MATH6510 Algebraic Topology

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Contents

Information

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Textbooks

- Allen Hatcher, Algebraic Topology
- Steven Weintraub, Foundations of Algebraic Topology

Part I
Basic Notions

1 Homotopy and Homotopy type

If X is a space, and A is a subspace, (X, A) is called a ***pair***. For a map $f : X \to Y$, write $f : (X, A) \to (Y, B)$ if $f(A) \subset B$.

In particular, $(X, \emptyset)$ can be identified with X.

f is a homeomorphism if both f and f^{-1} are continuous. Write $X \cong Y$, or $(X, A) \cong (Y, B)$ if $A \cong B$.

The product $(X, A) \times (Y, B) := (X \times Y, (X \times B) \cup (A \times Y))$.

Example 1.1. The product $S^1 \times S^1 = T^1$.

Figure 1: Torus

Definition 1.2. Two maps $f_0, f_1 : X \to Y$ are ***homotopic*** if there exists a ***homotopy*** (a family of) $f_t : X \to Y, t \in I$, such that the associated map $F : X \times I \to Y$ given by $F(x, t) = f_t(x)$ is continuous and $F(x, 0) = f_0(x)$, $F(x, 1) = f_1(x)$ for any $x \in X$. Write $f_0 \sim f_1$.

If $f_t(A) \subseteq B$ for any $t \in I$, write $f_0, f_1 : (X, A) \to (Y, B)$ and $F : (X, A) \times I \to (Y, B)$.

Lemma 1.3. *" $\sim$ " is an equivalence relation on maps $(X, A) \to (Y, B)$.*

Definition 1.4. f_0, f_1 are homotopic relative to A if there exists a homotopy between them which fixes A pointwise, i.e.

$$F(a, t) = f_0(a) = f_1(a), \forall a \in A, t \in I.$$

Write $f_0 \sim_A f_1$.

Definition 1.5. A map $f : (X, A) \to (Y, B)$ is called a ***homotopy equivalence*** if there is a map $g : (Y, B) \to (X, A)$ such that $fg \simeq \mathbf{1}_{(Y,B)}$ and $gf \simeq \mathbf{1}_{(X,A)}$. The spaces (X, A) and (Y, B) are said to be ***homotopy equivalent*** or to have the same ***homotopy type***. Write $(X, A) \simeq (Y, B)$.

Definition 1.6. A ***deformation retraction*** of X onto a subspace A is a homotopy from the identity map of X to a retraction of X onto A, a map $g : X \to X$ such that $g(X) = A$ and $g|_A = \mathbf{1}_A$, i.e.

$$g \simeq \mathbf{1}_X,\ g(X) = A,\ g|_A = \mathbf{1}_A.$$

Additionally, if $g \sim_A \mathbf{1}_X$, it is called a ***strong deformation retraction***.

Example 1.7. $\mathbb{S}^{n-1}$ is a strong deformation retraction of $X = \mathbb{R}^n \backslash \{0\}$ via

$$\begin{array}{cccc} F: & X \times I & \to & X \\ & (x, t) & \mapsto & x \cdot \frac{\|x\|^t}{\|x\|} \end{array},$$

here $F(x, 0) = \frac{x}{\|x\|} \in \mathbb{S}^{n-1}$, and $F(x, 1) = x$.

Definition 1.8. A space X having the homotopy type of a point is called ***contractible***, say $X \simeq *$. In other words, the identity map of the space is ***nullhomotopic***, that is, homotopic to a constant map.

Remark 1.9. In general, this is slightly weaker than saying the space deformation retracts to a point; see the exercise 6, Chap 0.

1.1 Cell Complexes

Definition 1.10. A space X constructed in the following way is called a ***cell complex*** or ***CW complex***:

1. Start with a discrete set X^0 , whose points are regarded as 0-***cells***.
2. Inductively, form the n-***skeleton*** X^n from X^{n-1} by attaching n-***cells*** e^n_α via maps $\varphi_\alpha : \mathbb{S}^{n-1} \to X^{n-1}$. This means that X^n is the quotient space of the disjoint union $X^{n-1} \coprod_\alpha D^n_\alpha$ of X^{n-1} with a collection of n-disks D^n_α under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$. Thus as a set, $X^n = X^{n-1} \coprod_\alpha e^n_\alpha$ where each e^n_α is an open n-disk.
3. One can either stop this inductive process at a finite stage, setting $X = X^n$ for some $n < \infty$, or one can continue indefinitely, setting $X = \cup_n X^n$. In the latter case X is given the weak topology: A set $A \subset X$ is open (or closed) if and only if $A \cap X^n$ is open (or closed) in X^n for each n.

Definition 1.11. A ***subcomplex*** of a cell complex X is a closed subspace $A \subset X$ that is a union of cells of X. A is a cell complex in its own right.

A pair (X, A) consisting of a cell complex X and a subcomplex A will be called a ***CW pair***.

1.2 Operations on spaces

Products. If X and Y are cell complexes, then $X \times Y$ has the structure of a cell complex with cells the products $e^m_\alpha \times e^n_\beta$ where e^m_α ranges over the cells of X and e^n_β ranges over the cells of Y.

Example. The cell structure on the torus $\mathbb{S}^1 \times \mathbb{S}^1$ is obtained in this way from the standard cell structure on $\mathbb{S}^1$.

Remark. The topology on $X \times Y$ as a cell complex is sometimes finer than the product topology, with more open sets than the product topology has, though the two topologies coincide if either X or Y has only finitely many cells, or if both X and Y have countably many cells.

Quotients. If (X, A) is a CW pair consisting of a cell complex X and a subcomplex A, then the quotient space X/A inherits a natural cell complex structure from X. The cells of X/A are the cells of $X - A$ plus one new 0-cell, the image of A in X/A. For a cell e^n_α of $X - A$ attached by $\varphi_\alpha : \mathbb{S}^{n-1} \to X^{n-1}$, the attaching map for the corresponding cell in X/A is the composition $\mathbb{S}^{n-1} \to X^{n-1} \to X^{n-1}/A^{n-1}$.

Example. Give $\mathbb{S}^{n-1}$ any cell structure and build D^n from $\mathbb{S}^{n-1}$ by attaching an n-cell, then the quotient $D^n/\mathbb{S}^{n-1}$ is $\mathbb{S}^n$ with its usual cell structure.

Suspension. For a space X , the suspension SX is the quotient of $X \times I$ obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point.

Example. $X = \mathbb{S}^n$, when $SX = \mathbb{S}^{n+1}$ with the two 'suspension points' at the north and south poles of $\mathbb{S}^{n+1}$, the points $(0, \cdots, 0, \pm 1)$.

Remark. A map $f : X \to Y$ suspends to $Sf : SX \to SY$, the quotient map of $f \times \mathbf{1} : X \times I \to Y \times I$.

Cones. For a space X , the cone $CX = (X \times I)/(X \times \{1\})$ is always contractible via

$$F: \begin{array}{ccc} CX \times I & \to & CX \\ ((x,s),t) & \mapsto & (x, \max\{s,t\}) \end{array}.$$

Here $F((x, s), 0) = (x, s)$ is the identity map, and $F((x, s), 1) = (x, 1)$ is the constant map.

Remark. If X is a CW complex, so are SX and CX as quotients of $X \times I$ with its product cell structure, I being given the standard cell structure of two 0-cells joined by a 1-cell.

Join. Given X and a second space Y the join $X * Y$ is the quotient space of $X \times Y \times I$ under the identifications $(x, y_1, 0) \sim (x, y_2, 0)$ and $(x_1, y, 1) \sim (x_2, y, 1)$.

Example. One simple example is $X = Y = I$, then we are collapsing two opposite faces of a cube onto line segments so that the cube becomes a tetrahedron.

Example. A very special case is when each X_i is just a point. For example, the join of two points is a line segment, the join of three points is a triangle, and the join of four points is a tetrahedron. In general, the join of n points is a convex polyhedron of dimension $n-1$ called a ***simplex***. Concretely, if the n points are the n standard basis vectors for $\mathbb{R}^n$, then their join is the $(n-1)$-dimensional simplex

$$\Delta_{n-1} = \{(t_1, \cdots, t_n) \in \mathbb{R}^n | t_1 + \cdots + t_n = 1, \text{ and } t_i \geq 0\}.$$

WedgeSum. Given spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$, then the wedge sum $X \vee Y$ is the quotient of the disjoint union $X \coprod Y$ obtained by identifying x_0 and y_0 to a single point.

Example. $\mathbb{S}^1 \vee \mathbb{S}^1$ is homeomorphic to the figure '8,' two circles touching at a point.

SmashProduct. The smash product $X \wedge Y$ is defined to be the quotient $X \times Y / X \vee Y$, where the wedge sum $X \vee Y$ is the union $(X \times \{y_0\}) \cup (\{x_0\} \times Y)$.

Remark. One can think of $X \wedge Y$ as a reduced version of $X \times Y$ obtained by collapsing away the parts that are not genuinely a product, the separate factors X and Y . The smash product X $\bigwedge$Y is a cell complex if X and Y are cell complexes with x_0 and y_0 0-cells, assuming that we give $X \times Y$ the cell-complex topology rather than the product topology in cases when these two topologies differ.

Example. $\mathbb{S}^m \wedge \mathbb{S}^n$ has a cell structure with just two cells, of dimensions 0 and $m+n$, hence $\mathbb{S}^m \wedge \mathbb{S}^n = \mathbb{S}^{m+n}$. In particular, when $m = n = 1$ we see that collapsing longitude and meridian circles of a torus to a point produces a 2-sphere.

Part II
The Fundamental Group

2 Paths and Homotopy

Definition 2.1. A ***path*** in a space X is a continuous map $f : I \to X$ where I is the unit interval $[0, 1]$.

Definition 2.2. A ***homotopy*** of paths in X is a family $f_t : I \to X, 0 \leq t \leq 1$, such that

1. The endpoints $f_t(0) = x_0$ and $f_t(1) = x_1$ are independent of t.

2. The associated map $F : I \times I \to X$ defined by $F(s, t) = f_t(s)$ is continuous.

Proposition 2.3. *The relation of homotopy on paths with fixed endpoints in any space is an equivalence relation.*

In particular, if we restrict attention to paths $f : I \to X$ with the same starting and ending point $f(0) = f(1) = x_0 \in X$. Such paths are called ***loops***, and the common starting and ending point x_0 is referred to as the ***basepoint***.

Definition 2.4. The set of all homotopy classes $[f]$ of loops $f : (I, \{0, 1\}) \to (X, x_0)$ (or equivalently $f : (\mathbb{S}^1, 1) \to (X, x_0)$) at the basepoint x_0 is denoted $\pi_1(X, x_0)$.

Proposition 2.5. . *$\pi_1(X, x_0)$ is a group with respect to the product $[f][g] = [fg]$. This group is called the **fundamental group** of X at the basepoint x_0.*

For a path f from x_0 to x_1, the i***nverse path*** f from x_1 back to x_0 is defined by $f(s) = f(1 - s)$. f is homotopic to a constant path.

We can define a ***change-of-basepoint*** map

$$\begin{array}{rccc} \Phi : & \pi_1(X, x_1) & \to & \pi_1(X, x_0) \\ & [f] & \mapsto & [\varphi \cdot f \cdot \bar{\varphi}] \end{array}$$

This is well-defined since if f_t is a homotopy of loops based at x_1 then $\varphi \cdot f_t \cdot \bar{\varphi}$ is a homotopy of loops based at x_0.

Proposition 2.6. *The map $\Phi : \pi_1(X, x_1) \to \pi_1(X, x_0)$ is an isomorphism.*

Remark 2.7. If $x_0 = x_1$, then φ represents an element in $\pi_1(X, x_0)$, and Φ is an inner automorphism. In fact, all the automorphisms arise this way.

$f : X \to Y$ with $f(x_0) = y_0$ induces

$$\begin{array}{rccc} f_* : & \pi_1(X, x_0) & \to & \pi_1(Y, y_0) \\ & [\gamma] & \mapsto & [f \circ \gamma] \end{array},$$

f_* is well-defined.

Theorem 2.8. *If $f : X \to Y$ with $f(x_0) = y_0$ is a homotopy equivalence, then the induced map f_* is an isomorphism.*

Now consider the spheres, we know that $\mathbb{S}^n$ is not contractible for any $n \geq 0$, but a surprising fact is that the infinite dimensional sphere $\mathbb{S}^\infty$ is contractible.

Lemma 2.9. *$\mathbb{S}^\infty$ is contractible.*

Proof. Define $f_t : \mathbb{R}^\infty \to \mathbb{R}^\infty, t \in I$ by $f_t(x_1, \cdots, x_n, \cdots) = (1-t)(x_1, \cdots, x_n, \cdots) + t(0, x_1, \cdots, x_n, \cdots)$. This maps non-zero vectors to non-zero vectors for any $t \in I$, so $\frac{f_t}{\|f_t\|}$ is a homotopy between $\mathrm{id}_{\mathbb{S}^\infty}$ and

$$f_1 : \begin{array}{ccc} \mathbb{S}^n & \to & \mathbb{S}^n \\ (x_1, \cdots, x_n, \cdots) & \mapsto & (0, x_1, \cdots, x_n, \cdots) \end{array}.$$

Define $g_t : \mathbb{R}^\infty \to \mathbb{R}^\infty, t \in I$ by $g_t(x_1, \cdots, x_n, \cdots) = (1-t)(0, x_1, \cdots, x_n, \cdots) + t(1, 0, \cdots, 0, \cdots)$. This also maps non-zero vectors to non-zero vectors for any $t \in I$, so $\frac{g_t}{\|g_t\|}$ is a homotopy between f_1 and a constant map.

Composing $\frac{f_t}{\|f_t\|}$ and $\frac{g_t}{\|g_t\|}$ gives a homotopy between $\mathrm{id}_{\mathbb{S}^\infty}$ and a constant map. □

Remark 2.10. We can regard $\mathbb{S}^\infty = \cup_{n \geq 0} \mathbb{S}^n$ via

$$\begin{array}{ccccccccc} \mathbb{S}^0 & \hookrightarrow & \mathbb{S}^1 & \hookrightarrow & \cdots & \hookrightarrow & \mathbb{S}^n & \hookrightarrow & \cdots \\ x & \mapsto & (x, 0) & & & & & & \end{array}$$

and the important property we are using is that every n-sphere is contractible when embedded in $(n+1)$-sphere.

Theorem 2.11. *If $\varphi : X \to Y$ is a homotopy equivalence with $\varphi(x_0) = y_0$, then $\forall x_0 \in X$, $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.*

To prove this theorem, we need the following lemma.

Lemma 2.12. *If $\varphi_t : X \to Y, t \in I$ is a homotopy and $x_0 \in X$, h is the path $t \mapsto \varphi_t(x_0)$, then*

$$\begin{array}{ccc} & {\scriptstyle (\varphi_1)_*} \nearrow & \pi_1(Y, \varphi_1(x_0)) \\ \pi_1(X, x_0) & & \downarrow \beta_h \\ & {\scriptstyle (\varphi_0)_*} \searrow & \pi_1(Y, \varphi_0(x_0)) \end{array}$$

where $\beta_h : [g] \mapsto [h \cdot g \cdot \bar{h}]$.

Proof. Let $h_t(s) = h(ts)$ be a homotopy. Suppose $[f] \in \pi_1(X, x_0)$, then $h_t \circ (\varphi_t \circ f) \circ \bar{h}_t$ is a homotopy of loops rel $\varphi_0(x_0)$ carrying $\varphi_0 \circ f$ to $h \circ (\varphi_1 \circ f) \circ \bar{h}$, so $(\varphi_0)_*([f]) = \beta_h((\varphi_1)_*([f]))$, i.e. the above diagram commutes. □

Now we can use this lemma to prove the above theorem.

Proof. **(Theorem 2.11.)** Let $\psi : Y \to X$ be a homotopy inverse of φ.

$$\pi_1(X, x_0) \xrightarrow[(1)]{\varphi_*} \pi_1(Y, \varphi(x_0)) \xrightarrow[(2)]{\psi_*} \pi_1(X, \psi\varphi(x_0)) \xrightarrow{\varphi_*} \pi_1(Y, \varphi\psi\varphi(x_0))$$

Since $\psi\varphi \simeq \mathrm{id}_X$, $\varphi_* \cdot \psi_* = (\psi\varphi)_* = \beta_h$ for some h as defined in the lemma, which is an isomorphism. Similar for $\varphi\psi \simeq \mathrm{id}_Y$, we know $\psi_* \cdot \varphi_*$ is an isomorphism. So φ_* is an isomorphism. □

3 Covering Spaces

Definition 3.1. Given a space X, a ***covering space*** of X consists of a space Y and a map $p : Y \to X$ satisfying that for each point $x \in X$ there is an open neighborhood U of x in X such that $p^{-1}(U)$ is a union of disjoint open sets each of which is mapped homeomorphically onto U by p. Such a U will be called ***evenly covered***. p is called a ***covering projection*** or ***covering map***.

Example 3.2. The most trivial case is when $Y = X$ and $p = \mathrm{id}_X$.

1. D is a discrete set,

$$p : \begin{array}{ccc} X \times D & \to & X \\ (x, d) & \mapsto & x \end{array}$$

2. $\mathbb{R}$ is a covering space of $\mathbb{S}^1$:

$$\begin{array}{rccc} p: & \mathbb{R} & \to & \mathbb{S}^1 \cong \mathbb{Z}\backslash\mathbb{R} \\ & t & \mapsto & e^{2\pi i t} \end{array}$$

3. $\mathbb{S}^1$ is also a covering space of $\mathbb{S}^1$:

$$\begin{array}{rccc} p: & \mathbb{S}^1 & \mapsto & \mathbb{S}^1 \cong C_n\backslash\mathbb{S}^n \\ & z & \mapsto & z^n \end{array}$$

   where C_n is the set of n-th roots of 1.

4. H is a discrete subgroup of a topological group G, then $p: G \to G/H$ is a covering map from G to the quotient space of left cosets.

5. Y is a topological space, G is a group action on Y that is ***properly disjointly***, i.e. for any $y \in Y$, there exists a neighborhood U of y such that for any $g \in G$, if $gU \cap U \neq \emptyset$,then $g = 1$. The $p: Y \to G\backslash Y$ is a covering map, where $G\backslash Y$ is the space of orbits with quotient topology.

   Proof. Let U be such a neighborhood, then GU is a neighborhood of Gy. GU is a disjoint union of gU for $g \in G$, and every $gU \xrightarrow{p} U$ is a homeomorphism. □

Lemma 3.3. *Here are some properties of the covering map p:*

1. *For any $x \in X$, $p^{-1}(x)$ is discrete subset of Y.*

2. *p is a local homeomorphism.*

3. *The topology on X is the quotient topology on Y from p.*

Proof. The disjointness of the preimages of p for an open neighborhood of x ensures the first property.

Need to show that p is an open map. For any open $V \subseteq Y$, for any $x \in p(V)$, there exists an open neighborhood U of x in X such that $p^{-1}(U)$ is a union of disjoint open sets each of which is mapped homeomorphically onto U by p, choose one V_i such that $V \cap V_i \neq \emptyset$, then

$$p|_{V_i} : V_i \to U$$

is a homeomorphism, so $p(V_i \cap V) = U \cap p(V)$ is open in U, therefore an open neighborhood of x. □

3.1 Lifting

Theorem 3.4. *(Unique Path Lifting)*

Let $p: Y \to X$ be a covering map, $x_0 \in X$, $y_0 = p(x_0) \in Y$. $f: I \to X$ is a path in X with $f(0) = x_0$. Then there exists a unique path lifting $\tilde{f}: I \to Y$ starting at y_0.

$$\begin{array}{ccc} & & Y \\ & \overset{\tilde{f}}{\nearrow} & \downarrow p \\ I & \underset{f}{\to} & X \end{array}$$

This a special case of the Unique Homopoty Lifting.

Theorem 3.5. *(Unique Homopoty Lifting)*

Let $p: Y \to X$ be a covering map, E be a space. $F: E \times I \to X$ is a map. Suppose we have $\tilde{f}: E \times \{0\} \to Y$ such that $p \circ \tilde{f}(e, 0) = f(e, 0), e \in E$, then $\tilde{f}$ extends uniquely to $\tilde{F}: E \times I \to Y$ such that $p \circ \tilde{F} = f$.

Proof. First construct a lift $\tilde{F} : U \times I \to Y$ for U some neighborhood in Y of a given point $e \in E$.

Since F is continuous, every point $(e, t) \in E \times I$ has a product neighborhood (U_t, I_t) such that $F(U_t \times I_t)$ is contained in an evenly covered neighborhood of $F(e, t) \in X$. By compactness of $e \times I$, finitely many such products (U_t, I_t) cover $e \times I$. Let U be the intersection of such finitely many U_t, and it follows that there exists a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that for each i, $F(U \times [t_i, t_{i+1}])$ is contained in an evenly covered neighborhood U_i. By induction, assume $\tilde{F}$ has been constructed on $U \times [0, t_i]$, then we have $F(U \times [t_i, t_{i+1}]) \subset U_i$, and $V_i \subset Y$such that $p|_{V_i} : V_i \to U_i$ is a diffeomorphism and $\tilde{F}(e, t_i) \in V_i$. After replacing U by a smaller neighborhood of e we may assume that $\tilde{F}(U \times \{t_i\})$ is contained in V_i, namely, replace $u \times \{t_i\}$ by its intersection with $(\tilde{F}|_{U \times \{t_i\}})^{-1}(V_i)$. (Why need to shrink?) Now we can define $\tilde{F}$ on $U \times [t_i, t_{i+1}]$ to be the composition of F with the homeomorphism $p|_{V_i}^{-1} : U_i \to V_i$. After a finite number of steps we eventually get a lift $\tilde{F} : U \times I \to Y$ for some neighborhood U of e.

Next we show the uniqueness in the special case that E is a point. In this case we can omit E from the notation. The proof is based on the "disjointness" of covering map.

Suppose $\tilde{F}$ and $\tilde{F}\prime$ are two lifts of $F : I \to X$ such that $\tilde{F}(0) = \tilde{F}\prime(0)$. As before, choose a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of I so that for each i, $F([t_i, t_{i+1}])$ is contained in some evenly covered neighborhood U_i. Assume inductively that $\tilde{F} = \tilde{F}\prime$ on $[0, t_i]$. Since $[t_i, t_{i+1}]$ is connected, so is $\tilde{F}([t_i, t_{i+1}])$, which must therefore lie in a single one of the disjoint open sets V_i projecting homeomorphically to U_i. By the same token, $\tilde{F}\prime([t_i, t_{i+1}])$ lies in a single V_i, in fact in the same one that contains $\tilde{F}([t_i, t_{i+1}])$ since $\tilde{F}\prime(t_i) = \tilde{F}(t_i)$. Because p is injective on V_i and $p \circ \tilde{F} = p \circ \tilde{F}\prime$, it follows that $\tilde{F} = \tilde{F}\prime$ on $[t_i, t_{i+1}]$, and the induction step is finished.

The last step is to observe that since the $\tilde{F}$'s constructed above on sets of the form $U \times I$ are unique when restricted to each segment $\{e\} \times I$, they must agree whenever two such sets $U \times I$ overlap. So we obtain a well-defined lift $\tilde{F}$ on all of $E \times I$. This $\tilde{F}$ is continuous since it is continuous on each $U \times I$. And $\tilde{F}$ is unique since it is unique on each segment $\{e\} \times I$. □

Definition 3.6. A space X is simply connected if X is path connected and has the trivial fundamental group $\pi_1(X, x_0) = 0$.

Proposition 3.7. *A space X is simply-connected iff there is a unique homotopy class of paths connecting any two points in X.*

Theorem 3.8. *Suppose Y is simply connected, G acts on Y properly disjointly. Let $X = G\backslash Y$, then for any $x_0 \in X$, $\pi_1(X, x_0) = G$.*

Proof. Let $p : Y \to X$ be the quotient map. p is a covering map. Take $y_0 \in Y$ with $p(y_0) = x_0$, then $g \mapsto gy_0$ is a bijection $G \to p^{-}(x_0)$ with $e \mapsto y_0$. For each $g \in G$, pick $\tilde{f}_g : I \to Y$ from y_0 to gy_0, and let $f_g = p \circ \tilde{f}_g$, then $[f_g] \in \pi_1(X, x_0)$.

Claim. $\varphi : G \to \pi_1(X, x_0), g \mapsto [f_g]$ is an isomorphism.

Proof. Well-defined. Suppose $\tilde{f}'_g : I \to Y$ is also a path from y_0 to gy_0,, then $\tilde{f}_g \simeq \tilde{f}'_g$ rel $\{0, 1\}$ because Y is simply connected. So composing with p gives that $f_g \simeq f'_g$ rel $\{0, 1\}$, hence $[f_g] = [f'_g]$.

Surjectivity. Let $[f] \in \pi_1(X, x_0)$, $f : (I, \{0, 1\}) \to (X, x_0)$ be a representative. By path lifting property, there exists a unique $\tilde{f} : (I, 0) \to (Y, y_0)$ with $p\tilde{f}(1) = x_0$, then $\tilde{f}(1) \in p^{-1}(x_0)$, so $\tilde{f}(1) = gy_0$ for some $g \in G$.

Injectivity. Suppose $f_g = p \circ \tilde{f}_g$, $[f_g] = 1 \in \pi_1(X, x_0)$, there exist $F : I \times I \to X$ such that $F(s, 0) = f_g(s), F(s, 1) = x_0, \forall s \in I$ and $F(0, t) = F(1, t) = x_0, \forall t \in I$. There exists a uniqe lifting $\tilde{F} : I \times I \to Y$, since $F(0, t) : I \to X$ and $F(1, t) : I \to X$ and $F(s, 1) : I \to X$ are the trivial paths starting, and ending x_0 and $p^{-1}(x_0)$ is a discrete subset of Y, so their lifts are all constant maps (or by uniqueness), hence $\tilde{F}(1, 0) = \tilde{F}(1, 1) = \tilde{F}(0, 1) = \tilde{F}(0, 0) = y_0$, so $gy_0 = y_0$, which means $g = e$.

Homomorphism. Suppose $g_1, g_2 \in G$, $\varphi(g_1)\varphi(g_2) = [f_{g_1}][f_{g_2}] = [f_{g_1} f_{g_2}]$. $\tilde{f}_{g_1} \cdot g_1\tilde{f}_{g_2}$ is the lift of $f_{g_1} \cdot f_{g_2}$ starting at y_0. It ends at $g_1 g_2 y_0$, which is also the endpoint of the lift of $f_{g_1 g_2}$, by the uniqueness of lifting, $\varphi(g_1 g_2) = [f_{g_1 g_2}]$. □

□

Example 3.9. $\pi_1(\mathbb{S}^1, 1) = \mathbb{Z}$ because $\mathbb{S}^1 = \mathbb{Z}\backslash\mathbb{R}$.

Definition 3.10. A space E is ***locally*** P means for any $e \in E$, for any neighborhood U of e, there exists an open neighborhood $V \subset U$ with V enjoys property P.

Proposition 3.11. ***(The Lifting Criterion.)*** *Let $p : (Y, y_0) \to (X, x_0)$ be a covering map. Suppose E is a locally path-connected and path-connected space. Suppose $f : (E, e_0) \to (X, x_0)$. A lift $\tilde{f}$ of f exists if and only if $f_*(\pi_1(E, e_0)) \subset p_*(\pi_1(Y, y_0))$.*

Proof. The 'only if' statement is obvious since $f_* = p_*\tilde{f}_*$.

For the converse, let $e \in E$ and γ be any path from e_0 to e, then path $f\gamma$ in X starting at x_0 has a unique lift $\widetilde{f\gamma}$ starting at y_0. Define $\tilde{f}(e) = \widetilde{f\gamma}(1)$. To show this is well defined, independent of the choice of γ, let γ' be another path from e_0 to e, then $(f\gamma')(\overline{f\gamma})$ is a loop h_0 at x_0 with $[h_0] \in f_*(\pi_1(E, e_0)) \subset p_*(\pi_1(Y, y_0))$. So there exists a homotopy h_t of h_0 to a loop h_1 that lifts to a loop $\tilde{h}_1$ in Y based at y_0. Apply the covering homotopy property to h_t to get a lifting $\tilde{h}_t$. Since $\tilde{h}_1$ is a loop at y_0, so is $\tilde{h}_0$. By the uniqueness of lifted paths, the first half of $\tilde{h}_0$ is $\widetilde{f\gamma\prime}$ and the second half is $\widetilde{f\gamma}$ traversed backwards, with the common midpoint $\widetilde{f\gamma}(1) = \widetilde{f\gamma\prime}(1)$. This shows that $\tilde{f}$ is well-defined.

To check that $\tilde{f}$ is continuous, let $U \subset X$ be an open neighborhood of $f(e)$ having a lift $V \subset Y$ containing $\tilde{f}(e)$ such that $p : V \to U$ is a homeomorphism. Choose a path-connected open neighborhood W of e with $f(W) \subset U$. For paths from e_0 to points $e\prime \in W$ we can take a fixed path γ from e_0 to e followed by paths η in W from e to the points $e\prime$. Then the paths $(f\gamma) \cdot (f\eta)$ in X have lifts $(\widetilde{f\gamma})(\widetilde{f\eta})$ where $\widetilde{f\eta} = p^{-1}(f\eta)$ and $p^{-1} : U \to V$ is the inverse of $p : V \to U$. Thus $\tilde{f}(W) \subset V$ and $\tilde{f}|_V = p^{-1}f$, hence $\tilde{f}$ is continuous at e. ☐

Proposition 3.12. ***(Unique Lifting.)*** *If E is connected and $p : Y \to X$ is a covering map, the lifts $\tilde{f}$ and $\tilde{f}'$ of a map $f : E \to X$ agree at a single point of E, then $\tilde{f} = \tilde{f}'$.*

Corollary 3.13. *Y, Y' are path-connected and locally path-connected. Two covering maps*

$$\begin{array}{ccc} (Y, y_0) & \overset{f}{\underset{\sim}{\to}} & (Y', y_0') \\ & \searrow p \quad p' \swarrow & \\ & (X, x_0) & \end{array}$$

are equivalent if and only if $p_*(\pi_1(Y, y_0)) = p'_*(\pi_1(Y', y_0'))$.

Proof. "Only if" $p = p' \circ f$ and $p' = p \circ f^{-1}$.

"If" lifting criterion implies p lifts to some $\tilde{p} : (Y, y_0) \to (Y', y_0')$ with $p = p' \circ \tilde{p}$, and p' lifts to some $\tilde{p}' : (Y', y_0') \to (Y, y_0)$ with $p' = p \circ \tilde{p}'$. Then $p' \circ \tilde{p} \circ \tilde{p}' = p'$ and $p \circ \tilde{p}' \circ \tilde{p} = p$. It follows that $\tilde{p}\tilde{p}'$and $\tilde{p}'\tilde{p}$ are lifts of p' and p respectively, But they fixed y_0', y_0 respectively, so by unique lifting, they are both the identity maps. Take $f = \tilde{p}$. ☐

Lemma. *$p : (Y, y_0) \to (X, x_0)$ is a covering map, then p_* is one-to-one.*

Proof. An element of $\ker(p_*)$ is represented by a loop $\tilde{f_0 : (I, \{0, 1}\}) \to (Y, y_0)$ with a homotopy f_t from $f_0 = p\tilde{f}_0$ to f_1 =constant path at x_0. But by homotopy lifting, $\tilde{f}_0$ is homotopic to the constant path at y_0. ☐

Definition 3.14. A covering map $\tilde{p} : \tilde{X} \to X$ is a ***universal cover*** if for any cover $p : Y \to X$, $\tilde{p}$ lifts to a map $q : \tilde{X} \to Y$,

$$\begin{array}{ccc} \tilde{X} & \overset{q}{\to} & Y \\ & \tilde{p} \searrow & \downarrow p \\ & & X \end{array}$$

then q will also be a covering map.

Proposition 3.15. *If $\tilde{X}$ is a simply connected, locally path-connected for X, it is a universal cover.*

Proof. Use the lifting criterion. ☐

3.2 Deck Transformations and Group Actions

Definition 3.16. $p : Y \to X$ is a covering map, $f : Y \xrightarrow{\cong} Y$ is a ***deck transformation*** when

$$\begin{array}{ccc} Y & \xrightarrow{f} & Y \\ & \searrow p \quad p \swarrow & \\ & X & \end{array}$$

commutes.

Example 3.17. $p : \mathbb{R} \to \mathbb{S}^1, r \mapsto e^{2\pi i r}$, $f : \mathbb{R} \to \mathbb{R}, r \mapsto r + n$, $n \in \mathbb{Z}$ is an isodeck transformation.

Lemma 3.18. *Assume Y is connected. If f and f' are deck transformation and $f(y) = f'(y)$ for some $y \in Y$, then $f = f'$.*

Proof. By unique lifting. □

Theorem 3.19. *Assume Y is connected. The group G of deck transformation acts on Y properly discretely.*

Proof. Only need to show that for any $y \in Y$ there exists a neighborhood V of y with the property that if $gV \cap V \neq \emptyset$, then $g = 1$. Take V to be an open neighborhood of y with

$$\tilde{p}|_V : V \xrightarrow{\cong} U$$

where $U = p(V)$ as for the lift of a covering map.

Suppose $g \in G$ and $gV \cap V \neq \emptyset$, then $gv_1 = v_2$ for some $v_1, v_2 \in V$. Then $p(v_1) = p(v_2)$ since g is a deck transformation. Therefore $v_1 = v_2$ as $p : V \xrightarrow{\cong} U$, so $g = 1$ by the lemma. (check?) □

$p : Y \to X$ is a covering map and Y is path-connected. $x_0 \in X$. For $y \in p^{-1}(x_0)$, $f : (I, \{0, 1\}) \to (X, x_0)$. Let $\tilde{f}_y$be the unique lift of f to a path $(I, 0) \to (Y, y_0)$.

Definition 3.20. Y is ***regular*** (or ***normal***) if for any such f, either

1. $\forall y \in p^{-1}(x_0), \tilde{f}_y(1) = y$
2. $\forall y \in p^{-1}(x_0)$, $\tilde{f}_y(1) \neq y$.

This condition does not depend on x_0.

A space is ***semilocally simply connected*** if $\forall x \in X$, $\exists$ open neighborhood U of x such that $U \hookrightarrow X$ induces the trivial map $\pi_1(U, x) \to \pi_1(X, x)$.

Example 3.21. Examples of a space that is not semilocally simply connected"

1. The plane with all the rational points removed.
2. The shrinking wedge of circles, the subspace $X \subset \mathbb{R}^2$ consisting of the circles of radius $1/n$ centered at the point $(1/n, 0)$ for $n = 1, 2, \cdots$.

Theorem 3.22. *Suppose X is*

1. *connected,*
2. *locally path connected, and*
3. *semilocally simply connected*

Let $x_0 \in X$. Suppose $p : Y \to X$ is a connected cover. (So is path-connected, locally path-connected and semilocally simply connected.) Then

1. *for any $H \leq \pi_1(X, x_0)$, there exists a unique such Y (up to equivalent) and $y_0 \in p^{-1}(x_0)$ such that $p_*(\pi(Y, y_0)) = H$.*

2. *$p^{-1}(x_0)$ is in one-to-one correspondence with right coset of H in $\pi_1(X, x_0)$. So in particular, the degree of the cover (or number of sheets) is the index of H in $\pi_1(X, x_0)$.*

3. *$H \triangleleft \pi_1(X, x_0)$ if and only if Y is regular. In this event, the group of deck transformations is isomorphic to $\pi_1(X, x_0)/H$.*

For all $y_1, y_2 \in p^{-1}(x_0)$, there exists an only deck transformation mapping y_1 to y_2. (Thus the cover is maximally symmetric.)

Corollary 3.23. *Under that same hypotheses, every X has a simply connected cover $p : \tilde{X} \to X$, unique up to equivalence (the universal cover). X is the quotient of $\tilde{X}$ by the group of deck transformations. If Y covers X, then $\tilde{X}$ covers Y.*

If $H \leq K \leq \pi_1(X, x_0)$, then Y_H covers Y_K by the lifting criterion.

$$\begin{tikzcd} Y_H \arrow[rr, dashed] \arrow[dr] & & Y_K \arrow[dl] \\ & X & \end{tikzcd}$$

The idea of constructing $\tilde{X}$. $\tilde{X} = \{[\gamma] | \gamma$ a path in X starting at $x_0\}$. $\gamma_1 \sim \gamma_2$ when there exists a homotopy classes of paths, keeping the endpoints fixed. The cover is

$$\begin{array}{cccc} p: & \tilde{X} & \to & X \\ & [\gamma] & \mapsto & \gamma(1) \end{array}$$

the points $\tilde{x}$ in $\tilde{X}$correspond to unique homotopy classes of paths in $\tilde{X}$since $\pi_1(\tilde{X}, x_0) = 1$, but homotopy classes of path upstairs correspond to homotopy classes of paths in X by homotopy lifting.

$\mathcal{U} = \{$open path connected $U \subseteq X | U \hookrightarrow X$ induces trivial map $\pi_1(U) \to \pi_1(X)\}$ is a basis for the topology on X (follows from semilocally simple connected).

Let γ be a path in X from x_0 to a point in U, then U 'lifts' to

$$U_{[\gamma]} \coloneqq \{[\gamma \cdot \eta] | \eta \text{ a path in } U \text{ with } \eta(0) = \gamma(1)\}$$

These $U_{[\gamma]}$ form a a basis for a topology on $\tilde{X}$, with $p : U_{[\gamma]} \xrightarrow{\cong} U$.

Then given $H \leq \pi_1(X, x_0)$, take $Y = \tilde{X}/\sim$ where $[\gamma] \sim [\gamma']$ when $\gamma(1) = \gamma'(1)$ and $[\gamma \cdot \bar{\gamma}'] \in H$.

3.3 Van Kampen's Theorem

Theorem 3.24. *Suppose $X = X_1 \cup X_2$ and $x_0 \in X_1 \cap X_2$, $X_1, X_2, \emptyset \neq A = X_1 \cap X_2$ are all path connected and open. Then*

$$\pi_1(X, x_0) = \pi_1(X_1, x_0) *_{\pi_1(A, x_0)} \pi_1(X_2, x_0)$$

That is, if $i_1 : A \hookrightarrow X_1, i_2 : A \hookrightarrow X_2$ are the inclusions of subspaces, then $\pi_1(X, x_0)$ is the free product $\pi_1(X_1, x_0) * \pi_1(X_2, x_0)$ modulo the relations $(i_1)_*(\alpha) = (i_2)_*(\alpha), \forall \alpha \in \pi_1(A, x_0)$.

In category theory, this is a push out:

$$\begin{tikzcd} & \pi_1(X_1, x_0) \arrow[dr] & & \\ \pi_1(A, x_0) \arrow[ur] \arrow[dr] & & G \arrow[r, dashed] & K \\ & \pi_1(X, x_0) \arrow[ur] & & \end{tikzcd}$$

Special cases

1. X_1, X_2 are simply connected, then X is simply connected. For instance, $\mathbb{S}^n (n > 1)$ is simply connected.

2. A is simply connected, then $\pi_1(X, x_0) = \pi_1(X_1, x_0) * \pi_1(X_2, x_0)$. For instance, $R_n = \vee_n \mathbb{S}^1$, or the rose with n petals, then $\pi_1(R_n, 0) = F_n = F(\alpha_1, \cdots \alpha_n)$, the free group with n generators. Induction on n, $X_1 \cap X_2$ is the broken n petals, simply connected.

3. $\pi_1(X_2, x_0) = 1$, then $\pi_1(X, x_0) = \pi_1(X_1, x_0)/ < \pi_1(A, x_0) >$. For instance, the torus with a disc glued along a meridian. X_1 flattened torus, X_2 flattened disc, $X_1 \cap X_2 \simeq \mathbb{S}^1$ is path connected. So $\pi_1(X, x_0) = \mathbb{Z}^2 / < (0, 1) >= \mathbb{Z}$.

4. If $P =< A|R >$ is a group representation for Γ, then

$$X = ((\bigvee_A \mathbb{S}^1) \bigsqcup (\bigsqcup_R \mathbb{D}^2)) / \sim$$

has $\pi_1(X, x_0) = \Gamma$. For instance, $< a, b | a^{-1} b^{-1} ab >= \mathbb{Z}^2$.

Proof. When $|A|, |R| < \infty$, note that when $P =< A| >$, $X = R_{|A|}$, then attach the discs one-by-one, each time killing $< r >$ where r describes the attaching map. $\square$

3.4 Complex

Definition 3.25. A ***1-complex*** (or ***graph***) λ is $C(V, E)/ \sim$ where $V = \{v_i\}$ is a discrete set of points (vertices), and $E = \{E_j\}$ is a disjoint collection of intervals $[0, 1]$ with $0, 1 \in E_j$ identified with same v_p and v_q. It is a finite complex if $|V|, |E| < \infty$.

For instance, R_n is the 1-compex with 1 vertex and n edges. An u-gon is a 1-complex with n vertices and n edges.

Definition 3.26. A ***tree*** is a contractible 1-complex.

Definition 3.27. A subcomplex $(V', E')/ \sim$ of C is a 1-complex with $V' \subseteq V$ and $E' \subseteq E$.

Theorem 3.28. *Let C be a connected 1-compelx, v a vertex of C, T a maximal tree in C. Then $\pi_1(C, v)$ is free with generators α_e in $1-1$ correspondence with edges $e \in C \backslash T$.*

Proof. Shrink T to a vertex, $(C, T) \simeq (R_{C \backslash T}, r_0)$.
$\alpha_e = v$ to the initial vertex of e in T, then e, then terminal vertex of e to v in T. $\square$

Corollary 3.29. *If $H \leq F$ where F is a free group, then H is free.*

Proof. $F = \pi_1(R, r_0)$ for some R whose petals correspond to some free basis. $H = \pi_1(\tilde{R}, \tilde{r}_0)$ for some cover $\tilde{R}$ of R. Note that the cover of a 1-complex is a 1-complex. (***Why?***) So H is free by the previous theorem. $\square$

Corollary 3.30. *If $F = F_k$ and $H \leq F$ of index n, then $H \cong F_{(k-1)n+1}$.*

Proof. $F = \pi_1(R_k, r_0)$. $H = \pi_1(\tilde{R}_k, \tilde{r}_0)$ for some cover $\tilde{R}_k$ of R_k of degree n.
$\tilde{R}_k$ has n vertices, and kn edges. So a maximal tree T in $\tilde{R}_k$ has $n-1$ edges, so $\tilde{R}_k - T$ has $kn - (n-1)$ edges. $\square$

Let $\pi_1(X)$ be the homotopy classes of maps $\mathbb{S}^1 \to X$ with no base point.

Theorem 3.31. *If X is path connected, then $\Phi : \pi_1(X, x_0) \to \pi_1(X)$ 'forget the base point' induces a bijection conjugate classes in $\pi_1(X, x_0)$ to $\pi_1(X)$.*

Proof. 'Onto'. Let $f : \mathbb{S}^1 \to X$ be a loop, $y = f(1)$, constrcut a path $h : (I, \{0,1\}) \to (X, \{x_0, y\})$ between x_0 and y, then $hf\bar{h} \simeq f$, and $h \cdot f \cdot \bar{h} \in \pi_1(X, x_0)$.

'Well-defined'. Suppose $[g_0]$ and $[g_1]$ are conjugate in $\pi_1(X, x_0)$, we want to show that g_0 and g_1 are freely homotopic $[h \cdot g_0 \cdot \bar{h}] = [g_1]$ in $\pi_1(X, x_0)$ for some loop h based at x_0. It's enough to show that g_0 is free homotopic to $h \cdot g \cdot \bar{h}$. Consider the homotopy

$$\begin{array}{rccc} F: & I \times I & \to & X \\ & (s,t) & \mapsto & \begin{cases} h(st) & 0 \le s \le \frac{1}{3} \\ & \frac{1}{3} \le s \le \frac{2}{3} \\ & \frac{2}{3} \le s \le 1 \end{cases} \end{array}$$

'One to one'. Suppose $\Phi([g_0])$ and $\Phi([g_1])$ are freely homotopic. Then $h \cdot g_1 \cdot \bar{h}$ is freely homotopic to g_0, and there exists

$$F : \mathbb{S}^1 \times I \to X$$

Fixed $x_0 \in X$, let $H : I \times I \to X$ be the revised version of F such that $H(0,t) = H(1,t) = x_0$. □

The Hawaiian Earning

H is a subspace of $\mathbb{R}^2$. Let C_n be the radius $\frac{1}{n}$ circle contered at $(\frac{1}{n}, 0)$, then $H = \bigcup_{n=1}^{\infty} C_n$.

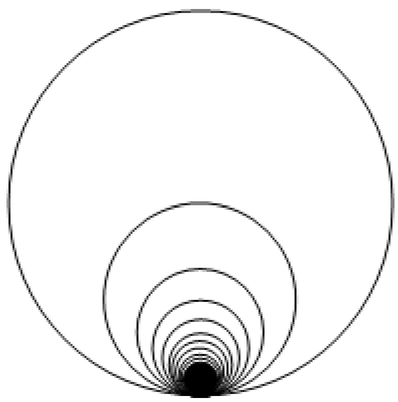

Figure 2: Hawaiian Earring

The Hawaiian earring can be given a complete metric and it is compact. It is path connected but not semilocally simply connected. The Hawaiian earring looks very similar to the wedge sum of countably infinitely many circles; that is, the rose with infinitely many petals, but those two spaces are not homeomorphic. The difference between their topologies is seen in the fact that, in the Hawaiian earring, every open neighborhood of the point of intersection of the circles contains all but finitely many of the circles. It is also seen in the fact that the wedge sum is not compact: the complement of the distinguished point is a union of open intervals; to those add a small open neighborhood of the distinguished point to get an open cover with no finite subcover.

Lemma 3.32. *$\pi_1(H, (0,0))$ is contractible.* ***(Check!)***

Proof. Define

$$\begin{array}{rccc} \rho: & \pi_1(H,(0,0)) & \to & \prod_{i=1}^{\infty} \mathbb{Z} \\ & f & \mapsto & (r_1 f, \cdots, r_n f, \cdots) \end{array}$$

where r_n is the retraction $H \twoheadrightarrow C_n$ which maps all $C_i (i \neq n)$ to $(0,0)$.

ρ is surjective.

Let f wrap k_n times around C_n in time $[1-\frac{1}{n}, 1-\frac{1}{n+1}]$, then $f \mapsto (k_1, \cdots, k_n \cdots)$ is continuous on $[0,l]$ for any $l < 1$, and continuous at 1 because the loops $C_n \to (0,0)$ as $n \to \infty$, this contrast with the fact $\bigvee_{\mathbb{N}} \mathbb{S}^1$ has $\pi_1 = F(a_1, \cdots, a_n, \cdots)$ which is countable.

Let $K_1 = K_2 = CH$, $K = K_1 \bigsqcup K_2/\sim$ where $*_1 \sim *_2$. CH is contractible, so simply connected. $K_1 \cap K_2 = *$ is simply connected (not open). □

Theorem 3.33. *K is not simply connected.*

Proof. Consider h goes around C_1 in K_1 and then C_1 in K_2, then C_2 in K_1, then C_2 in K_2, etc. f is not null-homotopic. □

Theorem 3.34. *K has no connected cover apart from itself. In particular, it has no simply connected cover.*

Part III
Homology

4 Intuition

We want to study:

1. (Chains) Start with formal sum of edges in the space X.
2. (Cycles) Consider only those with empty boundary.
3. (Boundaries) And only up to boundaries $\partial\sigma$.

We want to consider things like this:

$$\frac{\text{Abelian groups of cycles}}{\text{Abelian groups of boundaries}} := H_1(X, \mathbb{Z})$$

5 Homology

Let Δ_i be a i-cell. Define

$$C_i(X) = \{\text{free abelian group generated by maps } \Delta_i \to X\}$$

and cycles

$$Z_i(X) = \{\beta \in C_i(X) | \partial\beta = \emptyset\}$$

and boundaries

$$B_i(X) = \{\beta \in C_i(X) | \partial\sigma = \beta, \sigma \in C_{i+1}(X)\}$$

Formally, the definition is given by the Eilenberg-Steenrod axioms.

Definition 5.1. A ***homology theory*** associates $A \hookrightarrow X$ to every pair of topological spaces (X, A),

1. a sequence of abelian groups $H_i(X, A), i \in \mathbb{Z}$,
2. and a sequence of homomorphisms $\partial_i : H_i(X, A) \to H_{i-1}(A, \emptyset) = H_{i-1}(A), i \in \mathbb{Z}$,
3. and to each map $f : (X, A) \to (Y, B)$, the associated homomorphisms $f_i : H_i(X, A) \to H_i(Y, B)$ satisfy:
   (a) If $f : (X, A) \to (X, A)$ is the identity, then $f_i = \mathbf{1}$.
   (b) The associative law holds,

$$\begin{array}{ccccc} (X, A) & \xrightarrow{f} & (Y.B) & \xrightarrow{g} & (Z, C) \\ H_i(X, A) & \xrightarrow{f_i} & H_i(Y.B) & \xrightarrow{g_i} & H_i(Z, C) \end{array}$$

   i.e. $(g \circ f)_i = g_i \circ f_i$.
   (c) The following diagram commutes:

$$\begin{array}{ccc} H_i(X, A) & \xrightarrow{f_i} & H_i(Y, B) \\ \partial_i \downarrow & & \downarrow \partial_i \\ H_{i-1}(A) & \xrightarrow{(f|_A)_i} & H_{i-1}(B) \end{array}$$

   (d) There is an exact sequence

$$\begin{array}{ccccccccc} \cdots & \to & H_i(A) & \xrightarrow{\alpha_i} & H_i(X) & \to & H_i(X, A) & \xrightarrow{\partial_i} & H_{i-1}(A) & \to & \cdots \\ & & A & \hookrightarrow & X & & & & & & \\ & & & & (X, \emptyset) & \hookrightarrow & (X, A) & & & & \end{array}$$

(e) If $f : (X, A) \to (Y, B)$ and $g : (X, A) \to (Y, B)$ are homotopic, then $f_i = g_i$.

(f) Suppose $\bar{U} \subset A^o$, and $f : (X \backslash U, A \backslash U) \hookrightarrow (X, A)$, then $\forall i$, $f_i : H_i(X \backslash U, A \backslash U) \to H_i(X, A)$ is an isomorphism.

(g) If $X = \{\text{point}\}$, then $H_i(X) = 0, \forall i \geq 1$, and $H_0(X)$ could be anything, and is called the group of coefficients.

Definition 5.2. $(X \backslash U, A \backslash U) \hookrightarrow (X, A)$ is callled an ***excisive map***, and $A, B \subset X$ are an ***excisive couple*** if $(A, A \cap B) \hookrightarrow (A \cup B, B)$ is an excisive map.

For instance, $(X \backslash U, A)$ are an excisive couple.

Definition 5.3. H is compactly supported if for any $\alpha \in H_i(X, A)$, there exists $(X_0, A_0) \overset{j_\alpha}{\hookrightarrow} (X, A)$ where X_0, A_0 are compact, such that $\alpha = j_*(\alpha_0)$ for some $\alpha_0 \in H_i(X_0, A_0)$.

Lemma 5.4. *We know that*

1. $H_i(\emptyset) = 0$,
2. *If $f : (X, A) \to (Y, B)$ is a homotopy equivalence, then $f_i : H_i(X, A) \to H_i(Y, B)$ is an isomorphism.*
3. *If A is a retract of X, $j : A \hookrightarrow X$, $r : X \to A$, then for any i, $j_i : H_i(A) \to H_i(X)$ is injective, $r_i : H_i(X) \to H_i(A)$ is surjective, and $H_i(X) \cong H_i(A) \oplus H_i(X, A)$, here $H_i(X, A) = \ker r_i$.*
4. *If X_1, X_2 are unions of components of X, $j^1 : X_1 \to X$, $j^2 : X_2 \to X$, then*

$$j_i^1 + j_i^2 : H_i(X_1) \oplus H_i(X_2) \to H_i(X)$$

is an isomorphism.

Theorem 5.5. *If $f : (X, A) \to (Y, B)$ with $\begin{cases} f : X \to Y \\ f|_A : A \to B \end{cases}$ both be homotopy equivalences, then $\forall i$, $f_i : H_i(X, A) \to H_i(Y, B)$ is an isomorphism.*

Proof. We have the exact sequence

$$\begin{array}{ccccccccccc} \cdots \to & H_i(A) & \to & H_i(X) & \to & H_i(X, A) & \to & H_{i-1}(A) & \to & H_{i-1}(X) & \to \cdots \\ & \downarrow\cong & & \downarrow\cong & & \downarrow f_i & & \downarrow\cong & & \downarrow\cong & \\ \cdots \to & H_i(B) & \to & H_i(Y) & \to & H_i(Y, B) & \to & H_{i-1}(B) & \to & H_{i-1}(Y) & \to \cdots \end{array}$$

So f_i is an isomorphism by the following 'Five Lemma'. $\square$

Lemma 5.6. *(Five Lemma)*
If

$$\begin{array}{ccccccccc} A_1 & \to & A_2 & \to & A_3 & \to & A_4 & \to & A_5 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\ B_1 & \to & B_2 & \to & B_3 & \to & B_4 & \to & B_5 \end{array}$$

is a commutative diagram of exact sequences of abelian groups and f_1, f_2, f_4, f_5 are isomorphisms, then f_3 is a isomorphism.

Proof. f_3 is injective. Suppose $f_3(x) = 0$, then x maps to 0 in B_3, so x maps to 0 in B_4, so x maps to 0 in A_4 because f_4 is an isomophism. By exactness, some $z \in A_2$ maps to x. Let $w = f_2(z)$, then w maps to 0 in B_3, so some $v \in B_1$ maps to w, and let $u = f_1^{-1}(v)$, u maps to some $t \in B_2$, $f_2(t) = w$, so $t = z$. hence u maps to x, so $x = 0$.

Similar proof for Surjection. $\square$

Definition 5.7. X is a space, let $f : X \to *$ be a map to a single point. Define reduced homology $\tilde{H}_i(X) = \ker(f_i : H_i(X) \to H_i(*))$.

Theorem 5.8. *If $X \neq \emptyset$, $x_0 \in X$, then for any i,*

1. $\tilde{H}_i(X) \cong H_i(X, x_0)$, *and*
2. $H_i(X) \cong H_i(x_0) \oplus \tilde{H}_i(X)$.

Proof. Apply the lemma 4(3) to the retraction $X \twoheadrightarrow x_0$. □

Theorem 5.9. *Suppose $A \subseteq X$ is closed and nonempty, and ∂A has either*

1. *an open neighborhood C in A such that*

$$\begin{cases} X \backslash A^o \hookrightarrow (X \backslash A) \cup C \\ \partial A \hookrightarrow C \end{cases}$$

are both homotopy equivalence, or

2. *an open neighborhood C in $X \backslash A^o$ such that*

$$\begin{cases} A \hookrightarrow C \cup A \\ \partial A \hookrightarrow C \end{cases}$$

are both homotopy equivalence.

Then $(X \backslash A^o, \partial A) \to (X, A)$ is excisive.

Proof. Case 1:Let $V = A \backslash C$, then $(X \backslash A^o, \partial A) \to (X \backslash V, A \backslash V) \to (X, A)$. Th second map is excisive because V is closed in A and is in A^o.

The first map is an isomophism because the inclusions $X \backslash A^o \hookrightarrow X$ and $\partial A \hookrightarrow A \backslash V$ are both homotopy equivalence, hence excisive. So the composition is excisive. □

Theorem 5.10. *Suppose $A \subseteq X$ is nonempty, let $Y = X \cup_A CA = X \bigsqcup CA/\sim$ where A is identified with $A \times \{0\}$ in CA. Then for any i, $H_i(X, A) \simeq \tilde{H}_i(Y)$.*

Proof. Let $V := \{(a, s) | s \geq \frac{1}{2}, a \in A\} \subseteq CA$. $(Y \backslash V, CA \backslash V) \to (Y, A)$ is excisive by the axioms, and $(Y, *) \to (Y, CA)$ is excissive because the inclusion $(* \to CA)$ is a homotopy equivalence.

$(Y \backslash V, CA \backslash V) \to (X, A)$ is excisive because of strong deformation retract. So each mao induces an isomophism, so $H_i(X, A) \simeq \tilde{H}_i(Y)$. □

Theorem 5.11. *If $A \subseteq X$ is nonempty and closed and A is a strong deformation retract of some neighborhood W of A, then $H_i(X, A) \cong \tilde{H}_i(X/A)$.*

Proof. Consider

$$\begin{array}{rcll} H_i(X, A) & \cong & H_i(X, W) & (X, A) \hookrightarrow (X, W), A \hookrightarrow W \text{ is a homotopy equivalence} \\ & \cong & H_i(X - A, W - A) & \text{excision} \\ & \cong & H_i(X/A - A/A, W/A - A/A) & \text{homeomorphism of pairs} \\ & \cong & H_i(X/A, W/A) & \\ & \cong & H_i(X/A, A/A) & \text{excision} \\ & \cong & H_i(X/A, *) & \\ & \cong & \tilde{H}_i(A) & \end{array}$$

Why cannot get □

5.1 Mayer-Vietoris Theorem

Theorem 5.12. *Mayer-Vietoris (Version 1)*

Suppose $X = X_1 \cup X_2$, $A = X_1 \cap X_2$, $(X_1, A) \to (X, X_2)$ is excisive. Then there exists a long exact sequence

$$\cdots \to H_i(A) \to H_i(X_1) \oplus H_i(X_2) \to H_i(X) \to H_{i-1}(A) \to \cdots$$

Proof. We have the long exact sequence

$$\begin{array}{cccccccccccc} \cdots \to & H_{i+1}(X_1, A) & \xrightarrow{\partial_{1(i-1)}} & H_i(A) & \xrightarrow{\alpha_{1i}} & H_i(X_1) & \xrightarrow{\gamma_{1i}} & H_i(X_1, A) & \xrightarrow{\delta_{1i}} & H_{i-1}(A) & \to \cdots \\ & \cong\downarrow \varepsilon_{i+1} & & \downarrow \alpha_{2i} & & \downarrow \beta_{1i} & & \downarrow \varepsilon_i & & \downarrow \alpha_{i-1} & \\ \cdots \to & H_{i+1}(X, X_2) & \xrightarrow{\partial_{2(i-1)}} & H_i(X_2) & \xrightarrow{\beta_{2i}} & H(X) & \xrightarrow{\gamma_{2i}} & H_i(X, X_2) & \xrightarrow{\delta_{2i}} & H_{i-1}(X_2) & \to \cdots \end{array}$$

Prove that $\mathrm{Im}\alpha_i \subseteq \ker \beta_i, \mathrm{Im}\beta_i \subseteq \ker \Delta_i, \mathrm{Im}\Delta_i \subseteq \ker \alpha_i$, and the opposite inclusion by diagram chasing.

$\ker \beta_i \subseteq \mathrm{Im}\alpha_i$. Suppose $\beta_i(r, s) = \beta_{1i}(r) - \beta_{2i}(s) = 0$, let $u = \beta_{1i}(r) = \beta_{2i}(s)$, then $\gamma_{2i}(u) = 0$ by exactness, so $\varepsilon_i\gamma_{1i}(r) = 0$ by commutivity , and hence $\gamma_{1i}(r) = 0$ because ε_i is an isomorphism. So $r = \alpha_{1i}(q_0)$ for some $q_0 \in H_i(A)$. Let $s_0 = \alpha_{2i}(q_0)$, then $\beta_{2i}(s_0) = \beta_{1i}(r) = \beta_{2i}(s)$ by commuting square, so $\beta_{2i}(s - s_0) = 0$, $s - s_0 = \partial_{2i}(v)$ for some $v \in H_i(A)$. Then $\alpha_{2i}(x) = s - s_0$ by the commuting square. Set $q = q_0 + x$, then $\alpha_{1i}(q) = r$ and $\alpha_{2i}(q) = s$, so $\alpha_i(q) = (r, s)$. □

Theorem 5.13. *Mayer-Vietoris (Version 2)*

If $A, B \subseteq X$, $(A, A \cap B) \to (A \cup B, B)$ is excisive, then there exists a long exact sequence

$$\cdots \to H_i(X, A \cap B) \to H_i(X, A) \oplus H_i(X, B) \to H_i(X, A \cup B) \to H_{i-1}(X, A \cap B) \to \cdots$$

Proof. Same. □

Theorem 5.14. *Mayer-Vietoris (Version 3)*

Suppose $X = X_1 \cup X_2$, $A = X_1 \cap X_2$, $(X_1, A) \to (X, X_2)$ is excisive. Suppose $B \subseteq A$, then here exists a long exact sequence

$$\cdots \to H_i(A, B) \to H_i(X_1, B) \oplus H_i(X_2, B) \to H_i(X, B) \to H_{i-1}(A, B) \to \cdots$$

Proof. Same, but use the long exact sequence for a triple. □

Suspensions

X is a space, the suspension of X is $\Sigma X = X \times [-1, 1]/\sim$, where $X \times \{-1\} \sim c_-$ and $X \times \{1\} \sim c_+$. Then $\Sigma X = C_+X \bigsqcup C_-X$. $f : X \to Y$ induces $\Sigma f : \Sigma X \to \Sigma Y$.

Theorem 5.15. *For any i, there exists an isomorphism $\Sigma : \tilde{H}_{i+1}(\Sigma X) \to \tilde{H}_i(X)$. Moreover, Σ is functorial.*

$$\begin{array}{ccc} \tilde{H}_{i+1}(\Sigma X) & \xrightarrow{\Sigma} & \tilde{H}_i(X) \\ (\Sigma f)_{i+1} \downarrow & & \downarrow (\Sigma f)_i \\ \tilde{H}_{i+1}(\Sigma Y) & \xrightarrow{\Sigma} & \tilde{H}_i(Y) \end{array}$$

Proof. The long exact sequence for (C_+X, X) is

$$\cdots \to H_{i+1}(C_+X, X) \to H_i(X) \underset{\theta}{\overset{\phi}{\leftrightarrows}} H_i(C_+X) \xrightarrow{\psi} \cdots$$

Suppose $x_0 \in X$, the inclusion $x_0 \hookrightarrow X$ gives a map

$$\begin{array}{ccc} H_i(x_0) & \to & H_i(X) \\ \downarrow\cong & \nearrow \phi & \\ H_i(C_+X) & & \end{array}$$

which splits θ, i.e. $\theta\phi = \mathrm{id}_{H_i(C_+X)}$, so θ is surjective, so $\psi = 0$. So we get a short exact sequence

So $H_i(C + X, X) = \ker() = \tilde{H}_i(X)$, so by the theorem we start with, $H_{i+1}(C + X, X) \cong \tilde{H}_{i+1}(\Sigma X)$. □

Part IV
Cohomology

6 Eilenberg-Steenord Axioms for Cohomology

A ***cohomology theory*** associates $A \hookrightarrow X$ to every pair of topological spaces (X, A),

1. a sequence of abelian groups $H^i(X, A), i \in \mathbb{Z}$,
   (a) and a sequence of homomorphisms $\delta^i : H^i(A) \to H^{i+1}(X, A), i \in \mathbb{Z}$,
   (b) and to each map $f : (X, A) \to (Y, B)$, the associated homomorphisms $\{f^i : H^i(Y, B) \to H^i(X, A)\}_{i\in\mathbb{Z}}$ satisfy:
      i. If $f : (X, A) \to (X, A)$ is the identity, then $f^i = \mathrm{id}_{H^i(X,A)}, \forall i \in \mathbb{Z}$.
      ii. $(g \circ f)^i = f^i \circ g^i$. I.e. if we have $(X, A) \xrightarrow{f} (Y.B) \xrightarrow{g} (Z, C)$, then $H^i(Z, C) \xrightarrow{g^i} H^i(Y, B) \xrightarrow{f^i} H^i(X, A)$.
      iii. The following diagram commutes:

$$\begin{array}{ccc} H^{i+1}(X, A) & \xleftarrow{f^i} & H^{i+1}(Y, B) \\ \delta^i \uparrow & & \uparrow \delta^i \\ H^i(A) & \xrightarrow{(f|_A)^i} & H^i(B) \end{array}$$

      iv. There is a long exact sequence

$$\cdots \leftarrow H^i(A) \leftarrow H^i(X) \leftarrow H^i(X, A) \leftarrow H^{i-1}(A) \leftarrow \cdots$$

      v. If $f : (X, A) \to (Y, B)$ and $g : (X, A) \to (Y, B)$ are homotopic, then $f^i = g^i, \forall i \in \mathbb{Z}$.
      vi. Suppose $\bar{U} \subset A^o$, and $f : (X\backslash U, A\backslash U) \hookrightarrow (X, A)$ is the inclusion, then $\forall i$, $f^i : H^i(X, A) \to H^i(X\backslash U, A\backslash U)$ is an isomorphism.
      vii. If $X = \{\text{point}\}$, then $H^i(X) = 0, \forall i \neq 0$.

If $X \neq \emptyset$, $f : X \to *$, define the reduced cohomology $\tilde{H}^i(X) = \mathrm{Coker}(f^i : H^i(*) \to H^i(X))$.

Theorem 6.1. *(Mayer-Vietors Theorem in Cohomology) If $X = X_1 \cup X_2$, $A = X_1 \cap X_2$, and $(X_1, A) \to (X, X_2)$ is excisive, then there exists a long exact sequence*

$$\cdots \leftarrow H^i(A) \leftarrow H^i(X_1) \oplus H^i(X_2) \leftarrow H^i(X) \leftarrow H^{i-1}(A) \leftarrow \cdots$$

Part V
Homology Theory

7 Ordinary Homology Theory

We assume now that we have an ordinary homology theory, i.e., one that satisfies the dimension axiom, and we assume in addition that the coefficient group is the integers $\mathbb{Z}$.

7.1 Homology Groups of Spheres, and Some Classical Applications

Lemma 7.1. *The homology groups of 0-spheres are*

1. $H_0(S^0) \cong \mathbb{Z} \oplus \mathbb{Z}$. More precisely, $H_0(S^0) = \{mp + nq | m, n \in Z\}$.
2. $\tilde{H}_0\left(S^0\right) \cong \mathbb{Z}$. More precisely, $\tilde{H}_0(S^0) = \{n(q-p) | n \in Z\}$.
3. $\tilde{H}_i(S^0) = H_i(S^0) = 0$for $i \neq 0$.

Theorem 7.2. *(Invariance of Domain) If $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ are non-empty and open and $f : U \to V$ is a homomorphism, then $m = n$.*

Proof. For $U = \mathbb{R}^m$ and $V = \mathbb{R}^n$, take $y \in \mathbb{R}^n$ and $x = f^{-1}(y)$, then $f : (\mathbb{R}^m, \mathbb{R}^m - \{x\}) \to (\mathbb{R}^n, \mathbb{R}^n - \{y\})$ is a homeomoephism of pairs, so induces an homotopy (isomorphism), but the inclusion $\varphi : (D^k, \mathbb{S}^{k-1}) \hookrightarrow (\mathbb{R}^k, \mathbb{R}^k - \{x\})$ induces an isomorphism

$$H_i(\mathbb{R}^k, \mathbb{R}^k - \{x\}) \cong H_i(D^k, \mathbb{S}^{k-1}) = \begin{cases} \mathbb{Z} & i = k \\ 0 & \text{o.w.} \end{cases}$$

for every i because both $D^k \to \mathbb{R}^k$ and $\mathbb{S}^{k-1} \to \mathbb{R}^k - \{x\}$ are homotopy equivalence. Hence $m = n$.

In general, Take $x \in U, y \in V$ with $y = f(x)$. Let N be an open ball in V around y, let $M = f^{-1}(N) \subseteq U$. Let B be an open ball in M around x. Let $C = f(B)$. $f : (B.B - \{x\}) \to (C, C - \{y\})$ is a homoemorphism of pairs. Meanwhile,

$$H_i(B, B - \{x\}) = H_i(\mathbb{R}^m, \mathbb{R}^m - \{x\}), \forall i.$$

Note that $\overline{N - C} \subset (N - \{y\})^o$ in N, so $H_i(C, C - \{y\}) \cong H_i(N, N - \{y\})$ for any i by excision. But $H_i(N, N = \{y\}) \cong H_i(\mathbb{R}^n, \mathbb{R}^n - \{y\}), \forall i$, so $m = n$. ☐

Degree

$f : \mathbb{S}^n \to \mathbb{S}^n$ induces

$$\begin{array}{cccc} f_n : & \tilde{H}_n(S^n) & \to & \tilde{H}_n(S^n) \\ & \| & & \| \\ & \mathbb{Z} & & \mathbb{Z} \end{array}$$

This f_n must be $\alpha \mapsto d\alpha$ for some $d \in \mathbb{Z}$, which we call the ***degree*** $\deg(f)$ of f.

Properties

1. $\deg(\mathrm{Id}) = 1$.
2. If f is not onto, then $\deg(f) = 0$.

   Proof. If f is not onto, it factors through $\mathbb{S}^n \to \mathbb{S}^n - \{P\} \hookrightarrow \mathbb{S}^n$, which induces $\tilde{H}_n : \mathbb{Z} \to 0 \to \mathbb{Z}$ for some $P \in \mathbb{S}^n$, so f_n is a map $\mathbb{Z} \to \mathbb{Z}$ through 0. ☐

3. If $f \simeq g$, then $\deg(f) = \deg(g)$. (For $n > 0$, the converse is true. – Hopf 1925)

4. $\deg(fg) = \deg(f)\deg(g)$ by the composition axiom.

5. The degree of the antipodal map
$$\begin{array}{rccc} \alpha: & \mathbb{S}^n & \to & \mathbb{S}^n \\ & x & \mapsto & -x \end{array}$$
is $(-1)^{n+1}$.

Proof. Case $n = 2m-1$ is odd. View $\mathbb{S}^n$ as the unit sphere in $\mathbb{C}^m$, then $\alpha : (z_1, \cdots, z_m) \mapsto (-z_1, \cdots, -z_m)$ is homotopic to $\mathrm{Id}_{\mathbb{C}^m}$ via $f_t(z_1, \cdots, z_m) = e^{\pi i t}(z_1, \cdots, z_m)$, so $\deg(\alpha) = \deg(\mathrm{Id}_{\mathbb{C}^m}) = 1$.

Case $n = 0$. $\alpha : \{\pm 1\} \to \{\mp 1\}$, this induces the map $p \mapsto q, q \mapsto p$ on $H_0(\mathbb{S}^0)$. $\tilde{H}_0(\mathbb{S}^0) =< p - q >$. So α induces the map $p - q \mapsto q - p$ on $\tilde{H}_0(\mathbb{S}^0)$, So $\deg(\alpha) = -1$.

Case $n = 2m > 0$ is even. $\mathbb{S}^n$ is the unit sphere in $\mathbb{R} \times \mathbb{C}^m$. $\beta : (x, z_1, \cdots, z_m) \mapsto (-x, -z_1, \cdots, -z_m)$. Let $\beta' : (x, z_1, \cdots, z_m) \mapsto (-x, z_1, \cdots, z_m)$, then $\beta \simeq \beta'$ via $f_t(x, z_1, \cdots, z_m) = (-x, e^{\pi i t} z_1, \cdots, e^{\pi i t} z_m)$. But $\beta' = \sum^n \alpha$ where α is the antipodal map on $\mathbb{S}^0$. $f : \mathbb{S}^0 \to \mathbb{S}^0$, then
$$\begin{array}{rccc} \sum f : & \sum \mathbb{S}^0 & \to & \sum \mathbb{S}^0 \\ & (x, r) & \mapsto & (f(x), r) \end{array}$$

So we have
$$\begin{array}{cccccccccc} \cdots \leftarrow & \tilde{H}_n(\mathbb{S}^n) & \underset{\cong}{\overset{\Sigma}{\longleftarrow}} \cdots \underset{\cong}{\overset{\Sigma}{\longleftarrow}} & \tilde{H}_2(\mathbb{S}^2) & \underset{\cong}{\overset{\Sigma}{\longleftarrow}} & \tilde{H}_1(\mathbb{S}^1) & \underset{\cong}{\overset{\Sigma}{\longleftarrow}} & \tilde{H_0(\mathbb{S}^0)} \\ & \downarrow (\sum^n \alpha)_n & & \downarrow (\sum^2 \alpha)_2 & & \downarrow (\sum \alpha)_1 & & \downarrow \alpha_0 \\ \cdots \leftarrow & \tilde{H}_n(\mathbb{S}^n) & \underset{\cong}{\overset{\Sigma}{\longleftarrow}} \cdots \underset{\cong}{\overset{\Sigma}{\longleftarrow}} & \tilde{H}_2(\mathbb{S}^2) & \underset{\cong}{\overset{\Sigma}{\longleftarrow}} & \tilde{H}_1(\mathbb{S}^1) & \underset{\cong}{\overset{\Sigma}{\longleftarrow}} & \tilde{H_0(\mathbb{S}^0)} \end{array}$$

So $\deg(\beta) = \deg(\beta') = \deg(\alpha) = -1$. □

6. If $f : \mathbb{S}^n \to \mathbb{S}^n$ has no fixed points, then $\deg(f) = (-1)^{n+1}$.

Proof. $f \simeq \alpha$ via $f_t(x) = \frac{(1-t)f(x) - tx}{|(1-t)f(x) - tx|}$, the dominant is non-zero because f has no fixed points. □

7. The degree of a homeomorphism $\mathbb{S}^n \to \mathbb{S}^n$ is ± 1.

Theorem 7.3. *If n is even, C_2 is the only non-trivial group acting freely on $\mathbb{S}^n$.*

Proof. $C_2 = \{1, g\}$ acts on $\mathbb{S}^n$ freely. Suppose there is another group G acting on $\mathbb{S}^n$ freely, $\deg : G \mapsto \{\pm 1\}$ by property 7. If the action is free, every non-identity element maps to $(-1)^{n+1}$. So G is non-trivial only when n is even and G is C_2 in this case. □

Theorem 7.4. *(The Hairy Ball Theorem) $\mathbb{S}^n$ has a continuous field of nonzero tangent vertors if and only if n is odd.*

Proof. When n is odd, $v(x_1, \cdots, x_{2k}) = (-x_2, x_1, \cdots, -x_{2k}, x_{2k-1})$ is such a vector field.

For instance, $\mathbb{S}^3 = \{(x, y, z, t)\,|x^2 + y^2 + z^2 + t^2 = 1\}$. For fixed $t \in (-1, 1)$, $\{(x, y, z, t)\,|x^2 + y^2 + z^2 = 1 - t^2\}$ is a 2-sphere of radius $\sqrt{1 - t^2}$. For $t \in [-1, 0)$, this gives a D^3.

Suppose $x \mapsto v(x)$ is such a vector field. Replacing $v(x)$ by $\frac{v(x)}{|v(x)|}$ we may assume $|v(x)| = 1, \forall x$. Let $f_t(x) = (\cos t)\, x + (\sin t)\, v(x), 0 \le t \le \pi$. This is a homotopy between the identity map and the antipodal map α on $\mathbb{S}^n$. (Key point: $\forall t, \forall x \in \mathbb{S}^n$ $f_t(x) \in \mathbb{S}^n$.) But then $1 = \deg(\mathrm{Id}) = \deg(\alpha) = (-1)^{n+1}$, so n is odd. □

Example 7.5. Gluing two solid 1-torus $D^2 \times \mathbb{S}^1$ via idetity map on the boundary gives $\mathbb{S}^2 \times \mathbb{S}^1$.

7.2 CW-complex

Definition 7.6. X is obtained from A by adjoining an n-cell when there is a map $f : \mathbb{S}^{n-1} \to A$, called ***attaching map***, with $X = A \bigsqcup D^n / \sim$ where $f(p) \sim p$ for all $p \in \mathbb{S}^{n-1}$.

Remark 7.7. There exists an obvious map $D^n \to X$ restricting to f on $\mathbb{S}^{n-1} = \partial D^n$ and to a homoemorphism on $\text{int}(D)$.

Definition 7.8. A CW-structure on X is $X = \bigcup_{i=-1}^{\infty} X^i$ where $\emptyset = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq \cdots \subseteq X^n \subseteq \cdots$ are subspaces of X and each X^n is obtained from X^{n-1} by adjoining n-cells. That is,

$$X^n = \left(X^{n-1} \bigsqcup \left(\bigsqcup_{\lambda \in \Lambda_n} D^n_\lambda \right) \right) / \sim$$

where $p \sim f_\lambda(p)$ for some map $f_\lambda : \mathbb{S}^{n-1} \to X^{n-1}$. X^n is called an n-skeleton.

X has the weak topology with respect to $\{X^n\}$, i.e. $A \subseteq X$ is closed if and only if $A \cap X^n$ is closed, for any n.

The image of $\text{int}(D^n_\lambda)$ is an n-cell in X. X is the disjoint union of its cells. Say X is n-dimensional (or X is an n-***complex***) when $X = X^n$ but $X \neq X^{n-1}$.

Example 7.9. CW-complexes.

1. 0-complexes = discrete sets.
2. 1-compexes = graphs.
3. $\mathbb{S}^n = e^0 \cup e^n, \forall n \geq 1$. I.e. $\mathbb{S}^n = (* \bigsqcup D^n) / \sim$ where $p \sim f(p) = *, \forall p \in \partial D^n$.
4. $\mathbb{S}^0 \subseteq \mathbb{S}^1 \subseteq \mathbb{S}^2 \subseteq \cdots$ gives another CW-complex structure where $\mathbb{S}^n$ has 2 i-cells, $\forall 0 \leq i \leq n$.
5. Polytopes.
6. $\bigvee_{i \in \mathbb{Z}} \mathbb{S}^1$.

Example 7.10. Non-examples.

1. Hawaiin earring (Not locally contractible).
2. Cantor set (Compact CW-complexes have finitely many connected components).
3. Topologist's sine curve.

Lemma 7.11. *For a CW-complex X, it has the following two properties:*

1. ***Closure finiteness***: the closure of each cell in X intersects only finitely many other cells in X.
2. ***Weak topology***: $A \subseteq X$ is closed if and onlt if $A \cap \overline{e^i}$ is closed for any i-cell e^i in X.

Proof. First part. The closure of a cell in X is compact because it is the image of an n-disc. If it met infinitely many other cells (which are open), we could pick a point in each and get an infinite set in a compact set with no accumulation point. This is impossible. □

Lemma 7.12. *Let X be obtained from A by adjoining an n-cell. Then*

$$H_i(X, A) = \begin{cases} \mathbb{Z} & i = n \\ 0 & o.w. \end{cases}$$

Proof. Let $C = \{x \in D^n | |x| \geq 1/2\}$. Then A is a strong deformation retract of $A \cup C$. Thus $H_i(X, A) \cong H_i(D^n, C) \cong H_i(D^n, S^{n-1}) \cong \tilde{H}_i(D^n/S^{n-1}) \cong \tilde{H}_i(S^n)$. □

Lemma 7.13. *For the n-skeleton X^n, $n \geq 0$,*

1. $H_i(X^n, X^{n-1}) = 0$ *for* $i \neq n$.
2. *For each* $\lambda \in \Lambda_n$, $f_\lambda : \left(D^n_\lambda, S^{n-1}_\lambda\right) \to \left(X^n, X^{n-1}\right)$ *induces monomorphisms on homology, and furthermore* $H_n(X^n, X^{n-1}) = \bigoplus_{\lambda \in \Lambda_n} H_n(D^n_\lambda, S^{n-1}_\lambda)$.
3. $H_i(X^n) = 0$ *for* $i > n$.
4. *The inclusion* $X^{n-1} \to X^n$ *induces maps* $H_i(X^{n-1}) \to H_i(X^n)$ *that are isomorphisms except possibly for* $i = n-1, n$.
5. *There is an exact sequence* $0 \to H_n(X^n) \to H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}) \to H_{n-1}(X^n) \to 0$.

Proof. Let $X' = X^{n-1}$ with closed annuli attached. Then

$$H_i(X^n, X^{n-1}) \cong H_i(X^n, X') \cong H_i(X^n - X^{n-1}, X' - X^{n-1}) \cong \bigoplus_{\lambda \in \Lambda_n} H_i(D^n_\lambda, S^{n-1}_\lambda).$$

When $i \neq n$, $H_i(X^n, X^{n-1}) = 0$. When $i = n$, $H_n(X^n, X^{n-1}) = \bigoplus_{\lambda \in \Lambda_n} H_n(D^n_\lambda, S^{n-1}_\lambda)$.
The long exact sequence for $\left(X^n, X^{n-1}\right)$ is

$$\cdots \to \quad H_{i+1}\left(X^n, X^{n-1}\right) \quad \to \quad H_i(X^{n-1}) \quad \to \quad H_i(X^n) \quad \to \quad H_i(X^n, X^{n-1}) \quad \to \cdots$$

so $H_i(X^{n-1}) \cong H_i(X^n)$ except possibly for $i = n-1, n$. And there is an exact sequence $0 \to H_n(X^n) \to H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}) \to H_{n-1}(X^n) \to 0$.
By induction, $H_i(X^n) \cong H_i(X^{n-1}) \cong H_i(X^0) = 0$. ∎

Definition 7.14. A ***chain complex*** is a family $\mathscr{A} = \{A_i, d_i\}_{i \in Z}$ of R-modules A_i and maps $d_i : A_i \to A_{i-1}$ such that $d_{i-1} d_i = 0$.
The i-th homology group is $H_i(\mathscr{A}) := \frac{\ker d_i}{\mathrm{Im} d_{i+1}}$.

Assemble portions of the less for $\left(X^{n+1}, X^n\right), \left(X^n, X^{n-1}\right), \left(X^{n-1}, X^{n-2}\right)$ as follows

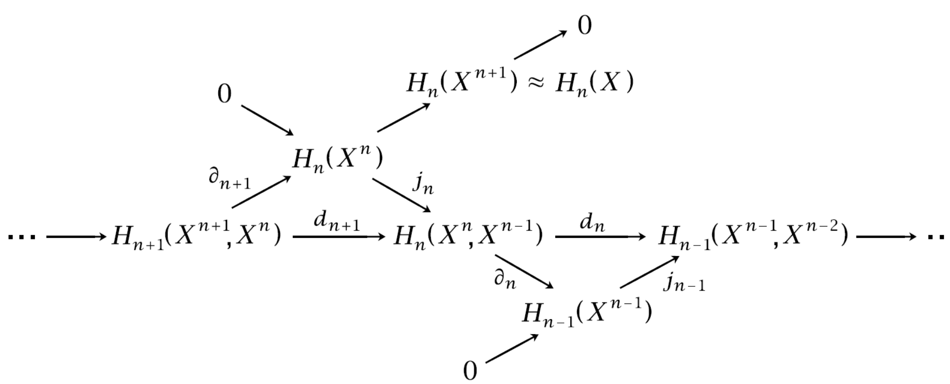

Figure 3: CW complex

Theorem 7.15. *The homology of cellular chain complex of X is $H_n(X)$*

Proof. From the diagram, $H_n(X) \cong H_n(X^n)/\mathrm{Im}(\partial_{n+1})$. j_n is one to one, so maps

$$\begin{cases} \mathrm{Im}(\partial_{n+1}) \xrightarrow{\cong} \mathrm{Im} d_{n+1} \\ H_n(X^n) \xrightarrow{\cong} \mathrm{Im} j_n = \ker \partial_n = \ker d_n \end{cases}$$

So $\ker \partial_n = \ker d_n$. So j_n induces

$$H_n(X) = \frac{H_n(X^n)}{\mathrm{Im}\partial_{n+1}} \xrightarrow{\cong} \frac{\ker d_n}{\mathrm{Im} d_{n+1}} = H_i(\mathscr{A})$$

□

Corollary 7.16. *We have*

1. *If X has no n-cells, then $H_n(X) = 0$.*
2. *If X has k n-cells, then $H_n(X)$ is generated by no more than k elements.*
3. *If X has no $(n-1)$-cells and no $(n+1)$-cells, $H_n(X)$ is free abelian with basis in 1-1 correspondence with the n-cells.*

Euler Characteristics

X is a finite CW complex. The Euler Characteristics is defined as

$$\chi(X) = \sum_n (-1)^n (\#n - \text{cells in } X).$$

Theorem 7.17. $\chi(X) = \sum_n (-1)^n \, rank H_n(X))$.

rank$H_n(X))$ is called the n-th Betti number, which is the number of $\mathbb{Z}$-summands when $H_n(X)$ is expressed as a direct sum of cyclic groups.

Corollary 7.18. *χ is a honotopy invariant of spaces that admit finite CW structures. Also χ does not depend on the particuar finite CW structure.*

Lemma 7.19. *If $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated abelian groups, then $rankA + rankC = rankB$.*

Proof. Tensor with $\mathbb{Q}$, then apply the Rank Nullity Theorem. □

Proof of the theorem. Suppose

$$0 \to C_k \xrightarrow{d_k} C_{k-1} \xrightarrow{d_{k-1}} \cdots \to C_1 \xrightarrow{d_1} C_0 \to 0$$

is a chain compelx of abelian groups with cycles $Z_n = \ker d_n$ and boundaries $B_n = \mathrm{Im} d_{n+1}$ and homology $H_n = Z_n / B_n$.

The short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & Z_n & \to & C_n & \to & B_{n-1} & \to & 0 \\ 0 & \to & B_n & \to & Z_n & \to & H_n & \to & 0 \end{array}$$

give rankC_n = rankZ_n + rankB_{n-1} and rankZ_n = rankB_n + rankH_n. So rankC_n = rankB_n + rankB_{n-1} + rankH_n. But then

$$\sum_n (-1)^n \mathrm{rank} C_n = (-1)^n \mathrm{rank} H_n.$$

Put $C_n = H_n(X^n, X^{n-1})$ (free abelian group m the n-cells , rank C_n is the number of cells) to get the theorem.

How to compute degrees?

Assume that $f : S^n \to S^n$ is surjective (else $\deg f = 0$).

Suppose $y \in \operatorname{Im}(f)$ such that $m = \left|f^{-1}(y)\right| \leq \infty$ with $f^{-1}(y) = \{x_1, \cdots x_m\}$, and such that there are open ball neighborhoods U_i of x_i and V of y with $f : U_i \to V$ and $U_i \cap U_j = \emptyset, \forall i \neq j$. $\tilde{H}_n(S^n) \cong \mathbb{Z}$.

The ***local degree*** d_i of f at x_i is

$$\begin{array}{ccc} H_n(U_i, U_i - x_i) & \xrightarrow{f^*} & H_n(V, V - y) \\ \alpha & \mapsto & d_i\alpha \end{array}$$

Theorem 7.20. $\deg f = d_1 + \cdots + d_m$.

Example 7.21. $f : S^1 \to S^1, z \mapsto z^k$ on $S^2 \to S^2$.

Proof. Take y any point in $\operatorname{Im}(f)$ apart from 0 or ∞. Then $|f^{-1}(y)| = k$. And all the U_i's and V can be taken with $f : U_i \to V$ a homeomorphism for all i. $d_i = 1$. hence $\deg f = k$.

$$\begin{array}{ccccccc} & \mathbb{Z} \cong & H_n(U_i, U_i - x_i) & \xrightarrow[\text{mult by } d_i]{f^*} & H_n(V, V - y) & \\ & \swarrow & k_i \downarrow & & \downarrow\cong & \\ \mathbb{Z} \cong H_n(S^n, S^n - x_i) & \overset{p_i}{\longleftarrow} & H_n\left(S^n, S^n - f^{-1}(y)\right) & \xrightarrow{f^*} & H_n(S^n, S^n - y) & = \mathbb{Z} \\ & & \cong\downarrow & & \uparrow & \\ & \nwarrow & \bigoplus_{i=1}^{m} H_n(U_i, U_i - x_i) & & \mid\cong & \\ & & j \uparrow & & \mid & \\ & \mathbb{Z} \cong & \tilde{H}_n(S^n) & \xrightarrow[\text{mult by } \deg f]{f^*} & \tilde{H}_n(S^n) & \end{array}$$

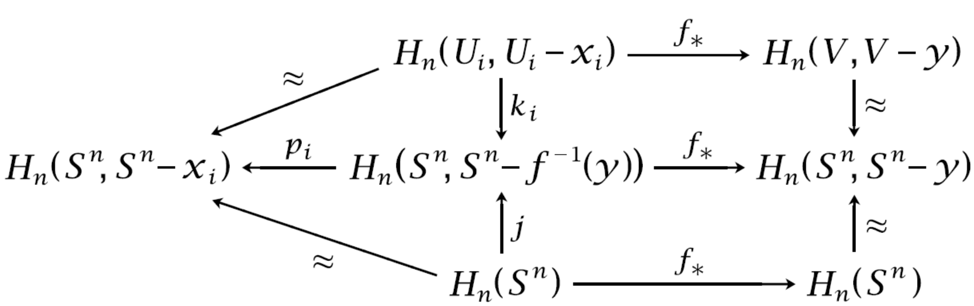

Figure 4: Homology of $f : S^n \to S^n$

$k_i(1) = (0, \cdots, 0, 1, 0, \cdots, 0)$, $p_i \circ j(1) = 1, \forall i$. So $j(1) = (1, \cdots, 1, 1 \cdots, 1)) = \sum_{i=1}^m k_i(1)$. So

$$\deg f = f^* j(1) = f^*\left(\sum_{i=1}^m k_i(1)\right) = \sum_{i=1}^m f^* k_i(1) = \sum_{i=1}^m d_i$$

where the first equality comes from the commutativity of the lower right diagram and the last equality comes from the commutativity of the upper right diagram. $\square$

Recall the cellular chain complex

$$\cdots \to H_{n+1}\left(X^{n+1}, X^n\right) \xrightarrow{d_{n+1}} H_n\left(X^n, X^{n-1}\right) \xrightarrow{d_n} H_{n-1}\left(X^{n-1}, X^{n-2}\right) \to \cdots$$

Its homology is $H_n(X)$.

Problem 7.22. What is d_n?

Solution. $d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$ for some coefficients $d_{\alpha\beta}$.

Theorem 7.23. *For $n > 1$, $d_{\alpha\beta}$ is the degree of the map $\Delta_{\alpha,\beta} : S^{n-1}_\alpha \to S^{n-1}_\beta$ where*

$$S^{n-1}_\alpha = \partial D^n_\alpha \xrightarrow[\text{attaching map}]{f_\alpha} X^{n-1} \xrightarrow[\text{quotient}]{q} X^{n-1}/X^{n-2} \xrightarrow[\text{collapse all cells but } e^{n-1}_\beta \text{ to a pt}]{q_\beta} aX^{n-1}/\left(X^{n-2} \coprod_{\gamma\neq\beta} e^{n-1}_\gamma\right) = S^{n-1}_\beta.$$

Remark 7.24. Consider X^1 = 1-skeleton, X^1/X^0 = wedges of circles, $X^1/\left(X^0 \coprod_{\gamma\neq\beta} e^1_\gamma\right)$ = a circle.

Corollary 7.25. *If $e^{n-1}_\beta \nsubseteq Im(f_\alpha)$, then $d_{\alpha\beta} = 0$.*

Proof. Consider the diagram

$$\begin{array}{ccccc}
H_n\left(D^n_\alpha, S^{n-1}_\alpha\right) & \to & H_n\left(S^{n-1}_\alpha\right) & \xrightarrow[\times d_{\alpha\beta}]{(\Delta_{\alpha\beta})_*} & H_n\left(S^{n-1}_\beta\right) \\
\downarrow (f_\alpha)_* & & \downarrow (f_\alpha)_* & & \uparrow \\
H_n\left(X^n, X^{n-1}\right) & \to & H_n\left(X^{n-1}\right) & \xrightarrow{q_*} & \tilde{H}_{n-1}\left(X^{n-1}/X^{n-2}\right) = \bigoplus_\beta \tilde{H}_{n-1}\left(D^{n-1}_\beta/S^{n-2}_\beta\right) \\
& \searrow d_n & \downarrow & & \| \\
& & H_{n-1}\left(X^{n-1}, X^{n-2}\right) & \xrightarrow[\cong]{q_*} & H_n\left(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2}\right)
\end{array}$$

Note that $X^{n-1}/X^{n-2} = \bigvee_\beta \left(D^{n-1}_\beta/S^{n-2}_\beta\right)$ and by Mayer-Vietoris for "good spaces", $\tilde{H}_{n-1}$ of a wedeg is a direct sum of its homologies. ☐

Example 7.26. The torus M_g with genus g = $4g$-gon with the identification:

$c_0 = 1, c_1 = 2g, c_2 = 1$.

The cellular chain complex is $0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \to 0$.

Claim: $d_1 = 0$. Follows from the fact that $X^0 = *$ and X^1 is connected.

$$H_1\left(X^1, X^0\right) \xrightarrow{d_1 = \partial} H_0\left(X^0\right)$$

So $d_1 = \partial = 0$.

Special case

$$\begin{array}{ccccccccccc}
\cdots \to & H_1\left(S^1, *\right) & \to & H_0(*) & \to & H_0\left(S^1\right) & \to & H_0(*) & \to & H_0\left(S^1, *\right) \\
& \| & & \| & & \| & & \| & & \| \\
& \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & 0
\end{array}$$

General case

$$\begin{array}{ccccccccccc}
\cdots \to & H_1\left(X^1, X^0\right) & \to & H_0\left(X^0\right) & \to & H_0\left(X^1\right) & \to & H_0\left(X^0\right) & \to & H_0\left(X^1, X^0\right) \\
& \| & & \| & & \| & & \| & & \| \\
& \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & 0
\end{array}$$

Claim $d_2 = 0$. $d_{ea_i} = 1 - 1 = 0$.

$$H_n(M_g) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z}^{2g} & n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Example 7.27. $N_g = 2g$-gon with the identification:
$N_1 =$ projective plane, $N_2 =$Klein bottle.
The cellular chain complex is $0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^g \xrightarrow{d_1} \mathbb{Z} \to 0$.
$d_1 = 0$ as before. $d_{ea_i} = 1 + 1 = 2$. $d_2(1) = (2, \cdots, 2)$.

$$H_n(N_g) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2 & n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Example 7.28. Identify $\mathbb{R}P^n = S^n / \sim, x \sim -x$. If we give $S^n = (e_0 \cup e_0') \cup \cdots \cup (e_n \cup e_n')$ the CW structure that contains 2 i-cells for $0 \le i \le n$, then $\mathbb{R}P^n = e_0 \cup e_1 \cup \cdots \cup e_n$. So the cellular chain complex for $\mathbb{R}P^n$ is

$$0 \to \mathbb{Z} \xrightarrow{d_n} \mathbb{Z} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \to 0$$

where d_k is the multiplication by degree of $\Delta : S^{k-1} \to \mathbb{R}P^{k-1} \to \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} = S^{k-1}$.
E.g. $k = 2$.

$$S^1 \xrightarrow{\text{attaching map}} \mathbb{R}P^1 \to \mathbb{R}P^1/\mathbb{R}P^0 = S^1$$

is multiplication by 1.
Same argument for $k = 3$.
Δ restricts to a homeomorphism on each (open) hemisphere, i.e. on each component of S^{k-1}/S^{k-2}. These homeomosphisms are obtained from each other by precomposing with the antipodal map. So by the local degree formula, $\deg \Delta = \deg(\mathrm{Id}) + \deg(\text{antipodal}) = 1 + (-1)^n$. So

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z}_2 & 0 < k < n \text{ odd} \\ \mathbb{Z} & k = 0, k = n(\text{odd}) \\ 0 & \text{otherwise.} \end{cases}$$

7.3 Weirdand Unmotivated Homological Algebra

Let $F : A \to B$ be a map between chain complexes, i.e.

$$\cdots \to \quad A_{i+1}$$

$$\begin{array}{ccccccccc} \cdots \xrightarrow{d} & A_{i+1} & \xrightarrow{d} & A_i & \xrightarrow{d} & A_{i-1} & \xrightarrow{d} \cdots \\ & f_{i+1} \downarrow & & \downarrow f_i & & \downarrow f_{i-1} & \\ \cdots \xrightarrow{d} & B_{i+1} & \xrightarrow{d} & B_i & \xrightarrow{d} & B_{i-1} & \xrightarrow{d} \cdots \end{array}$$

Lemma 7.29. *The induced map*

$$\begin{array}{cccc} F_* : & H_*(A) & \to & H_*(B) \\ & [a] & \mapsto & f_i([a]) := [f_i(a)] \end{array}$$

is well-defined.

Proof. $H_i(A) = \ker d_i / \mathrm{Im} d_{i+1}$. Let $[a] = [a'] \in H_1(A)$, then $a - a' = da''$ for some $a'' \in A_{i+1}$, so $f_i(a) - f_i(a') = f_i(da'') = df_{i+1}a''$, hence $[f_i(a)] = [f_i(a')] \in H_i(B)$. ☐

Definition 7.30. A chain homotopy between maps $F, G : A \to B$ is a collection $\Phi = \{\varphi_i : A_i \to B_{i+1}\}_{i \in \mathbb{Z}}$ of maps such that for any i, $d\varphi_i + \varphi_{i-1}d = f_i - g_i : A_i \to B_i$.

Lemma 7.31. *Given such a chain homotopy, $F_* = G_*$.*

Proof. Suppose $a \in \ker(d_i : A_i \to A_{i-1})$, then

$$\begin{aligned} [f_i(a)] - [g_i(a)] &= [d\varphi_i(a) + \varphi_{i-1}d(a)] \\ &= [d\varphi_i(a)]\,(\in \mathrm{Im}(d_{i+1} : B_{i+1} \to B_i)) + [\varphi_{i-1}d(a)] \\ &= 0 \end{aligned}$$

□

Lemma 7.32. *(The Sanke Lemma) If* $0 \to A \xrightarrow{F} B \xrightarrow{G} C \to 0$ *is a short exact sequence of chain complexes, then there exists a long exact sequence*

$$\cdots \to H_i(A) \xrightarrow{f_i} H_i(B) \xrightarrow{g_i} H_i(C) \xrightarrow{\partial_i} H_{i-1}(A) \to \cdots$$

Proof. We can define ∂ by the following steps.

$$\begin{array}{ccccccc} & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & A_i & \xrightarrow{f_i} & B_i & \xrightarrow{g_i} & C_i & \to 0 \\ & d\downarrow & & d\downarrow & & d\downarrow & \\ 0 \to & A_{i-1} & \xrightarrow{f_{i-1}} & B_{i-1} & \xrightarrow{g_{i-1}} & C_{i-1} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \end{array}$$

Suppose $c_i \in C_i$ is a cycle, representing an element of $H_i(C)$. Take $b_i \in B_i$ mapping to c_i. Let $b_{i-1} = db_i$, then $g_{i-1}(b_{i-1}) = 0$ since $dc_i = 0$. So there exists a unique $a_{i-1} \in A_{i-1}$ mapping to b_{i-1} because f_{i-1} is injective. Define $\partial_i[c_i] = [a_{i-1}] \in H_{i-1}(A)$.

Now check we get a long exact sequence by lots of diagram chasing. □

Exercise 7.33. Check that this is well-defined (does not depend on the choice of b_i) and exactness.

8 Singular Homoloty

Definition 8.1. An ***n-simplex*** is $\Delta^n = \left\{(t_0, \cdots, t_n) \in \mathbb{R}^{n+1} | \sum t_i = 1, t_i \geq 0, \forall i\right\}$. $\Delta^n := [v_0, \cdots, v_n]$ denotes an n-simplex with an ordering on its vertices.

A face is a subsimplex with vertices same subset of $v_0, \cdots, v_n$, inheriting the ordering.

Let X be a space. A ***singular n-simplex*** X is a map $\sigma : \Delta^n \to X$.

$C_n(X) :=$ free abelian group on all such σ. Elements of $C_n(X)$ are called n-chains. And define

$$\begin{array}{rccc} \partial_n : & C_n(X) & \to & C_{n-1}(X) \\ & \sigma & \mapsto & \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0, \cdots, \hat{v}_i, \cdots, v_n]} \end{array}$$

Lemma 8.2. $\partial_{n-1} \cdot \partial_n = 0$.

Proof. Calculate

$$\begin{array}{rcl} \partial_{n-1} \cdot \partial_n(\sigma) & = & \sum_{j<i} (-1)^i (-1)^j \sigma|_{[v_0, \cdots, \hat{v}_j, \cdots, \hat{v}_i, \cdots, v_n]} \\ & & + \sum_{j>i} (-1)^i (-1)^{j-1} \sigma|_{[v_0, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_n]} \\ & = & 0 \end{array}$$

□

So $\{C_n(X), \partial_n\}$ is a chain complex and defines

$$H_n(X) = \frac{\ker \partial_n}{\mathrm{Im}\partial_{n+1}}$$

the n-th singular homology group of X, where $\ker \partial_n$ denotes the cycles and $\mathrm{Im}\partial_{n+1}$ denotes the boundaries.

Let $C_n(X, A) := \frac{C_n(X)}{C_n(A)}$. ∂_n is induced by the previous ∂_n.

Lemma 8.3. *A map $f : (X, A) \to (Y, B)$ induces*

$$\begin{array}{rl} f_n : & C_n(X) \to C_n(Y), \sigma \mapsto f \circ \sigma, \\ f_n : & C_n(X, A) \to C_n(Y, B), \sigma \mapsto f \circ \sigma, \\ f_n : & H_n(X) \to H_n(Y) \\ f_n : & H_n(X, A) \to H_n(Y, B) \end{array}$$

(The Snake Lemma) If $0 \to A \xrightarrow{F} B \xrightarrow{G} C \to 0$ is a short exact sequence of chain complexes, then there exits a long exact sequence

$$\cdots \to H_i(A) \xrightarrow{f_i} H_i(B) \xrightarrow{g_i} H_i(C) \xrightarrow{\partial_i} H_{i-1}(A) \to \cdots$$

Theorem 8.4. *The Singular Homology is an ordinary homology theory.*

Proof. The 1,2,3 axioms are obvious.

4. Apply the snake lemma to the short exact sequence $0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0$.

$$\begin{array}{ccccccc} 0 \to & C_i(A) & \to & C_i(X) & \to & C_i(X, A) & \to 0 \\ & \partial \downarrow & & \partial \downarrow & & \partial \downarrow & \\ 0 \to & C_{i-1}(A) & \to & C_{i-1}(X) & \to & C_{i-1}(X, A) & \to 0 \end{array}$$

So

$$\begin{array}{rccc} \partial_i : & H_i(X, A) & \to & H_{i-1}(A) \\ & [\alpha] & \mapsto & [\partial \alpha] \end{array}$$

Consider

$$\cdots \xrightarrow{\partial_{i+1}} \frac{C_i(X)}{C_i(A)} \xrightarrow{\partial_i} \frac{C_{i-1}(X)}{C_{i-1}(A)} \xrightarrow{\partial_{i-1}} \cdots$$

And $H_i(X,A)=\frac{\ker\partial_i}{\mathrm{Im}\partial_{i+1}}$. So $\alpha\in C_i(X)$ represents an element of $H_i(X,A)$ when $\partial\alpha\in C_{i-1}(A)$, then $\partial\alpha$ represents an element of $H_{i-1}(A)$ because $\partial^2\alpha=0$.

6. excision. (***Check!***)

7. For any i, there exists $\sigma_i:\Delta\to *$ and

$$\partial\sigma_i=\underbrace{(1-1+1+\cdots)}_{i+1}=\begin{cases}0 & i\text{ odd},\\ \sigma_{i-1} & i\text{ even}.\end{cases}$$

So the chain complex

$$\cdots\to C_n(X)\to\cdots C_1(X)\to C_0(X)\to 0$$

is

$$\cdots\to\mathbb{Z}\xrightarrow{\mathrm{id}}\mathbb{Z}\xrightarrow{0}\mathbb{Z}\xrightarrow{\mathrm{id}}\mathbb{Z}\xrightarrow{0}\mathbb{Z}\to 0$$

So

$$H_i(*)=\begin{cases}\mathbb{Z} & i=0\\ 0 & \text{otherwise}.\end{cases}$$

5. Show that if $f,g:X\to Y$ are homotopic, then $f_*=g_*$.

$\Delta^n\times I$ is the union of $n+1$-simplices $\Delta_i^{n+1}\coloneqq[v_0,\cdots,v_i,w_i,\cdots,w_n]$ where

$$\Delta_i^{n+1}\cap\Delta_{i+1}^{n+1}=[v_0,\cdots,v_i,w_{i+1},\cdots,w_n]$$

which is an n-simplex. And

$$\begin{aligned}\Delta^n\times\{0\}&=[v_0,\cdots,v_n],\\ \Delta^n\times\{1\}&=[w_0,\cdots,w_n].\end{aligned}$$

Suppose $F:X\times I\to Y$ is a homotopy equivalence from f to g. Define $P:C_n(X)\to C_{n+1}(Y)$ by $\sigma\mapsto\sum_i(-1)^iF\circ(\sigma\times\mathrm{id})|_{[v_0,\cdots,v_i,w_i,\cdots,w_n]}$. Then $\partial P+P\partial=g_n-f_n$. because

$$\partial P\sigma = \sum_{j\le i}(-1)^i(-1)^jF\circ(\sigma\times\mathrm{id})|_{[v_0,\cdots,\hat{v}_j,\cdots,v_i,w_i,\cdots,w_n]}+\sum_{j\ge i}(-1)^i(-1)^{j+1}F\circ(\sigma\times\mathrm{id})|_{[v_0,\cdots,v_i,w_i,\cdots,\hat{w}_j,\cdots,w_n]}$$

Terms with $i=j$ cancel except $F\circ(\sigma\times\mathrm{id})|_{[\hat{v}_0,w_0,\cdots,w_n]}=g_n(\sigma)$ and $-F\circ(\sigma\times\mathrm{id})|_{[v_0,\cdots,v_n,\hat{w}_n]}=-f_n(\sigma)$.
Terms with $i\neq j$ are $-P\partial(\sigma)$. (***Check!***)
∂P: boundary of the 'prison'.
$P\partial$: sides of the 'prison'.
g_n-f_n: top of the prison - bottom of the prison. □

Proposition 8.5. *If $\{X_i\}_{i\in I}$ are components of X, that is, $X=\bigsqcup_{i\in I}X_i$ and X_i is open in X for any i, then $H_n(X)=\bigoplus_{i\in I}H_n(X_i)$.*

Proof. If $c=\sum_{j=1}^k m_j\sigma_j\in C_n(X)$, then each σ_j is supported in one X_i. Thus $C_n(X)=\bigoplus_{i\in I}C_n(X_i)$. And as for any i, $\partial:C_n(X_i)\to C_{n-1}(X_i)$ as chain complexes, $C_*(X)=\bigoplus_{i\in I}C_*(X_i)$. It follows what we want. □

Theorem 8.6. *$H_0(X)$ is the free abelian group on the path components of X.*

Proof. By the previous result it suffices to prove this when X is path connected.

Identifying $C_0(X)$ with the free abelian group on X.

$$H_0(X) = \frac{Z_0(X)}{B_0(X)} = \frac{C_0(X)}{B_0(X)}$$

Define

$$\begin{array}{rccc} \varepsilon: & C_0(X) & \to & \mathbb{Z} \\ & \sum n_i x_i & \mapsto & \sum n_i \end{array}$$

the augmentation map induced by $X \to *$ with kernel $\tilde{H}_0(X)$.

Claim: $B_0(X) \subseteq \ker \varepsilon$.

$B_0(X)$ is generated by the boundary of 1-simplices, i.e. map $f: I \to X$, $\partial f = f(1) - f(0)$, and $\varepsilon(\partial f) = 1 - 1 = 0$.

Claim: $\ker \varepsilon \subseteq B_0(X)$.

Suppose $\varepsilon\left(\sum n_i x_i\right) = \sum n_i = 0$, then

$$\sum_{i=0}^{m} n_i x_i = \sum_{i=0}^{m} n_i (x_i - x_0)$$

which is a sum of boundaries of singular 1-simplices because X is path connected. □

Theorem 8.7. *The map $h: \pi_1(X, x_0) \to H_1(X)$ that maps the equivalent classes of loops $f: \Delta^1 = [0,1] \to X$ with $f(0) = f(1)$ to 1-cycles is a well-defined homomorphism. If X is path connected, h is onto and $\ker h = [\pi_1(X, x_0), \pi_1(X, x_0)]$ whence the abelianization $\pi_1^{ab}(X, x_0) = H_1(X)$.*

Proof. Notation: $f \simeq g$ for homotopic rel base point. $f \sim g$ for "f and g are homologous", i.e. $f - g = \partial I$ for some $I \in C_2(X)$.

1. If f is a constant map, $f \sim 0$.

   Take σ a singular 2-simplex $[v_0, v_1, v_2]$ with $\mathrm{Im} f = \mathrm{Im}\sigma$, then

   $$\partial\sigma = \sigma|_{[v_1,v_2]} - \sigma|_{[v_0,v_2]} + \sigma|_{[v_0,v_1]} = f - f + f = f.$$

2. If $f \simeq g$ then $f \sim g$.

   Let $F: I \times I \to X$ be the homotopy from f to g. Here $I \times I$ is a 3-simplex $[v_0, v_1, v_2, v_3]$.

   $$\partial\left(F|_{[v_0,v_1,v_2]} - F|_{[v_0,v_2,v_3]}\right) = f - g + \text{two constant singular 1-simplices}$$

   So $f \sim g$.

3. The cancatenation of loops based at x_0 $f \cdot g \sim f + g$.

   $$\begin{array}{ccccc} & v_2 & & & \\ f\cdot g \nearrow & & \nwarrow g & \xrightarrow{\sigma} & X \\ v_0 & \xrightarrow{f} & v_1 & & \end{array}$$

   Define σ to be constant in the interior of the simplex, then $\partial\sigma = f + g - f \cdot g$. So h is a homomorphism.

4. Backwards $\tilde{f} \sim -f$.

   $0 \sim f \cdot \tilde{f} \sim f + \tilde{f}$.

Assume X is path connected. Why h is onto? Any element of $H_1(X)$ can be represented by $\sum_i \sigma_i$ where each σ_i is a loop. Suppose $\sum n_i \sigma_i$ is a 1-cycle, reexpress the sum such that each $n_i = \pm 1$. Replace σ_i with $\tilde{\sigma}_i$ if $n_i = -1$ so that all the n_i are positive. Join into loops using 3 and the fact that $\partial\left(\sum_i \sigma_i\right) = 0$. Let γ_i

be a path from x_0 to $\sigma_i(0)$,then $\gamma_i \cdot \sigma_i \cdot \tilde{\gamma}_i \sim \sigma_i$ by 3. Concatenate these to give a single loop based at x_0 mapped by h to $\sum \sigma_i$.

Since $H_1(X)$ is abelian, $[\pi_1(X, x_0), \pi_1(X, x_0)] \leq \ker h$.

Suppose $[f] \in \ker h$, then

$$h = \partial \sum_i n_i \sigma_i = \sum_{i,j} (-1)^j n_i \tau_{ij}$$

for some 2-chain $\sum n_i \sigma_i$. We may assume that each $n_i = \pm 1$. Then all but one of the $(-1)^j n_i \tau_{ij}$ pair off and cancel, the remaining are being f. Assemble a 2-complex K (a closed orientable surface with a disc removed) from the Δ_i by identifying edges according to these pairings (preserving the edge orientations). The σ_i then assemble to give a map $\sigma : K \to X$. Homotopy σ, keeping the f-edges fixed to get a new 2-chain $\sum_i n_i \sigma_i'$ with boundary and all the τ_{ij}' are loops at x_0. In $\pi^{ab}(X, x_0)$ written addtively,

$$[f] = \sum_{ij} (-1)^j n_i \left[\tau_{ij}'\right] = \sum_i n_i \left[\partial \sigma_i'\right] = \sum_i n_i [0] = 0$$

because τ_{ij}' are boundaries of 2-simplices. □

9 (Singular) Homology with Coeffcients

9.1 Introduction

Let G be an abelian group, $C_n(X;G)$ the group of finite sums $\sum_i n_i \sigma_i$ with $n_i \in G$, σ_i singular n-simplices. ∂ as before, $\partial^2 = 0$ as before.

$C_n(X;G), C_n(X,A;G) = \frac{C_n(X;G)}{C_n(A;G)}$ form chain complexes. $H_n(X;G), H_n(X,A;G)$ are homotopy groups with coefficients in G.

Ordinary homology theory with coefficients in G.

Cellular homology generalization similarly and agrees with the singular homology on CW-complexes.

$C_n(X;G) \cong C_n(X) \otimes G$.

Theorem 9.1. *The Universal Coeffcient Theorem for Homology*

There exists a split shor exact sequence

$$0 \to H_n(X) \otimes G \to H_n(X;G) \to Tor(H_{n-1}(X), G) \to 0$$

where $Tor(H,G)$ is an algberaic gadget measuring the common torsion of H and G. (Def A.3.7, Weintraub). So

- *$H_*(X)$ and G determine $H_*(X;G)$.*
- *$H_n(X;G) = (H_n(X) \otimes G) \oplus Tor(H_{n-1}(X;G))$*

Fact 9.2. *$Tor(H,G) = 0$ when H is free and G is a field.*

Proof. *Check!* □

9.2 Homology of Products

Theorem 9.3. *(Künneth Formula) For spaces X, Y there exists a short exact sequence*

$$0 \to (H_*(X) \otimes H_*(Y))_n \to H_n(X \times Y) \to Tor(H_*(X), H_*(Y))_{n-1} \to 0$$

where $(H_*(X) \oplus H_*(Y))_n = \bigoplus_{i+j=n} (H_i(X) \otimes H_j(Y))$ and $Tor(H_*(X), H_*(Y))_{n-1} = \bigoplus_{i+j=n-1} Tor(H_i(X), H_{n-j-1}(Y))$.

Proof. *Check!* □

Example 9.4. $X = Y = S^1$ where

$$H_n(S^1) = \begin{cases} \mathbb{Z} & n = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

And we know

$$H_n(S^1 \times S^1) = \begin{cases} \mathbb{Z} & n = 0, 2, \\ \mathbb{Z}^2 & n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Part VI
Cohomology

Chain complex of free abelian groups $\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$

G an abelian group. Apply the contravarient functor $\mathrm{Hom}\left(-, G\right)$ to get

$$\cdots \leftarrow \mathrm{Hom}\left(C_{n+1}, G\right) \xleftarrow{\delta^n} \mathrm{Hom}\left(C_n, G\right) \xleftarrow{\delta^{n-1}} \mathrm{Hom}\left(C_{n+1}, G\right) \leftarrow \cdots$$

where δ^n is defined by:

for $f \in \mathrm{Hom}\left(C_n, G\right)$, $\delta^n\left(f\right) := f\partial_{n+1} : C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{f} G$.

We can see that $\delta \circ \delta = 0$ from $\partial^2 = 0$, so we get a ***cochain complex***. Define $Z^n = \ker \delta^n$ as ***cocycles***, $B^n = \mathrm{Im}\delta^{n-1}$ as ***coboundaries***. The n-th cohomology group is $H^n\left(X; G\right) = \frac{Z^n}{B^n}$.

The case $C_n = C_n(X)$ gives ***singular cohomology*** $H_n\left(X; G\right)$ with G-coefficents.

Explicitly, $\delta^n f$ is, for a singular $(n+1)$-simplex $\sigma : \Delta^{n+1} \to X$, $\left(\delta^n f\right)\sigma = \left(f\partial_{n+1}\right)\sigma = f\left(\partial_{n+1}\sigma\right) = \sum_i \left(-1\right)^i f\left(\sigma|_{[v_0, \cdots, \hat{v_i}, \cdots, v_{n+1}]}\right)$.

Theorem 9.5. *Singular cohomology is an ordinary cohomology theory with coefficient group G.*

Theorem 9.6. *If X is a CW complex, $H^n\left(X; G\right)$ is the homology of the dual (i.e. apply $Hom\left(-, G\right)$) of the cellular chain complex*

$$\cdots \xrightarrow{d} H_n\left(X^n, X^{n-1}\right) \xrightarrow{d} H_{n-1}\left(X^n, X^{n-1}\right) \to \cdots$$

Proposition 9.7. *$H^0\left(X; G\right)$ is the direct product of copies of G, one for each path connected component.*

Proof. $H^0\left(X; G\right) = \frac{Z^0}{B^0} = Z^0 = \ker\left(\delta : C^0\left(X; G\right) \to C^1\left(X; G\right)\right)$. $\delta f = 0$ if and only if $\left(\delta f\right)\sigma = \left(f\partial\right)\sigma = 0$ for any singular 1-simplcies $\sigma : \Delta^1 \to X$, i.e. $\left(f\partial\right)\sigma = f\sigma\left(1\right) - f\sigma\left(0\right) = 0$, if and only if f is constant on path connected components. $\square$

Theorem 9.8. *If $\left(X, x_0\right)$ is path connected, and G is an abelian group, then $H^1\left(X; G\right) = Hom\left(\pi_1\left(X, x_0\right), G\right)$.*

Example 9.9. If $X = \mathbb{R}P^2$, $\pi_1\left(\mathbb{R}P^2\right) = \mathbb{Z}_2$.

If $G = \mathbb{Z}_2$, then $H^1\left(\mathbb{R}P^2, G\right) = \mathbb{Z}_2$.

If $G = \mathbb{Z}$, then $H^1\left(\mathbb{R}P^2, G\right) = 0$.

10 A Geometric Interpretation

Let G be an abelian group. X is a "triangulated space" of Δ-complex. $\Delta_i(X;G)$ = finite sums of 1-simplices with G-coefficients.

$$\begin{array}{rrcl} d: & \Delta_n(X;G) & \to & \Delta_{n-1}(X;G) \\ & [v_0,\cdots,v_i] & \mapsto & \sum_j (-1)^j [v_0,\cdots,\hat{v}_j,\cdots,v_i] \end{array}$$

This is the ***simplicial homology***, a special case of cellular homology.

$\Delta^i(X;G) = \mathrm{Hom}(\Delta_i(X;G),G)$ = abelian group of assignments of elements of G to the i-simplices of X.

For $f \in \Delta^i(X;G)$, σ an $(i+1)$-simplex, then $(\delta f)\sigma = f(d\sigma)$.

Example 10.1. X is a surface. $f \in \Delta^1(X;G)$ represents an element of $H^1(X;G)$ when $\delta f = 0$, i.e. the sum around each triangle is 0. $(\delta f)\sigma = 0$ for all 2-simplices $\sigma = [v_0,v_1,v_2]$.

$$(\delta f)\sigma = f(d\sigma) = f([v_1,v_2]-[v_0,v_2]+[v_0,v_1]) = f[v_1,v_2]+f[v_2,v_0]+f[v_0,v_1]$$

Suppose $G=\mathbb{Z}_2$, then $f \in \Delta^1(X;G)$ is an assignment of 0/1 to the edges. And $\delta f = 0$ when around each triangle there are zero or two 1's. Draw curves transverse to the edges assigned 1's. $f = \delta g$ for some $g \in \Delta^0(X;G)$, i.e. $f \in H^1(X;G)$ of and only if the curves subdivide X into regions X_0, X_1 where g assigns 0's to all the vertices in X_0 and 1's to vertices in X_1.

$H^1(S^2;\mathbb{Z}) = 0, H^1(\mathbb{R}P^2,\mathbb{Z}_2) = \mathbb{Z}_2$.

Theorem 10.2. *The Universal Coefficient Theorem for cohomology*

There exists a split short exact sequence

$$0 \to Ext^1(H_{n-1}(C);G) \to H^n(C;G) \to Hom(H_n(C);G) \to 0$$

Fact 10.3. *H, G are abelian groups, $Ext(H;G)$ is $H^1(F;G)$ where F is any chain complex*

$$\cdots \to 0 \to F_1 \to F_0 \to H \to 0 \to \cdots$$

with F_0, F_1 free abelian groups.

Classifies extentions $0 \to G \to J \to H \to 0$.

11 Calculations of cohomology groups

Theorem 11.1. *Universal Coefficient Theorem for cohomology*

If a chain complex C of free abelian groups has homology groups $H_n(C)$, then the cohomology groups $H_n(C;G)$ of the cochain complex $Hom(C_n,G)$ are determined by split exact sequences

$$0 \to Ext(H_{n-1}(C),G) \to H^n(C;G) \to Hom(H_n(C),G) \to 0$$

Remark 11.2. Computing $\mathrm{Ext}(H,G)$ for finitely generated H is not difficult using the following three properties:

$\mathrm{Ext}(H \oplus H\prime, G) \cong \mathrm{Ext}(H,G) \oplus \mathrm{Ext}(H\prime,G)$.

$\mathrm{Ext}(H,G) = 0$ if H is free.

$\mathrm{Ext}(\mathbb{Z}_n,G) \cong G/nG$.

Example 11.3. We know that the homology groups of M_g are

$$H_n(M_g) = \begin{cases} \mathbb{Z} & n = 0,2, \\ \mathbb{Z}^{2g} & n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

So

$$H^k(M_g;\mathbb{Z}_2) = \begin{cases} \hom(H_0(M),\mathbb{Z}_2)\oplus 0 = \mathbb{Z}_2 & k=0,\\ \hom(H_1(M),\mathbb{Z}_2)\oplus \mathrm{Ext}(H_0(M),\mathbb{Z}_2) = \mathbb{Z}_2^{2g} & k=1,\\ \hom(H_2(M),\mathbb{Z}_2)\oplus \mathrm{Ext}(H_1(M),\mathbb{Z}_2) = \mathbb{Z}_2 & k=2,\\ 0 & \text{otherwise.}\end{cases}$$

Example 11.4. We know that the homology groups of N_g are

$$H_n(M_g) = \begin{cases} \mathbb{Z} & n=0,2,\\ \mathbb{Z}^{g-1}\oplus\mathbb{Z}_2 & n=1,\\ 0 & \text{otherwise.}\end{cases}$$

So

$$H^k(N_g;\mathbb{Z}_2) = \begin{cases} \hom(H_0(N),\mathbb{Z}_2)\oplus 0 = \mathbb{Z}_2 & k=0,\\ \hom(H_1(N),\mathbb{Z}_2)\oplus \mathrm{Ext}(H_0(N),\mathbb{Z}_2) = \mathbb{Z}_2^{g} & k=1,\\ \hom(H_2(N),\mathbb{Z}_2)\oplus \mathrm{Ext}(H_1(N),\mathbb{Z}_2) = \mathbb{Z}_2^2 & k=2,\\ 0 & \text{otherwise.}\end{cases}$$

11.1 The Cup Product

Consider the cross product on homology. Let X, Y be CW-complexes, we have a well-defined map

$$\begin{array}{ccc} H_i(X)\times H_j(Y) & \to & H_{i+j}(X\times Y)\\ (e^i,e^j) & \mapsto & e^i\times e^j \end{array}$$

But this does not lead to a map

$$H^i(X)\times H^j(X)\to H^{i+j}(X)$$

Consider the diagonal map $X\hookrightarrow X\times X, x\mapsto(x,x)$, it induces a natural product on $H^i(X)$,

$$H^i(X)\times H^j(X)\to H^{i+j}(X\times X)\to H^{i+j}(X).$$

Definition 11.5. X is a space, R is a ring, $\varphi\in C^k(X;R), \phi\in C^l(X;R)$, the cup product $\varphi\cup\phi\in C^{k+l}(X;R)$ is the cochain whose value on a singular simplex $\sigma:\Delta^{k+1}\to X$,

$$(\varphi\cup\phi)\,\sigma = \varphi\left(\sigma|_{[v_0,\cdots,v_k]}\right)\phi\left(\sigma|_{[v_k,\cdots,v_{k+l}]}\right).$$

Lemma 11.6. $\delta(\varphi\cup\phi) = \delta\varphi\cup\phi + (-1)^k\varphi\cup\delta\phi$.

Proof. For $\sigma:\Delta^{k+l+1}\to X$,

$$\begin{aligned}
(\delta\varphi\cup\phi)\,\sigma &= \sum_{i=0}^{k+1}(-1)^i\varphi\left(\sigma|_{[v_0,\cdots,\hat{v}_i,\cdots,v_{k+1}]}\right)\phi\left(\sigma|_{[v_{k+1},\cdots,v_{k+l+1}]}\right),\\
\left((-1)^k\varphi\cup\delta\phi\right)\sigma &= \sum_{i=k}^{k+l+1}(-1)^i\varphi\left(\sigma|_{[v_0,\cdots,v_k]}\right)\phi\left(\sigma|_{[v_k,\cdots,\hat{v}_i,\cdots,v_{k+l+1}]}\right),\\
(\delta(\varphi\cup\phi)) = (\varphi\cup\phi)(\partial\sigma) &= \sum_{i=0}^{k+l+1}(\varphi\cup\phi)\left(\sigma|_{[v_0,\cdots,\hat{v}_i,\cdots,v_{k+l+1}]}\right).
\end{aligned}$$

∎

Corollary 11.7. *The cup product induces an associative distributive product*

$$H^i(X;R)\times H^j(X;R)\to H^{i+j}(X;R)$$

making $H^*(X;R)=\bigoplus_i H^i(X;R)$ *a graded ring. If* $1\in R$*, then the cocycle* $1\in H^0(X;R)$ *taking value 1on every singular 0-simplex, is an identity for* $H^*(X;R)$.

Proof. If φ, ϕ are cocycles, then so is $\varphi \cup \phi$ be the lemma. And if φ is a cocycle and $\delta\phi$ a coboundary, then $\varphi \cup \delta\phi$ is a coboundary as it equals $\pm\left(\delta\left(\varphi \cup \phi\right) - \delta\varphi \cup \phi\right)$. Likewise if ϕ is a cocycle and $\delta\varphi$ is a coboundary, then $\delta\varphi \cup \phi$ is a coboundary. $\square$

Theorem 11.8. *$f : X \to Y$ induces a ring homomorphism*

$$H^*\left(Y; R\right) \to H^*\left(X; R\right)$$

If X and Y are homotopic, then their cohomology rings are isomorphism.

If R is commutative, $\varphi \cup \phi = (-1)^{kl}\, \phi \cup \varphi$.

Example 11.9. $M = M_g$, $R = \mathbb{Z}_2$. $H^1\left(M; \mathbb{Z}_2\right) \cong \hom\left(H_1\left(M\right), \mathbb{Z}_2\right) = \mathbb{Z}_2^{2g}$. $H^2\left(M; \mathbb{Z}_2\right) = \mathbb{Z}_2$.

$$H^1\left(M; \mathbb{Z}_2\right) \times H^1\left(M; \mathbb{Z}_2\right) \to H^2\left(M; \mathbb{Z}_2\right)$$

$H^1\left(M; \mathbb{Z}_2\right)$ has basis $\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g$ dual to the basis $a_1, b_1, \cdots, a_g, b_g$ for $H_1\left(M\right)$. $\alpha_i\left(a_j\right) = \delta_{ij}, \alpha_i\left(b_j\right) = 0, \beta_i\left(a_j\right) = 0, \beta_i\left(b_j\right) = \delta_{ij}$.

$$\left(\alpha_1 \cup \beta_1\right)\left[v_0, v_1, v_2\right] = \alpha_1\left(\left[v_0, v_1\right]\right)\beta\left(\alpha\left[v_1, v_2\right]\right) = \begin{cases} 1 & \text{on } \sigma \\ 0 & \text{otherwise.} \end{cases}$$

So $\gamma = \alpha_1 \cup \beta_1$ generates $H^2\left(M; \mathbb{Z}_2\right) = \mathbb{Z}_2 = \{0, \gamma\}$. In general, $\alpha_i \cup \beta_j = -\beta_j \cup \alpha_i = \delta_{ij}\gamma$, $\alpha_i \cup \alpha_j = \beta_i \cup \beta_j = 0$, $\forall i, j$.

Example 11.10. $N = N_g$, $R = \mathbb{Z}_2$. $a_1, \cdots, a_g$ a basis for $H_1\left(X; \mathbb{Z}_2\right)$, dual basis $\alpha_1, \cdots, \alpha_g$ for $H^1\left(X; \mathbb{Z}_2\right) = \mathbb{Z}_2^g$.

$H^2\left(X; \mathbb{Z}_2\right) = \mathbb{Z}_2$. $\left(\alpha_1 \cup \alpha_1\right)\left(\sigma\right) = 1$. $\alpha_i \cup \alpha_j = \delta_{ij}$.
Case $n = 1$ is S^1. Case $n = 2$ is N_1.
$H^*\left(\mathbb{R}P^n; \mathbb{Z}_2\right) = \mathbb{Z}_2\left[\alpha\right] / \left(\alpha^{n+1}\right)$.

$$H^k\left(\mathbb{R}P^n; \mathbb{Z}_2\right) \cong \operatorname{Hom}\left(H_k\left(\mathbb{R}P^n\right), \mathbb{Z}_2\right) \oplus \operatorname{Ext}\left(H_k\left(\mathbb{R}P^n\right), \mathbb{Z}_2\right) \cong \begin{cases} \mathbb{Z}_2 & 0 < k \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Example 11.11. $H^*\left(\mathbb{C}P^n; \mathbb{Z}_2\right) \cong \mathbb{Z}\left[\alpha\right] / \left(\alpha^{n+1}\right)$.

Example 11.12. $H^*\left(T^n; R\right) = \bigwedge_R\left[\alpha_1, \cdots, \alpha_n\right]$ is the exterior algebra, free R-module with basis the products $\alpha_{i_1} \cdots \alpha_{i_k}$ $\left(1 \le i_1 < \cdots < i_k \le n\right)$ subject to $\alpha_i\alpha_j = -\alpha_j\alpha_i, \alpha_i^2 = 0$.
$T^n = \underbrace{S^1 \times \cdots \times S^1}_{n}$.

$$H_k\left(T^2; \mathbb{Z}\right) = \begin{cases} \mathbb{Z} & n = 0, \\ \mathbb{Z}^2 & n = 1, \\ \mathbb{Z} & n = 2, \\ 0 & \end{cases}$$

$\bigwedge_{\mathbb{Z}}\left[\alpha_1, \alpha_2\right]$ has a basis $1, \alpha_1, \alpha_2, \alpha_1\alpha_2$.

Example 11.13. $H^*\left(\bigsqcup_\alpha X_\alpha; R\right) = \prod_\alpha H^*\left(X_\alpha; R\right)$.

$\tilde{H}^*\left(\bigvee_\alpha X_\alpha; R\right) = \prod_\alpha \tilde{H}^*\left(X_\alpha; R\right)$ if the base points are neighborhood retracts.

Example 11.14. $\mathbb{C}P^2 = e^0 \cup e^2 \cup e^4$ and $S^2 \bigvee S^4$ are not homotopy equivalent because their cohomologies have different ring structures.

Corollary 11.15. *There exists a map $S^3 \to S^2$ not homotopic to a constant map, specifically the attaching map of e^4 in $\mathbb{C}P^2$.*

Part VII
Manifolds

12 Orientability

Definition 12.1. An n-***manifold*** is a second countable Hausdorff space M in which every point has a neighborhood homeomorphic to $\mathbb{R}^n$.

Example 12.2. $S^n, T^n, \mathbb{R}^n, \mathbb{C}^n, M_g, N_g, \mathbb{R}P^n, \mathbb{C}P^n$, Lens space.

Let $G = \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}$.

Lemma 12.3. *For any point $x \in M$, $H_n(M, M - \{x\}; G) \cong G$.*

Proof. Note that there exists a neighborhood U of x such that $(U, U - \{x\}) \cong (\mathbb{R}^n, \mathbb{R}^n - \{0\})$, and the inclusion $(U, U - \{x\}) \hookrightarrow (M, M - \{x\})$ is excisive, so

$$H_n(M, M - \{x\}; G) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}; G) \cong H_n\left(D^n, S^{n-1}; G\right) \cong \tilde{H}_n(S^n; G) \cong G.$$

□

Definition 12.4. A ***local orientation*** of M at x is a choice μ_x of generator of $H_n(M, M - \{x\}; \mathbb{Z}) \cong \mathbb{Z}$.

If $B \subseteq M^n$ is homeomorphic to an n-ball and contains x and y, then we get a local orientation at y from a local orientation at x via

$$H_n(M, M - \{x\}) \cong H_n(M, M - B) \cong H_n(M, M - \{y\}).$$

We say $x \mapsto \mu_x$ is ***locally consistent*** if for any $x \in M$, there exists $B \subseteq M$ with $B \cong D^n$ and $x \in B$ such that for any $y \in B$, μ_y is that determined by μ_x.

We say $x \mapsto \mu_x$ is an ***orientation*** of M if the 'local consistency' condition holds for every $x \in M$, and we say M is ***orientable***.

The obstruction to orientability for surfaces is a Möbius band subspace.

Proposition 12.5. *Every manifold M has an orientable 2-sheeted covering $\tilde{M}$.*

Proof. Let $\tilde{M} = \{\mu_x | x \in M, \mu_x \text{ is a local orientation at } x\}$. Define a topology on $\tilde{M}$ using the balls gives a covering

$$\begin{array}{ccc} \tilde{M} & \to & M \\ \mu_x & \mapsto & x \end{array}$$

□

Example 12.6. $S^n \to \mathbb{R}P^n$ and $T^2 \to K$ Klein bottle.

Proposition 12.7. *If M is connected, then M is orientable if and only if $\tilde{M}$ has two components.*

Proof. Since M is connected, $\tilde{M}$ has 1 or 2 components.

If it has two components, each is homeomorphic to M, so M is orientable.

Conversely, if M is orientable, it has 2 orientations and so $\tilde{M}$ has two components. □

Corollary 12.8. *If M is connected and orientable, then $\pi_1(M)$ has no index 2 subgroup.*

Proof. ***Check!*** □

Proposition 12.9. *If a manifold is simply connected, then it is orientable.*

Example 12.10. The fundamental group of Klein bottle is $\pi_1(K) = \left\langle a, b | a^2 b^2 \right\rangle$ with an index 2 subgroup $\left\langle a^{-1} b^{-1} ab | a^2 b^2 \right\rangle$.

If we replaced $\mathbb{Z}$ by $\mathbb{Z}_2$ we would find there is never an obstruction to "$\mathbb{Z}_2$-orientability". There is only one generator of $\mathbb{Z}_2$, so you can't fail to make a consistent choice.

Theorem 12.11. *Suppose M is a closed (compact, with boundary) connected n-manifold*

1. $H_n(M;\mathbb{Z}) = \begin{cases} \mathbb{Z} & M \text{ is orientable,} \\ 0 & M \text{ is not orientable.} \end{cases}$

2. $H_n(M;\mathbb{Z}_2) = \mathbb{Z}_2$.

Proof. (Sketch) Assume M has a Δ-structure (or CW structure). In fact, a simplicial structure.

1. A non-zero element of $H_n(M;\mathbb{Z})$ must be represented by $\tau = \sum_i n_i \sigma_i$ for some n-simplicies σ_i, $n_i \in \mathbb{Z}\setminus\{0\}$. But $\partial\tau = 0$. Every $(n-1)$-simplices in M is the face of exact 2 n-simplices, so for every $(n-1)$-simplex either both or neither of these simplices are present among these σ_i. And if both are present, their contributions cancel. The connectedness of M indicates that it's actually "both" in every case, and if $\sigma_1, \cdots, \sigma_k$ are all the n-simplices, then $n_i = \pm n_j = \pm 1, \forall i \neq j$. Whether their contributions to $\partial\tau$ cancel depends on orientability.

2. The sum of all n-simplices is a generator of $H_n(M;\mathbb{Z}_2)$.

□

Definition 12.12. An element of $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$ or $H_n(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$ whose image in $H_n(M, M-\{x\}; G)$ is a generator for all x is called a ***fundamental class***.

We see from earlier, this exists and is a sum of all the n-simplices.

12.1 Poincaré Duality

Let $R = \mathbb{Z}$ or $\mathbb{Z}_2$.

Theorem 12.13. *If M is a closed R-orientable (i.e. if $R = \mathbb{Z}$, orientable; if $R = \mathbb{Z}_2$, no condition) n-manifold with fundamental class $[m] \in H_n(M;R) \cong R$, then for any k,*

$$\begin{array}{ccc} H^k(M;R) & \to & H_{n-k}(M;R) \\ \alpha & \mapsto & [m] \frown \alpha \end{array}$$

is an isomorphism.

Corollary 12.14. *Modulo their torsion subgroup, $H_k(M;\mathbb{Z}) \cong H_{n-k}(M;\mathbb{Z})$, for any k.*

Example 12.15. $\mathbb{R}P^n$ is orientable if and only if n is odd.

12.2 Dual cell structure

Remark 12.16. This explains Poincaré Duality in all differentiable manifold but it takes a bit of work.

Definition 12.17. Let C be a cell structure on M. C^* is the dual cell structure with an $(n-i)$-cell dual to each i-cell in C.

Example 12.18. A surface, S^2 for instance. $C_0^* \cong C_2, C_1^* \cong C_1, C_2^* \cong C_0$. Canonical isomorphism if we use $\mathbb{Z}_i$ coefficients. More generally one has to make sign changes.

Claim 12.19. $H_i(C;\mathbb{Z}_2) \cong H^{2-i}(C^*;\mathbb{Z}_2)$. Hence $H_i(M;\mathbb{Z}_2) \cong H^{2-i}(M;\mathbb{Z}_2)$.

Proof. $\partial : C_i \to C_{i-1}$ assigning to each cell the sum of its faces becomes $\delta : C_{2-i}^* \to C_{2-i+1}^*$ assigning to each cell the sum of cells of which it is a face. □

12.3 Cap product

Definition 12.20. Let X be a space, R be a ring. The cap product $\frown: C_k(X;R)\times C^l(X;R)\to C_{k-l}(X;R)$ for $k\geq l$ is, for $\sigma:\Delta^k\to X$ and $\varphi\in C^l(X;R)$,

$$\sigma\frown\varphi=\varphi\left(\sigma|_{[v_0,\cdots,v_l]}\right)\sigma|_{[v_l,\cdots,v_k]}$$

where $\Delta^k=[v_0,\cdots,v_k]$, and then extend bilinearly.

Proposition 12.21. *The cap product induces a map*

$$H_k(X;R)\times H^l(X;R)\to H_{k-l}(X;R).$$

Proof. Similar to cup product. ***(Check!)*** □

Example 12.22. $M=M_g$. Fundamental class $[m]$ generating $H_2(M;R)$ represented by the sum of the $4g$ simplices with the signs indicated in the case of $R=\mathbb{Z}$.

Poincaré Duality tells us that $H^1(M;R)\cong H_1(M;R)$.

$$[m]\frown\alpha_1=\sum_{4g\text{ simplices }\alpha}\pm\alpha_1\left(\sigma|_{[v_0,v_1]}\right)\sigma|_{[v_1,v_2]}=b_1$$

More generally, we see that $[m]\frown\alpha_i=b_i$ and $[m]\frown\beta_i=-a_i$.

Example 12.23. $N=N_g$. Fundamental class $[m]$ generating $H_2(N;\mathbb{Z}_2)$ represented by the sum of the $2g$ simplices. Poincaré Duality tells us that $H^1(N;\mathbb{Z}_2)\cong H_1(N;\mathbb{Z}_2)$.

In general, we see that $[m]\frown\alpha_i=a_i$.

Proposition 12.24. $\psi(\alpha\frown\varphi)=(\varphi\smile\psi)\alpha$ *for* $\alpha\in C_{k+k}(X;R),\varphi\in C^k(X;R),\psi\in C^l(X;R)$.

But this does not carry through to (co)homology. because $H^l(X;R)\ncong Hom(H_l(X;R),R)$.

But when M is a closed R-orientable n-manifold we do get from Poincaré Duality that

$$\begin{array}{ccc} H^k(M;R)\times H^{n-k} & \to & R \\ (\varphi,\psi) & \mapsto & (\varphi\smile\psi)[m]=\psi([m]\frown\varphi)\end{array}$$

is non-singular when $R=\mathbb{Z}_2$ or $R=\mathbb{Z}$ and there is no torsion.

Definition 12.25. $A\times B\to R$ is non-singular when the induces $A\to\operatorname{Hom}(B,R)$ and $B\to\operatorname{Hom}(A,R)$ are both isomorphisms.

Corollary 12.26. $H^*(\mathbb{R}P^n;\mathbb{Z}_2)\cong\mathbb{Z}_2[\alpha]/\left(\alpha^{n+1}\right)$ *where* $|\alpha|=1$.

Proof. Induction on n. Let α be the nonzero element in $H^1(\mathbb{R}P^n;\mathbb{Z}_2)\cong\mathbb{Z}_2$. The inclusion map

$$\mathbb{R}P^{n-1}\hookrightarrow\mathbb{R}P^n$$

induces an isomophism on H^i for $1\leq i\leq n-1$.

So by induction H^n is given by $\alpha^i=\underbrace{\alpha\smile\cdots\smile\alpha}_{i}$ for all $1\leq i\leq n-1$.

Consider

$$\begin{array}{ccc} H^1(\mathbb{R}P^n;\mathbb{Z}_2)\times H^{n-1}(\mathbb{R}P^n;\mathbb{Z}_2) & \to & \mathbb{Z}_2 \\ \left(\alpha,\alpha^{n-1}\right) & \mapsto & 1\end{array}$$

because of non-singularity. □

Part VIII
Presentation

13 Simplicial Approximation

Definition 13.1. A map $f:K\to L$ is ***simplicial*** if it sends each simplex of K to a simplex of L by a linear map taking vertices to vertices.

In barycentric coordinates, a linear map of a simplex $[v_0, \cdots, v_n]$ has the form $\sum_i t_i v_i \mapsto \sum_i t_i f(v_i)$. Since a linear map from a simplex to a simplex is uniquely determined by its values on vertices, this means that a simplicial map is uniquely determined by its values on vertices.

It is easy to see that a map from the vertices of K to the vertices of L extends to a simplicial map iff it sends the vertices of each simplex of K to the vertices of some simplex of L.

Definition 13.2. The star $\mathrm{St}\sigma$ of a simplex σ in a simplicial complex X is defined to be the subcomplex consisting of all the simplices of X that contain σ. Closely related to this is the open star $\mathrm{st}\sigma$, which is the union of the interiors of all simplices containing σ, where the interior of a simplex τ is by definition $\tau - \partial\tau$. Thus $\mathrm{st}\sigma$ is an open set in X whose closure is $\mathrm{St}\sigma$.

Lemma 13.3. *For vertices $v_1, \cdots, v_n$ of a simplicial complex X, the intersection $stv_1 \cap \cdots \cap stv_n$ is empty unless $v_1, \cdots, v_n$ are the vertices of a simplex σ of X, in which case $stv_1 \cap \cdots \cap stv_n = st\sigma$.*

We can use this lemma to show the ***Simplicial Approximation Theorem***:

Theorem 13.4. *If K is a finite simplicial complex and L is an arbitrary simplicial complex, then any map $f : K \to L$ is homotopic to a map that is simplicial with respect to some iterated barycentric subdivision of K.*

Proof. Choose a metric on K that restricts to the standard Euclidean metric on each simplex of K. For example, K can be viewed as a subcomplex of a simplex Δ^N whose vertices are all the vertices of K, and we can restrict a standard metric on Δ^N to give a metric on K. Let ε be a Lebesgue number for the open cover $\left\{f^{-1}(\mathrm{st}w)\,|w \text{ is a vertex of } L\right\}$ of K. After iterated barycentric subdivision of K we may assume that each simplex has diameter less than $\varepsilon/2$. The closed star of each vertex v of K then has diameter less than ε, hence this closed star maps by f to the open star of some vertex$g(v)$ of L. The resulting map $g : K^0 \to L^0$ thus satisfies $f(\mathrm{St}v) \subset \mathrm{st}g(v)$ for all vertices v of K. □