}, \mathcal{W}) \to (\mathcal{D}, \mathcal{W})$. and $G : (\mathcal{D}, \mathcal{W}) \to (\mathcal{C}, \mathcal{W})$

- relative natural transformations

$$\alpha : F \circ G \dashrightarrow \mathrm{Id}_{\mathcal{D}}$$

$$\beta : \mathrm{Id}_{\mathcal{C}} \dashrightarrow G \circ F$$

such that the triangle identities

$$F \overset{F\beta}{\dashrightarrow} FGF \overset{\alpha F}{\dashrightarrow} F$$

$$G \overset{\beta G}{\dashrightarrow} GFG \overset{G\alpha}{\dashrightarrow} G$$

are identity 1-cells.

Example 7.25. 1. Usual adjoint functors are relative adjunctions

$$F : (\mathcal{C}, \mathrm{Iso}) \rightleftarrows (\mathcal{D}, \mathrm{Iso}) : G$$

2. More generally, any relative adjoint functors for a relative adjunction.

3. A Quillen adjunction between two cofibrantly generated model categories

$$F : \mathcal{M} \rightleftarrows \mathcal{N} : G$$

with functorial fibrant and cofibrant replacements $Q : \mathcal{M} \to \mathcal{M}$ and $R : \mathcal{N} \to \mathcal{N}$, then

$$F \circ Q : (\mathcal{M}, \mathcal{W}) \rightleftarrows (\mathcal{N}, \mathcal{W}) : G \circ R$$

is a genuine relative adjunction.

4. (Conjecturally) Model approximation of Quillen Adjunctions

Example 7.26 (Thomason subdivision)**.** $\mathcal{N}^{op} : \mathbf{Cat}^{op} \rightleftarrows \mathbf{sSet}^{op} : c^{op}$ is a relative adjunction with respect to Thomason subdivision.

Example 7.27. Let G be an affine algebraic group (scheme) over a field k, e.g. $G = GL_n$. There is an adjunction

$$(-)_G : \mathbf{Gr} \rightleftarrows \mathbf{Comm}_k : G$$

$$\Gamma \longmapsto \mathcal{O}\left[\mathrm{Rep}_G(\Gamma)\right]$$

where $\mathcal{O}\left[\mathrm{Rep}_G(\Gamma)\right]$ is the affine scheme of G-representations of Γ $\rho_A : \Gamma \to G(A), \forall A \in \mathbf{Comm}_k$. This adjunction extends to an adjunction in simplicial setting

$$(-)_G : \mathbf{sGr} \rightleftarrows \mathbf{sComm}_k : G$$

which however is not a Quillen pair. Nevertheless, it behaves like a Quillen pair.

1. $\mathbb{L}(-)_G$ and $\mathbb{R}G$ exists.
2. $\mathbb{L}(-)_G : \mathbf{sGr} \rightleftarrows \mathbf{sComm}_k : \mathbb{R}G$) is an adjunction between homotopy categories.
3. $\mathbb{L}(-)_G$ commutes with homotopy colimits.

Conjecture 7.28. $\mathbb{L}(-)_G$ *is a relative Quillen adjunction.*

It then follows $\mathbb{L}(-)_G$ commutes with realizable colimits.

Lemma 7.29.
- *If*
$$F : (\mathcal{C}, \mathcal{W}) \rightleftarrows (\mathcal{D}, \mathcal{W}) : G$$
is a relative adjunction, then
$$F : \mathcal{C}\left[\mathcal{W}^{-1}\right] \rightleftarrows \mathcal{D}\left[\mathcal{W}^{-1}\right] : G$$
is a classical adjunction.
- *For any small category* I*, if*
$$F : (\mathcal{C}, \mathcal{W}) \rightleftarrows (\mathcal{D}, \mathcal{W}) : G$$
is a relative adjunction, then
$$F^I : (\mathcal{C}^I, \mathcal{W}^I) \rightleftarrows (\mathcal{D}^I, \mathcal{W}^I) : G^I$$
is a relative adjunction.

Definition 7.30 (Rodügues-Gonzalez)**. Realizable homotopy colimit** is a relative colimit.

Given $I \in \mathbf{Cat}$, and a relative category $(\mathcal{C}, \mathcal{W})$, consider the constant diagram functor
$$c_I : \mathcal{C} \longrightarrow \mathcal{C}^I$$
a realizable hocolimit is a relative adjunction

Model approximation Chacholski-Schrer (Dwyer) Homotopy theory of diagrams:
If $(\mathcal{C}, \mathcal{W})$ is a relative category, we define (left) model approximation of $(\mathcal{C}, \mathcal{W})$ to be an adjunction
$$l : \mathcal{M} \rightleftarrows \mathcal{C} : r$$
where M is a model category.

1. r is a relative functor $r(\mathcal{W}_{\mathcal{C}}) \subseteq \mathcal{W}_{\mathcal{M}}$.
2. l is a relative functor on $\mathcal{W}_{\mathcal{M}}|_{\mathcal{M}^{cof}}$.
3. for any cofibrant $X \in \mathrm{Ob}(\mathcal{M}^{cof})$ and any $A \in \mathrm{Ob}(\mathcal{C})$, $f : X \to G(A)$ is a weak equivalence if and only if $f^{\#} : F(X) \to A$ is a weak equivalence.

Lemma 7.31. *There is a derived adjunction*

$$\mathbb{L}l : \mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{C}) : \bar{r}$$

and $\bar{r} : \mathrm{Ho}(\mathcal{C}) \hookrightarrow \mathrm{Ho}(\mathcal{M})$ *is fully faithful.*

Example 7.32 (CS)**.** Let $\mathcal{M}$ be a(ny) model category and I a small category. $\mathcal{M}^{I}$ is not a model category but a relative category with pointwise weak equivalences.

Theorem 7.33. *There is a natural left model approximation*

$$l : \mathcal{M}_{b}^{\Delta I} \rightleftarrows \mathcal{M}^{I} : r$$

where $\Delta I h \downarrow \mathcal{N}_{*} I$ *and* $\mathcal{M}_{b}^{\Delta I} \subset \mathcal{M}^{\Delta I}$ *subcategory of bounded diagrams (i.e.*$F : \Delta I \to \mathcal{M}$ *maps degeneracy maps* s^{j} *to weak equivalences in* $\mathcal{M}$*.*

Question 7.34. *What left model approximation of an adjoint pair of functors*

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

where $\mathcal{C}, \mathcal{D}$ *are relative functor but functors are not.*

Definition 7.35 (B-Ramadoss)**.** A **left model approximation** of (F, G) is a Quillen pair $\widehat{F} : \mathcal{M} \rightleftarrows \mathcal{N} : \widehat{G}$ such that

$$\begin{tikzcd} \mathcal{M} \arrow[r, shift left, "l"] \arrow[d, shift left, "\widehat{F}"] & \mathcal{C} \arrow[l, shift left, "r"] \arrow[d, shift left, "F"] \\ \mathcal{N} \arrow[r, shift left, "p"] \arrow[u, shift left, "\widehat{G}"] & \mathcal{D} \arrow[l, shift left, "q"] \arrow[u, shift left, "G"] \end{tikzcd}$$

1. $l : \mathcal{M} \rightleftarrows \mathcal{C} : r$ is a left model approximation.
2. $p : \mathcal{N} \rightleftarrows \mathcal{D} : q$ is a left model approximation.
3. $p \circ \widehat{F} \overset{\sim}{\dashrightarrow} F \circ l$ is a relative isomorphism of functors such that "left-looking" arrows are natural weak equivalences on $\mathcal{M}^{cof}$, i.e. $\mathrm{Im}(\mathbb{R}G \circ \overline{q}) \subseteq \mathrm{Im}(\bar{r}$.

Theorem 7.36. *If* $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ *is an adjoint pair between two relative categories that admits a left model approximation* $\widehat{F} : \mathcal{M} \rightleftarrows \mathcal{N} : \widehat{G}$ *in the above sense. Then both* F *and* G *have total left (right) derived functors and they are adjoint*

$$\mathbb{L}F : \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D}) : \mathbb{R}G$$

given by

$$\begin{aligned} \mathbb{L}F &\cong \mathbb{L}p \circ \mathbb{L}\widehat{F} \circ \overline{r} \\ \mathbb{R}G &\cong \mathbb{L}l \circ \mathbb{R}\widehat{G} \circ \overline{q} \end{aligned}$$

Corollary 7.37. $\mathbb{L}F$ *commutes with (small) colimits.*

Question 7.38. *In this setting,* (F, G) *defines a relative adjunction.*

Example 7.39. Special case: $(-)_G : \mathbf{sGr} \rightleftarrows \mathbf{sComm} : G$.

8 Stable Homotopy Theory

8.1 Motivation Results

S. Schwede "Spectra in model categories"

Theorem 8.1. *"Stabilization = Abelianization" Theorem (over* $\mathbb{Q}$*)*

Theorem 8.2. $\Sigma^\infty_+ \cong \mathbb{L}(-)_{ab}$ *derived cotangent complex.*

Let $\mathcal{C}$ be a proper (simplicial) model category, for instance $\mathcal{C} = \mathbf{sComm}_k$.
Define $\mathcal{C}//B$ to be the augmented category over B with

- objects are $B \overset{s}{\hookrightarrow} X \xrightarrow{r} B$.
- morphisms are $f : X \to Y$ such that the following diagram commutes.

$$\begin{tikzcd} B \arrow[r, hook, "s"] \arrow[d, equal] & X \arrow[r, "r"] \arrow[d, "f"] & B \arrow[d, equal] \\ B \arrow[r, hook, "s'"] & Y \arrow[r, "r'"] & B \end{tikzcd}$$

Lemma 8.3. $\mathcal{C}//B$ *inherits naturally a model structure from* $\mathcal{C}$.

Note $\mathcal{C}//B$ is pointed with $\emptyset = * = \{B \xrightarrow{\mathrm{Id}} B \xrightarrow{\mathrm{Id}} B\}$. So $(\mathcal{C}//B)^\infty$ is the category of spectra over $\mathcal{C}//B$ ($\simeq \mathbf{Top}_*$).

There is a Quillen pair

$$\begin{aligned} \Sigma^\infty : \mathcal{C}//B &\to (\mathcal{C}//b)^\infty : \Omega^\infty \\ X_0 &\longleftarrow \{X_n, \sigma_n\} \\ X &\longmapsto \Sigma^\infty X = \{X, \mathrm{Id}\} \end{aligned} \tag{8.1}$$

{{eq:stab}}

Factorization Category Fix $f : A \to B$ in $\mathcal{C}$, define the category $A \backslash \mathcal{C}/B)_f$ with

- objects are $(A \xrightarrow{\alpha} P \xrightarrow{\beta} B)$ such that $f = \beta \circ \alpha$.

Note when $f = \mathrm{Id}_B$, this is $\mathcal{C}//B$.

Lemma 8.4 (Schwede). *$(A \backslash \mathcal{C}/B)_f$ is a simplicial category.*

There is a natural pair of adjoint functors

$$\begin{aligned} f^* = (-)_+ : (A \backslash \mathcal{C}/B)_f &\longrightarrow \mathcal{C}//B : f_* \\ (A \xrightarrow{sf} X \xrightarrow{r} B) &\longleftarrow (B \xrightarrow{s} X \xrightarrow{r} B) \\ ((A \xrightarrow{\alpha} X \xrightarrow{\beta} B) &\longmapsto (B \hookrightarrow B \textstyle\bigcup_A P \xrightarrow{(\mathrm{Id},\beta)} B) \end{aligned} \tag{8.2}$$ {{eq:f}}

Combining (8.1) and (8.2) define the Quillen pair

$$\Sigma^\infty_+ : (A \backslash \mathcal{C}/B)_f \underset{f_*}{\overset{(-)_+}{\rightleftarrows}} (\mathcal{C}//B) \underset{\Omega^\infty}{\overset{\Sigma_\infty}{\rightleftarrows}} (\mathcal{C}//B)^\infty \tag{8.3}$$ {{eq:stab-adj}}

and we get an adjunction between homotopy categories

$$\mathbb{L}\Sigma^\infty_+ : \mathrm{Ho}((A \backslash \mathcal{C}/B)_f) \rightleftarrows \mathrm{Ho}((\mathcal{C}//B)^\infty) : \mathbb{R}f_{*,0}.$$

Definition 8.5. The suspension spectrum of unreduced suspension of A over B is given by

$$\Sigma^\infty_+(A \xrightarrow{f} B) := \mathbb{L}\Sigma^\infty_+(A \xrightarrow{f} B \xrightarrow{\mathrm{Id}} B).$$

Suppose $\mathcal{C} = \mathbf{sComm}_k$ where $k \supseteq \mathbb{Q}$.

Theorem 8.6. *For any $B \in \mathrm{Ob}(\mathbf{sComm}_k)$ fixed, the composition*

$$B - \mathbf{s}\,\mathrm{Mod}_k \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} (B - \mathbf{s}\,\mathrm{Mod}_k)^\infty \underset{i^\infty}{\overset{\mathrm{Sym}^\infty_B}{\rightleftarrows}} \mathbf{sComm}_k//B)^\infty$$

is a Quillen equivalence.

Hence

$$\mathrm{Ho}(B - \mathbf{s}\,\mathrm{Mod}_k) \underset{\mathbb{R}\Omega^\infty}{\overset{\mathbb{L}\Sigma^\infty}{\rightleftarrows}} \mathrm{Ho}((B - \mathbf{s}\,\mathrm{Mod}_k))^\infty \underset{\mathbb{R}i^\infty}{\overset{\mathbb{L}\mathrm{Sym}^\infty_B}{\rightleftarrows}} \mathrm{Ho}(\mathbf{sComm}_k//B)^\infty)$$

are equivalences of homotopy categories.

Theorem 8.7. *Given a map* $f : A \to B$ *in* $\mathbf{sComm}_k$*, the following diagram commutes.*

$$\begin{tikzcd}
(A\backslash \mathbf{sComm}_k/B)_f \arrow[r, "(-)_+"] \arrow[dr, "\mathrm{ab}_{A/B}"'] & \mathbf{sComm}_k//B \arrow[r, "\Sigma^\infty"] \arrow[d, "\mathrm{ab}_B = (B\backslash(-)/B)_{ab}"] & (\mathbf{sComm}_k//B)^\infty \arrow[d, "\mathrm{ab}_B^\infty"] \\
 & B - \mathbf{s}\,\mathrm{Mod}_k \arrow[r, "\Sigma^\infty"'] & (B - \mathbf{s}\,\mathrm{Mod}_k)^\infty
\end{tikzcd}$$

The Quillen derived cotangent functor

$$\mathbb{L}\mathrm{ab}_{A/B} : \mathrm{Ho}(A\backslash \mathbf{sComm}_k/B) \longrightarrow \mathrm{Ho}(B - \mathbf{s}\,\mathrm{Mod}_k)$$

is isomorphic to $\mathbb{L}\Sigma^\infty_+(A \xrightarrow{f} B)$.

Recall Quillen abelianization (or Quillen homology, 1970)

$$\mathrm{ab} : \mathcal{C} \rightleftarrows \mathcal{C}_{ab} : i$$

where $\mathcal{C}_{ab}$ is the category of abelian group objects in a model category $\mathcal{C}$. Then $\mathbb{L}(-)_{ab} : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C}_{ab})$ gives Quillen homology.

Example 8.8. Consider

$$\Omega(-/B) : \mathbf{Alg}_k \downarrow B \rightleftarrows (\mathbf{Alg}_k \downarrow B)_{ab} \cong B - \mathbf{Bimod} : B \ltimes (-)$$

$$B \ltimes M \longleftarrow\!\!\shortmid\; M$$

$$(A \xrightarrow{f} B) \longmapsto \Omega^1_{Kahl}(A) \otimes_A B$$

then $\mathbb{L}\Omega(-/B)(A \xrightarrow{f} B) = \Omega^1_{Kahl}(P) \otimes_P B$ where P is a cofibrant replacement of A.

Question 8.9. *What happens to* $A \neq S$ *Theorem if* $k \not\supseteq \mathbb{Q}$.

S. Schwede Stable homotopy of Algebraic theories.
Feng, Galatius, Venkatesh

8.2 Γ-Spaces and Homotopy-theoretic generalization of Quillen homology

Γ-Spaces (G. Segal) are simplest models for connective spectra.
Survey [DGM].

Definition 8.10. Let Γ^{op} be the skeleton of the category of finite pointed sets.

- objects are $n_+ = \{0, 1, \cdots, n\}_{n \geq 0}$ with basepoint 0.
- morphisms are based morphisms $f : n_+ \to m_+$ such that $f(0) = 0$.

A Γ-space is a pointed functor $X : \Gamma^{op} \to \mathbf{sSet}_*$ such that $X(0_+) = \Delta[0]_*$.

The idea is to replace simplicial abelian groups

Fix an algebraic theory $\mathcal{T}$ and let $\mathcal{A}_{\mathcal{T}}$ be the category of $\mathcal{T}$-algebras. $\mathbf{s}\mathcal{A}_{\mathcal{T}}$ is the category of (strict) simplicial $\mathcal{T}$-algebras/

Quillen's abelianization (restricted to $\mathcal{A}_{\mathcal{T}}$) is

$$(\mathcal{A}_{\mathcal{T}})_{ab} \cong \mathrm{Mod}(\mathcal{T}^{ab})$$

Theorem 8.11 (Schwede)**.** *For any algebraic theory, there exists a Gamma-ring T^s such that $\mathcal{A}_{\mathcal{T}}^{\infty} = \mathrm{Sp}_{\mathcal{T}} \simeq \mathrm{Mod}(T^s)$.*

Γ-spaces as (simple) model for spectra

1. Quillen
2. THH

algebraic theories

(Schwede's) stabilization theorem v.s. (Quillen) abelianization (linearization) -theorem

The category $A\backslash\backslash\mathbf{sComm}_k//A$ of augmented algebras $\{A \to B \to A\}$ has a model structure with initial and terminal object $A \to A \to A$, thus a stable model category, and by Quillen's abelianization get a functor $A\backslash\backslash\mathbf{sComm}_k//A \to \mathbf{s}\,\mathrm{Mod}_k - A$.

$\Sigma^{\infty} : A\backslash\backslash\mathbf{sComm}_k//A \hookrightarrow \mathbf{Sp}_0(\mathbf{sComm}_k)$.

References

[B] Bernard Badzioch, *Algebraic theories in homotopy theory*, Annals of Mathematics **155** (2002), 895–913.

[BK] C Barwick and DM Kan, *Relative categories: Another model for the homotopy theory of homotopy theories, arxiv. org math*, AT, November (2010).

[DGM] Bjørn Ian Dundas, Thomas G Goodwillie, and Randy McCarthy, *The local structure of algebraic k-theory*, Vol. 18, Springer Science & Business Media, 2012.

[DK] Tobias Dyckerhoff and Mikhail Kapranov, *Crossed simplicial groups and structured surfaces*, Stacks and categories in geometry, topology, and algebra **643** (2015), 37–110.

[FH] Emmanuel D Farjoun and Kathryn Hess, *Normal and conormal maps in homotopy theory*, Homology, Homotopy and Applications **14** (2012), no. 1, 79–112.

[G] Beatriz Rodriguez Gonzalez, *Realizable homotopy colimits*, arXiv preprint arXiv:1104.0646 (2011).

[H] Philip S Hirschhorn, *Model categories and their localizations*, American Mathematical Soc., 2003.

[J] John F Jardine, *Categorical homotopy theory*, Homology, Homotopy and Applications **8** (2006), no. 1, 71–144.

[M] J Peter May, *Simplicial objects in algebraic topology*, Vol. 11, University of Chicago Press, 1992.

[P] Matan Prezma, *Homotopy normal maps*, Algebraic & Geometric Topology **12** (2012), no. 2, 1211–1238.