MATH7520 Principal G-Bundles and Clasifying Spaces

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Taught by Brian Hwang

Notes by Yun Liu

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Part I

Principal G-bundles

1 Fiber Bundle

1.1 Fiber Bundle

Definition 1.1. A *fiber bundle* is a structure (E, B, π, F), where E, B, and F are topological spaces and $\pi : E \to B$ is a continuous surjection satisfying a local triviality condition: for every $x \in E$, there is an open neighborhood $U \subseteq B$ of $\pi(x)$ such that there is a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ in such a way that π agrees with the projection onto the first factor. That is, the following diagram should commute:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

The set of all $\{(U_i, \varphi_i)\}$ is called a *local trivialization* of the bundle.Thus for any $p \in B$ in B, the preimage $\pi^{-1}(p)$ is homeomorphic to F and is called the *fiber* over p. Every fiber bundle $\pi : E \to B$ is an open map, since projections of products are open maps. Therefore B carries the quotient topology determined by the map π. The space B is called the *base space* of the bundle, E the total space, and F the fiber. The map π is called the projection map (or bundle projection). A fiber bundle (E, B, π, F) is often denoted

$$\begin{tikzcd} F \arrow[dr, hook] & \\ & E \arrow[d, "\pi"] \\ & B \end{tikzcd}$$

Example 1.1. (Trivial bundle) Let $E = B \times F$ and let $\pi : E \to B$ be the projection onto the first factor. Then E is a fiber bundle (of F) over B. Here E is not just locally a product but globally one. Any such fiber bundle is called a *trivial bundle*. Any fiber bundle over a contractible CW-complex is trivial.

Example 1.2. (Covering space) A covering space is a fiber bundle such that the bundle projection is a local homeomorphism. It follows that the fiber is a discrete space.

Definition 1.2. A *fibration* (or *Hurewicz fibration* or *Hurewicz fiber space*) is a continuous mapping $p\colon E \to B$ satisfying the homotopy lifting property with respect to any space, i.e. for any space X and for any homotopy $f\colon X \times [0,1] \to B$, and for any map $\tilde{f}_0\colon X \to E$ lifting $f_0 = f|_{X\times\{0\}}$ (i.e., so that $f_0 = \pi \circ \tilde{f}_0$), there exists a homotopy $\tilde{f}\colon X \times [0,1] \to E$ lifting f, (i.e., so that $f = \pi \circ \tilde{f}$,) which also satisfies $\tilde{f}_0 = \tilde{f}|_{X\times\{0\}}$. In other words, we have the following diagram

$$\begin{tikzcd} X \arrow[r, "\tilde{f}_0"] \arrow[d, "X\times\{0\}"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "f"] \arrow[ur, dotted, "\tilde{f}"] & B \end{tikzcd}$$

Example 1.3. Fibre bundles are important examples of fibrations.

Definition 1.3. A *vector bundle* is a fibre bundle with whose fibers are vector spaces.

Example 1.4. *Tangent* and *cotangent bundles* of smooth manifolds are vector bundles.

Example 1.5. A *sphere bundle* is a fiber bundle whose fiber is an n-sphere. Given a vector bundle E with a metric (such as the tangent bundle to a Riemannian manifold) one can construct the associated unit sphere bundle, for which the fiber over a point x is the set of all unit vectors in E_x. When the vector bundle in question is the tangent bundle TM, the unit sphere bundle is known as the unit tangent bundle, and is denoted UTM.

Fiber bundles often come with a group of symmetries which describe the matching conditions between overlapping local trivialization charts.

Specifically, let G be a topological group which acts continuously on the fiber space F on the right. We lose nothing if we require G to act faithfully on F so that it may be thought of as a group of homeomorphisms of F. A G-atlas for the bundle (E, B, π, F) is a local trivialization such that for any two local trivializations for the overlapping charts (U_i, φ_i) and (U_j, φ_j) the function

$$\varphi_i \varphi_j^{-1}\colon\ (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$$

is given by

$$\varphi_i \varphi_j^{-1}(x, \xi) = (x, t_{ij}(x)\, \xi)$$

where $t_{ij} : U_i \cap U_j \to G$ is a continuous map called a *transition function.* Two G-atlases are equivalent if their union is also a G-atlas. A G-bundle is a fiber bundle with an equivalence class of G-atlases. The group G is called the *structure group* of the bundle.

Definition 1.4. A *bundle map* (or *bundle morphism*) consists of a pair of continuous functions $\varphi \colon E \to F, f \colon M \to N$ such that $\pi_F \circ \varphi = f \circ \pi_E$. That is, the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "\pi_E"'] & F \arrow[d, "\pi_F"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

2 Principal G-bundles

Definition 2.1. A *principal G-bundle* is a fiber bundle $\pi : E \to B$ with a free and transitive (right) action by G on the fibers, such that the local trivializations intertwine the right G-action with right translation (i.e. if $\varphi : \pi^{-1}(U) \to U \times G$ and $h(p) = (u, g)$, then $h(p \cdot g') = (u, gg')$).

Equivalently, E is equipped with a free right G-action such that $\pi : E \to E/G \cong B$ and it satisfies some appropriate local trivialization condistion.

Warning. Principal G-bundles are not just G-fibre bundles. There is no canonical identity on fibre F. More specifically, fibres of principal G-bundles are G-tosors, i.e. principal homogeneous spaces.

Example 2.1. (unit tangent vectors on S^2) $E = \left\{(x, v) \in TS^2 | \, ||v|| = 1\right\} \to S^2$. This is a nontrivial principal S^1-bundle because it contains no global sections.

Example 2.2. (Hopf fibration) $S^3 \to S^2$ is a principal S^1-bundle.

Question. How to tell the difference between the two?

Principal G-bundles are much more rigid than fibre bundles.

Proposition 2.1. *If $E \to B, E' \to B'$ are principal G-bundles and $\phi : E \to E'$ is a principal bundle morphism such that the underlying map is identity $Id_B : B \to B$, then ϕ is an isomorphism.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
& E' \arrow[dr, dashed] \arrow[dd, dashed] & \\
E \arrow[ur, dotted, "\phi"] \arrow[rr] \arrow[dd] & & EG \arrow[dd] \\
& B \arrow[dl, equal, "Id"] \arrow[dr, dashed] & \\
B \arrow[rr] & & BG
\end{tikzcd}$$

Since $\mathrm{Bun}_G(B) \cong [B, BG]$, homotopy classes of maps $B \to BG$ corresponds to isomorphism classes of principal G-bundles on B. $\square$

2.1 Sections of Principal G-bundles

Recall a section of $E \xrightarrow{\pi} B$ over $U \subseteq B$ is a map $s : U \to E$ such that $\pi \circ s = \mathrm{Id}_U : U \to U$ and a trvialization over U is a (continuous) map $h : \pi^{-1}(U) \to U \times G$ which intertwines the right G-action with the right translation.

Proposition 2.2. *There is a bijection*

$$\{\text{sections } s \in \Gamma(U,E)\} \leftrightarrow \left\{\text{trivialization } h : \pi^{-1}(U) \to U \times G\right\}.$$

Proof. Given a section $s \in \Gamma(U,E)$, we can define a trivialization as follows. For any $p \in \pi^{-1}(U)$, let $b = \pi(p) \in U$ and $g = \tau(s(b), p) \in G$ is the element which sends $s(b)$ to p, then we have a map

$$\begin{array}{rccc} h: & \pi^{-1}(U) & \to & U \times G \\ & p & \mapsto & (\pi(p), g) \end{array}$$

which intertwines the G-action $\tau(s(b), p \cdot g') = \tau(s(b), p) \cdot g'$, and the inverse map is given by $(b,g) \mapsto s(b) \cdot g$.

Conversely, given a trivialization $h : \pi^{-1}(U) \to U \times G$ we have a canonical section

$$\begin{array}{rccc} s: & U & \to & E \\ & b & \mapsto & h^{-1}(b,e) \end{array}$$

which is an element in $\Gamma(U,E)$. □

Corollary 2.1. *A principal G-bundle is trivial if and only if it admits global sections.*

Remark 2.1. The key point is, sections $s \in \Gamma(U,E)$ gives a choice of identity on fibre over $b \in U$.

2.2 Associated Bundles

Given a fibre bundle, we can construct a pricipal G-bundle and vice versa.

Given a principal G-bundle, it comes with a group homomorphism $\rho : G \to \mathrm{Aut}(F)$ which is a left G-action on F, then we have a canonical (right) G-action on $E \times F$

$$\begin{aligned} G &\to \operatorname{Aut}(E \times F) \\ g &\mapsto (p, f) \cdot g = \left(p \cdot g, \rho\left(g^{-1}\right) \cdot f\right) \end{aligned}$$

and we will show that the quotient $(E \times F)/G$ is a fibre bundle.

Proposition 2.3. *$(E \times F)/G = E \times^{G} F$ is a fibre bundle over B with fibre F,*

$$\begin{aligned} \pi^{G}: \ E \times^{G} F &\to B \\ [p, f] &\mapsto \pi(p) \end{aligned}$$

this fibre bundle $E \times^{G} F$ is called the associated fibre bundle to $\pi : E \to B$ via the G-action on F.

Proof. First, we want to show that the projection map is well-defined. Let $(p', f') \in [p, f]$, then $p' = p \cdot g, f' = g^{-1} f \cdot f$ for some $g \in G$, then $\pi(p \cdot g) = \pi(p)$ because π is a principal G-bundle.

Second, we need to show that fibres are homeomophic to F. Fix $b \in B$ and choose $p_o \in \pi^{-1}(b)$, we have a continuous map

$$\begin{aligned} F &\to \left(\pi^{G}\right)^{-1}(b) \\ f &\mapsto [p_0, f] \end{aligned}$$

with inverse $[p, f] \mapsto \tau(p_0, p) \cdot f$. Then

$$\begin{aligned} \pi^{-1}(b) \times F &\to F \\ (p, f) &\mapsto \tau(p_0, p) \cdot f \end{aligned}$$

is invariant with respect to the G-action,

$$(p, f) \cdot g = \left(p \cdot g, g^{-1} f\right) = \tau(p_0, p \cdot g) \cdot g^{-1} \cdot f = \tau(p_0, p) \cdot g \cdot g^{-1} \cdot f = \tau(p_0, p) f$$

thus we get the induced map on quotient $\left(\pi^{-1}(b) \times F\right)/G = \left(\pi^{G}\right)^{-1}(b) \xrightarrow{\cong} F$.

Last, we need to show local triviality. Assume without lose of generality, $E = B \times G$ by local trvialization-section correspondence, $(E \times F)/G = (B \times G \times F)/G = \{[b, g, f], b \in B, g \in G, f \in F\}$.

We have isomoprhisms of bundles

$$\begin{aligned} (B \times G \times F)/G &\to B \times F \\ [b, g, f] &\mapsto (b, g \cdot f) \end{aligned}$$

with inverse map

$$\begin{aligned} B \times F &\to (B \times G \times F)/G \\ (b, f) &\mapsto [b, e, f] \end{aligned}$$

□

Proposition 2.4. *Given a fibre bundle $F \hookrightarrow E \xrightarrow{\pi} B$ with fibre F and choice of structure group $G = Aut(F)$, there is a principal $Aut(F)$-bundle P such that $E = P \times^G F$.*

Proof. For $b \in B$, let $P_b =$ set of G-isomorphisms $\phi : F \to \pi^{-1}(b)$, then P_b has a right G-action since for any $g \in G = \mathrm{Aut}(F)$, $\phi \circ g \in P_b$. This is a free and transitive action because if $\phi, \phi' : F \to \pi^{-1}(b)$ are two G-isomorphisms, then $\phi^{-1} \circ \phi' \in \mathrm{Aut}(F) = G$. The bundle is defined as $P = \bigsqcup_{b \in B} P_b \to B$.

Case1: If $E = B \times F$, $\pi^{-1}(b) = \{b\} \times F$, so we have a canonical isomorphism $P_b \cong G$, then $P = \bigsqcup_{b \in B} P_b = \bigsqcup_{b \in B} \{b\} \times G = B \times G$.

Case2: do the above construction for local trivialization.

Then it remains to show that $P \times^G F \cong E$. Note that any element in $P \times^G F$ is an equivalence class $[b, \phi, f]$ where $b \in B, \phi : F \to \pi^{-1}(b), f \in F$, then we have a map

$$\begin{aligned} P \times^G F &\to E \\ [b, \phi, f] &\mapsto \phi(f) \end{aligned}$$

this is well-defined because $\left[b, \phi \cdot g, g^{-1} f\right] \mapsto \phi\left(g g^{-1} f\right) = \phi(f)$. And since ϕ is an isomorphism, this map is a piecewise isomorphism, with inverse given by $p \mapsto [\pi(p), \phi_0, \tau(p, \phi_0(f_0)) f_0]$ where $\phi_0 : F \to \pi^{-1}(b)$ and $f_0 \in F$ are fixed. □

Remark 2.2. We've shown that for any principal G-bundle $\pi : E \to B$, there is an associated fibre bundle $\pi^G : E \times^G F \to B$, and for each fibre bundle $F \hookrightarrow E \xrightarrow{\pi} B$ with fibre F and choice of structure group $G = \mathrm{Aut}(F)$, there is a principal G-bundle whose associated fibre bundle is isomorphic to the one given. In particular, trivial principal G-bundles corresponds to trivial fibre bundles.

Example 2.3. (Vector bundles as principal bundles) If $F = V$ is a vector space and $\rho : G \to GL(V)$ is a linear G-action (e.g. representation) on V, the $P \times^G V$ is the associated fibre bundle. Conversely, each real/complex vector bundle $\pi : E \to B$ is associated to a principal $GL(n, \mathbf{R}) / GL(n, \mathbf{C})$-bundle. By chossing an inner product on F, we can take a principal $O(n) / U(n)$ bundle.

Example 2.4. (Associated self-principal bundle) Let $\rho : G \to \mathrm{Aut}(G)$ be an adjoint representation of G on itself, this does not preserves the group structure of G but it commutes with right multiplication and preserves the G-torsor structure. Given a principal G-bundle $G \hookrightarrow P \xrightarrow{\pi} B$, what is $P \times_\rho^G G$? This is also a principal G-bundle $[p, g] g' = [p, gg']$ and is isomorphic to P. Consider the map

$$\begin{aligned} P \times_\rho^G G &\to P \\ [p, g] &\mapsto p \cdot g \end{aligned}$$

This map is well-defined $\left[p \cdot h, h^{-1} g\right] \mapsto p \cdot hh^{-1} g = p \cdot g$, with inverse $p \mapsto [p.e]$.

(Variant) Let $\phi \in \mathrm{Aut}(G)$ be an automorphism, consider the left action $g \cdot h = \phi(g) h$, what is $P \times_\rho^G G$? This is already a principal G-bundle. We have two cases.

1. If $\phi(g) = \gamma g \gamma^{-1}$ for some $\gamma \in G$ is an inner automorphism, then we have

$$P \times_\phi^G G \xrightarrow{\cong} P, [p, g] \mapsto p\gamma g.$$

This is well-defined $\left[p \cdot h, \gamma h^{-1} \gamma^{-1} g\right] \mapsto ph\gamma\gamma^{-1} h^{-1} \gamma g = p\gamma g$, with inverse

$$P \times_\phi^G G \xleftarrow{\cong} P, \left[p \cdot \gamma^{-1}, e\right] \leftarrow\!\shortmid p.$$

2. If ϕ is not an inner product, $P \times_\rho^G G \ncong P$.

(General) Given a group homomorphism $\phi : G \to H$, we have a left G-action on H, $g \cdot h = \phi(g) h$, so we can form a principal H-bundle $P \times_\phi^G H$.

2.3 Sections of Associated Bundles

Key. relate to equivariant functions on P.

If F has a left G-action, we say $f : P \to F$ is *equivariant* if $f(p \cdot g) = g^{-1} f(p), g \in G$, denote the set of all equivariant maps $P \to F$ by $\mathbf{Map}(P, F)^G$.

Proposition 2.5. *Let $\pi : P \to B$ be a principal G-bundle, and F has a left G-action. Let $E = P \times^G F$. There is a bijection $\Gamma(U, E) = \mathbf{Map}\left(\pi^{-1}(U), F\right)^G$.*

Proof. We use the local section/trivialization correspondence to identify $\pi^{-1}(U) \cong U \times F$.

Given an equivariant map $\tilde{s} : \pi^{-1}(U) \to F$, define a section $s : U \to E$ by

$$b \mapsto [p, \tilde{s}(p)], p \in \pi^{-1}(b).$$

By equivariance, $[p \cdot g, \tilde{s}(p \cdot g)] = \left[p \cdot g, g^{-1}\tilde{s}(p)\right] = [p, \tilde{s}(p)]$, so we obtain a local section.

Conversely, given $s \in \Gamma(U, E)$, suppose $s(\pi(p)) = [p, f]$, then we can define

$$\begin{array}{rccc} \tilde{s}: & \pi^{-1}(U) & \to & F \\ & p & \mapsto & f \end{array}$$

note $\tilde{s}(p \cdot g) = g^{-1} f$ because $s(\pi(p \cdot g)) = s(\pi(p)) = [p, f] = \left[p \cdot g, g^{-1} f\right]$. □

Proposition 2.6. *Fix G, let $\pi : P \to B$ be a principal G-bundle. There is a bijection morphisms $\left(\phi : P \to Q, \bar{\phi} : B \to B'\right)$ of principal bundles and sections of $P \times^G Q$ where Q is a left G-space with action $g \cdot q = qg^{-1}$.*

Proof. Any principal bundle morphism $\phi : P \to Q$ is equivalent to a G-equivariant map $\phi : P \to Q$, so we have

$$\mathbf{Mor}(P, Q) = \mathbf{Map}(P, Q)^G \cong \Gamma\left(B, P \times^G Q\right).$$

□

2.4 Homotopy Classes of Principal G-bundles

Recall we have $\mathbf{Prin}_G(X) = [X, BG]$ obtained by pulling back the universal bundle

$$\begin{tikzcd} P \cong f^*(EG) \arrow[r] \arrow[d, "\pi"'] & EG \arrow[d] \\ X \arrow[r, "f"] & BG \end{tikzcd}$$

Proposition 2.7. *If $\pi : P \to B'$ is a principal G-bundle and if $f_0, f_1 : B \to B'$ are homotopic, then $f_0^*(P) = f_1^*(P)$ as G-bundles over B.*

Proof. Consider the pullback by homotopy $f_t : B \times I \to B'$, it suffices to show that for any principal G-bundle $Q \to B \times I$, the restrictions $Q_0 := Q|_{B\times\{0\}} \to B \times \{0\} \cong B$ and $Q_1 : Q|_{B\times\{1\}} \to B \times \{1\} \cong B$ are isomorphisms and so we can produce an isomorphism $Q \cong Q_0 \times I$ of bundles over $B \times I$,then restricts to $B \times \{1\}$ to get an isomorphism $Q_1 \cong Q_0$.

We want to show that Q and $Q_0 \times I$ are isomorphic as bundles over $B \times I$. Assuming that $Q \to B \times I$ is a principal G-bundle, by Proposition 2.1 it's enough to give a morphsim $Q \to Q_0 \times I$ lying over $\mathrm{Id} : B \times I \to B \times I$. Equivalently, we need to find a section of $Q \times^G (Q_0 \times I) \to B \times I$. Note $Q \times^G (Q_0 \times I)$ has a section over $B \times \{0\}$ because $Q|_{B\times\{0\}}$ and $Q_0 \times I|_{B\times\{0\}} \cong Q_0$ are isomorphic. To extend this section, we use the homotopy lifting property. Since $Q \times^G (Q_0 \times I) \to B \times I$ is a fibration, we have a lift in the following commuting diagram.

$$\begin{tikzcd} B \times \{0\} \arrow[r] \arrow[d] & Q \times^G (Q_0 \times I) \arrow[d] \\ B \times I \arrow[r, "\mathrm{Id}"] \arrow[ur, dotted, "\exists"] & B \times I \end{tikzcd}$$

∎

2.5 Classifying Space BG

Note that we have a functor

$$\begin{array}{ccc} \mathbf{Space} & \longrightarrow & \mathbf{Set} \\ B & \longmapsto & \mathbf{Prin}_G(B) = \mathbf{Bun}_G(G) \\ \left(A \xrightarrow{f} B\right) & \longmapsto & \left(\begin{array}{ccc} \mathbf{Prin}_G(B) & \to & \mathbf{Prin}_G(A) \\ P & \mapsto & f^*P \end{array}\right) \end{array}$$

Since homotopic maps gives isomorphic principal G-bundles, we can look at $\mathbf{Prin}_G : \mathbf{hTop} \to \mathbf{Set}$ where in $\mathbf{hTop}$, morphisms are homotopy classes of maps in $\mathbf{Space}$. We wang to show this contravariant functor is representable.

Definition 2.2. A principal G-bundle $EG \to BG$ is *universal* if the total space EG is contractible.

There are many ways to realize the classifying space.

- Moduli space.
$$[X, BG] = \mathbf{Prin}_G(X) = \mathbf{Bun}_G(X).$$
If the moduli functor is representable, it is represneted by BG.
- Infinitely dimensional realization. For example, line bundles are represented by $\mathbb{C}P^{\infty}$, vector bundles are represented by Grassmannians.
- Simplicial realization.

Lemma 2.1. *If (B, A) is a CW pair and F is a space such that $\pi_k(F) = 0$ if $B \backslash A$ has cells of dimension $k+1$, then every map $f : A \to F$ extends to a map $\tilde{f} : B \to F$ such that $\tilde{f}|_A = f$.*

Proof. Induction on k. We can assume that without lose of generality that f has been extended to the k-skeleton B^k of the pair (B, A). For each $(k+1)$-cell $e^{k+1} \in B$ with attaching map $\phi : \partial I^{k+1} \to B^k$, by assumption, the composition $f \circ \phi : \partial I^{k+1} \to B^k \to F$ is null-homotopic, hence we can extend to $B^k \cup e^{k+1}$. Do the same for each $(k+1)$-cell. □

Corollary 2.2. *If (B, A) is a CW pair and $F \hookrightarrow E \xrightarrow{\pi} B$ is a fibre bundle, and if $\pi_k(F) = 0$ for all k where $B \backslash A$ has $(k+1)$-cells, then every section over A can be extended to global section. In* *particular, if* $(B, A) = (B, \phi)$*, then* $E \to B$ *admits a global section if and only if* F *is* $\dim(B)$*-connected.*

Proof. If $E = B \times F$ is trivial, then a section is equivalent to a map $B \to F$, then we are done by previou lemma.

In general, refine CW structure on B to open coverings corresponds to local trvializations. Namely, we proceed by induction on k, assuming that a section s has been extended to k-skeleton, so $s \in \Gamma\left(B^k, E\right)$. Let $e^{k+1} \subseteq B$ be a $(k+1)$-cell. It may not be in the domain of one trivialization, but then subdivide $e^{k+1} \cong I^{k+1}$ into small cubes, then we can reduce to $e^{k+1} \subseteq U_\alpha$ where $\{U_\alpha\}$ is an open covering of B and $\pi^{-1}(U_\alpha) \cong U_\alpha \times F$. □

Theorem 2.1. *Let* $EG \to BG$ *be a universal bundle, i.e.* EG *is weakly contractible. Then for any CW complex* B*, we have a bijection* $[B, BG] \longleftrightarrow \mathbf{Prin}_G(B)$ *where*

$$\begin{array}{rcl} [-, BG] & \to & \mathbf{Prin}_G(-) \\ [f] & \mapsto & f^*(EG) \end{array}$$

is an equivalence of functors $\mathbf{hTop} \to \mathbf{Set}$.

Proof. Surjectivity. Let $Q \to B$ be a principal G-bundle, then by Corollary 2.2 the associated fibre bundle $Q \times^G EG$ has a global section over B because EG is contractible, thus by Proposition 2.6 we have a morphism of G-bundles

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & EG \arrow[d] \\ B \arrow[r, "f"] & BG \end{tikzcd}$$

Such a moprhism is equivalent to a morphism

$$\begin{tikzcd} Q \arrow[r, "\cong"] \arrow[d] & f^*(EG) \arrow[d] \\ B \arrow[r, equal, "\mathrm{Id}"] & B \end{tikzcd}$$

which is an isomorphism.

Injectivity. Suppose $f_0, f_1 : B \to BG$ gives $f_0^*(EG) \cong f_1^*(EG)$, we want to show that $f_0 \sim f_1$. Consider the principal G-bundle $P = f_0^*(EG) \times I \to B \times I$ such that $p|_{B\times\{0\}} \cong f_0^*(EG) \cong p|_{B\times\{1\}}$.

Consider the G-bundle morphism

$$\begin{tikzcd} P|_{B\times\{0\}} \arrow[r, "\cong"] \arrow[dr] & f_0^*(EG) \arrow[r] \arrow[d] & EG \arrow[d] \\ & B \arrow[r, "f_0"] & BG \end{tikzcd}$$

it corresponds to a section $s_0 \in \Gamma\left(B \times \{0\}, P \times^G EG\right)$. Similarly, if we look over $B \times \{1\}$, we get a local section $s_1 \in \Gamma\left(B \times \{1\}, P \times^G EG\right)$. So we obtain a section $s_0 \cup s_1 \in \Gamma\left(B \times \{0,1\}, P \times^G EG\right)$. Since EG is contractible, this extends to a global section $s \in \Gamma\left(B \times I, P \times^G EG\right)$, which corresponds to a principal bundle morphism

$$\begin{tikzcd} P \arrow[r] \arrow[d] & EG \arrow[d] \\ B \times I \arrow[r, "h"] & BG \end{tikzcd}$$

and induces a map $h : B \times I \to BG$ which is a homotopy from f_0 to f_1. □

Fact 2.1. *[?] Given a topological group G, there exists a universal principal bundle $EG \to BG$.*

Corollary 2.3. *We can take BG to have the structure of a CW complex when BG admits a simplicial realization and such a BG is unique up to homotopy.*

Remark 2.3. We really should think of the equivalence as equivalences between bifunctors $[-, B(\bullet)] \simeq \mathbf{Prin}_\bullet(-)$.

2.6 Classifying Space Functor B

We want to show that the classifying space construction

$$\begin{aligned} B: \ \mathbf{Gp} &\longrightarrow \mathbf{hCW} \\ G &\longmapsto BG \end{aligned}$$

is functorial (up to homotopy). Note there is no canonical construction of BG.

Proposition 2.8. *For each homomorphism $\phi \in Hom_{\mathbf{Gr}}(G, H)$, there exists a homotopy class $B\phi \in [BG, BH]$ such that $B(\phi \circ \psi) = B(\phi) \circ B(\psi)$, $B(Id) = Id$, and $B(G \times H) = BG \times BH$, i.e. B respects products.*

Proof. We have a universal bundle $EG \to BG$. Consider the associated bundle $EG \times_{\phi}^{G} H$ which is a principal H-bundle over BG, thus it correpsonds to a map $B\phi \in [BG, BH]$. We get the composition property because

$$\left(EG \times_{\phi}^{G} H\right) \times_{\psi}^{H} K \cong EG \times_{\psi\phi}^{H} K.$$

Furthermore, $EG \times^{G} G \cong EG$, so $B\left(\mathrm{Id}\right) = \mathrm{Id}$.

For products, note that $EG \times EH$ is a contractible space with a $G \times H$-action,

$$\left(EG \times EH\right)/G \times H \cong BG \times BH.$$

□

Proposition 2.9. *Let $i : H \hookrightarrow G$ be a subgroup such that $G \to G/H$ is a principal H-bundle (e.g. H is a closed subgroup such that G/H is a homogeneous space). Then we can view $Bi : BH \to BG$ as a fibre bundle with fibre G/H.*

Proof. Under our hypothesis, EG is a contractible space with a free (right) H-action, so EG/H is a classifying space of H. Furthermore, EG is universal bundle of H.

We have $EG/H \cong \left(EG \times^{G} G\right)/H \cong EG \times^{G} \left(G/H\right)$, thus we have a morphism of principal G-bundles

$$\begin{tikzcd}
EG \arrow[r, "\mathrm{Id}"] \arrow[d] & EG \arrow[d] \\
BH = EG/H \arrow[r] & BG = EG/G
\end{tikzcd}$$

so the induced map can be identified with $BH \cong EG \times^{G} \left(G/H\right) \to BG$, this is a G/H-fibre bundle. □

Proposition 2.10. *If $\phi : G \to G$ is an inner automorphism, then $B\phi : BG \to BG$ is homotopic to Id.*

Proof. It follows from the fact that $EG \times^{G} G = EG$. □

2.7 Grassmannians and Classifying Spaces

Let V be a finite-dimensional vector space over a field F, say $\mathbf{R}$, or $\mathbf{C}$. Let $0 \le k \le \dim V$.

Definition 2.3. The Grassmannians are defiend as $\mathrm{Gr}_k(V) = \{W \subseteq V | \dim W = k\}, \mathrm{Gr}^k(V) = \{W \subseteq V | \dim W + k = \dim V\}$.

Grassmannians can be viewed as smooth manifolds, or algebraic varieties.

Over $\mathrm{Gr}_k(V)$ we have a tautological exact sequence $0 \to S \to V \to Q \to 0$ of vector bundles.

- fibre of universal subbundle $S_W \cong W, W \in \mathrm{Gr}_k(V)$.
- fibre of universal quotient bundle $Q_W \cong V/W$.
- $V_W \cong V$.

Fact 2.2. *Any vector bundle of rank k $\pi : E \to M$ for M compact can be expressed as pullback of the universal quotient bundle/ subbundle over $Gr^k(V)/Gr_k(V)$, i.e. there exists finite dimensional vector space V and smooth maps*

$$\begin{tikzcd} E \arrow[r] \arrow[d] & Q \arrow[d] \\ M \arrow[r, "f"] & Gr^k(V) \end{tikzcd} \quad \Bigg| \quad \begin{tikzcd} E \arrow[r] \arrow[d] & Q \arrow[d] \\ M \arrow[r, "f"] & Gr_k(V) \end{tikzcd}$$

Proof. Since $E \to M$ is locally trivial and M is compact, there exists finite open cover $\{U_\alpha\}_{\alpha \in A}$ of M and basis $s_1^\alpha, \cdots, s_k^\alpha \in \Gamma(U_\alpha, E)$ of local trivializations. Let $\{\rho^\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$, then $\tilde{s}_i^\alpha = \rho^\alpha s_i^\alpha \in \Gamma(M, E)$ which vanishes outside U_α. Define a finite dimensional vector space $V = \mathrm{Span}_F \{\tilde{s}_i^\alpha\}_{\alpha \in A, 1 \le i \le k}$, the for any $p \in M$, there is a linear map

$$\begin{array}{rccc} \mathrm{ev}_p: & V & \to & E_p \\ & \tilde{s}_i^\alpha & \mapsto & \tilde{s}_i^\alpha(p) \end{array}$$

which is surjective. Thus $V/\mathrm{Ker}(\mathrm{ev}_p) \cong E_p$. By taking the inverse of this isomorphism, we obtain a map $\tilde{f} : E \to Q$ with

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d] & Q \arrow[d] & \\ M \arrow[r, "f"] & \mathrm{Gr}^k(V) & p \arrow[r, mapsto] & \mathrm{Ker}(\mathrm{ev}_p) \end{tikzcd}$$

$\square$

There are natural questions as follows.

Quetsion. What if M is not compact? What about all vector bundles instead of just k-dimensional vector bundles?

It motivates us to study infinite Grassmannians.

Infinite Grassmannians

Given an integer $k \geq 1$, consider the sequence of clsed inclusions

$$\begin{array}{ccccccc} \mathbf{R}^q & \hookrightarrow & \mathbf{R}^{q+1} & \hookrightarrow & \mathbf{R}^{q+2} & \hookrightarrow & \cdots \\ (x_1, x_2, \cdots) & \mapsto & (0, x_1, x_2, \cdots) & \mapsto & (0, 0, x_1, x_2, \cdots) & & \end{array}$$

which gives corresponding closed inclusions

$$\begin{array}{ccccccc} \mathrm{Gr}_k\left(\mathbf{R}^q\right) & \hookrightarrow & \mathrm{Gr}_k\left(\mathbf{R}^{q+1}\right) & \hookrightarrow & \mathrm{Gr}_k\left(\mathbf{R}^{q+2}\right) & \hookrightarrow & \cdots \\ W & \mapsto & 0 \oplus W & \mapsto & 0 \oplus 0 \oplus W & & \end{array}$$

and

$$\begin{array}{ccccccc} \mathrm{Gr}^k\left(\mathbf{R}^q\right) & \hookrightarrow & \mathrm{Gr}^k\left(\mathbf{R}^{q+1}\right) & \hookrightarrow & \mathrm{Gr}^k\left(\mathbf{R}^{q+2}\right) & \hookrightarrow & \cdots \\ W & \mapsto & \mathbf{R} \oplus W & \mapsto & \mathbf{R}^2 \oplus W & & \end{array}$$

We can get a corresponding sequence of pullback maps of universal quotient bundle

$$\begin{tikzcd} Q_q \arrow[r] \arrow[d] & Q_{q+1} \arrow[r] \arrow[d] & Q_{q+2} \arrow[r] \arrow[d] & \cdots \\ \mathrm{Gr}_k\left(\mathbf{R}^q\right) \arrow[r] & \mathrm{Gr}_k\left(\mathbf{R}^{q+1}\right) \arrow[r] & \mathrm{Gr}_k\left(\mathbf{R}^{q+2}\right) \arrow[r] & \cdots \end{tikzcd}$$

By taking colimit of this diagram we get a vector bundle $\pi : Q^{\mathrm{univ}} \to B_k$. B_k is a topological space, but not a manifold.

Question. If we want to make $\pi : Q^{\mathrm{univ}} \to B_k$ continuous, what is the best topology to impose here?

Fact 2.3. *If X is metrizable and $\pi : E \to X$ is a vector bundle over X, then there exists a classfying* *diagram*

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & Q^{univ} \arrow[d] \\ X \arrow[r, "f"] & B_k \end{tikzcd}$$

Question. What does it mean for Q^{univ} to be contractible?

One model for B_k is Hilbert manifold, a manifold modeled on Hilbert spaces. Thus it is a separable Hausdorff space in which each point has a neighbourhood homeomorphic to an infinite dimensional Hilbert space.

Let $\mathcal{H}$ be a seperable Hilbert space over $\mathbf{R}$ or $\mathbf{C}$. The topological on $\mathcal{H}$ is induced by inner product $\langle -, - \rangle$ on $\mathcal{H}$. Given an integer $k \geq 1$, $\mathrm{Gr}_k(\mathcal{H}) = \{W \subseteq H | \dim W = k\}$. Usually $\mathrm{Gr}^k(\mathcal{H}) = \{W \subseteq H | \dim H/W = k\}$ is called a Hilbert manifold, or Grassmannian. Since $\mathcal{H}$ is separable, we can pick an orthogonal basis $\{e_1, \cdots, e_n, \cdots\}$ of $\mathcal{H}$. Define $\mathbf{R}^q \subseteq \mathcal{H}$ by $\mathbf{R}^q \cong \mathrm{Span}_{\mathbf{R}}\{e_1, \cdots, e_q\}$, then we have a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{Gr}^k(\mathbf{R}^q) \arrow[rr] \arrow[dr] & & \mathrm{Gr}^k\left(\mathbf{R}^{q+1}\right) \arrow[r] \arrow[dl] & \cdots \\ & & \mathrm{Gr}^k(\mathcal{H}) & & \end{tikzcd}$$

taking colimit we can get a map $i : B_k \to \mathrm{Gr}^k(\mathcal{H})$.

Proposition 2.11. *The map $i : B_k \to Gr^k(\mathcal{H})$ is a homotopy equivalence.*

Fact 2.4. *Let $F \hookrightarrow E \xrightarrow{p} B$ be a fibre bundle such that F is contractible and metrizable topological manifold. If B is metrizable, then π admits a section.*

Corollary 2.4. *Given the conditions above, and suppose that F, E, B are all homotopic to CW complexes, then π is a homotopy equivalence.*

Proof. As F is contractible, the long exact sequence in homotopy groups shows that $p_n : \pi_n(E) \to \pi_n(B)$ is always an isomorphism. The Whitehead theorem (that a weak homotopy equivalence of CW-complexes is an homotopy equivalence) now shows that p is a homotopy equivalence. □

Let $\mathcal{H}$ be a Hilbert space, we want to understand how to get a universal $GL_k(\mathbf{R})$-bundle over its classifying space.

Consider the *Stiefel manifold* $St_k(\mathcal{H}) = \left\{(e_1, \cdots, e_k) \in \mathcal{H}^k | \langle e_i, e_j \rangle = \delta_{ij}, \forall 1 \leq i, j \leq k\right\}$ which is the set of all orthonormal k-frames in $\mathcal{H}$, this is an infinite dimensional vector space which is a Hilbert manifold. We have a projection map

$$\begin{array}{cccc} \pi : & St_k(\mathcal{H}) & \to & Gr_k(\mathcal{H}) \\ & (e_1, \cdots, e_n) & \mapsto & \mathrm{Span}\{e_1, \cdots, e_n\} \end{array}$$

which is smooth (involves work) and is a principal $GL_k(\mathbf{R})$-bundle.

Theorem 2.2. *$St_k(\mathcal{H})$ is contractible.*

Corollary 2.5. *$\pi : St_k(\mathcal{H}) \to Gr_k(\mathcal{H})$ is a universal $GL_k(\mathbf{R})$-bundle.*

Remark 2.4. Usually theorem can be proved like this for complete, metrizable, locally convex vector spaces. Alternatively, we can prove it via an induction argument for $St_k(\mathbf{R}^\infty)$ with a colimit topology.

Part II

Classifying Spaces

3 Construction of Universal Bundles

In previous lectures, we have shown that for given topological group G and topological space X, the homotopy classes of principal G-bundles over X can be uniquely determined by a map $f : X \to BG$ from X to the classifying space of G, i.e. $\mathbf{Prin}_G(X) = [X, BG]$ can be obtained by pulling back the universal bundle

$$\begin{array}{ccc} P \cong f^*(EG) & \longrightarrow & EG \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ X & \xrightarrow{f} & BG \end{array}$$

We ssen that classifying space characterizes the principal G-bundles on X, so it is appealing to know how to construct BG and whether such construction is canonical.

Question1. How to construct a classifying space of G, and in particular, when will this construction be a CW complex? Is this construction unique (up tp homotopy)?

Question2. For any CW complex X, when will it be the classifying space for some group G? Is such group unique?

3.1 Mail Result

Definition 3.1. A topological group G is called a *countable CW-group* if G is a countable CW complex such that the inverse map $G \to G, g \mapsto g^{-1}$ and the product map $G \times G \to G, (f, g) \mapsto fg$ are both cellular (i.e. carry the k-skeleton into the k-skeleton).

Theorem 3.1.

1. *Any countable CW-group G has a countable CW complex X as classifying space. If there is another CW complex X' which is a classifying space for G, then X, X' are of the same homotopy type.*

2. *Any countable connected CW complex X is the classifying space for some countable CW-group G. If there is another countable CW-group G' with the same classifying space X, then $G' = G$.*

3.2 Topology of Joins

Definition 3.2. The *join* $A_1 \circ \cdots \circ A_n$ of n topological spaces is defined with the following data:

- n real numbers $t_i, 1 \leq i \leq n$ satisfying $t_i \geq 0, \sum t_i = 1$, and
- a point $a_i \in A_i$ for each i if $t_i \neq 0$.

Such a point in $A_1 \circ \cdots \circ A_n$ will be denoted by $t_1 a_1 \oplus \cdots \oplus t_n a_n$ where the element a_i may be chosen randomly or omitted whenever the corresponding $t_i = 0$.

Definition 3.3. The *strong topology* in $A_1 \circ \cdots \circ A_n$ is the initial topology with respect to the coordinate functions

$$t_i : A_1 \circ \cdots \circ A_n \to [0,1] \quad \text{and} \quad a_i : t_i^{-1}(0,1] \to a_i \ ,$$

i.e. any $f : X \to A_1 \circ \cdots \circ A_n$ is continuous if and only if the compositions $\left\{ t_i f : X \to [0,1], a_i f : X_i \to A_i | X_i = f^{-1} (\right.$ are continuous.

Remark 3.1. The join of infinitely many topological spaces in the strong topology can be defined in the same manner, with the restriction that all but a finite number of t_i should vanish.

This operation of finite/infinite joins is associative and commutative (up to isomorphism).

Proposition 3.1. *Let $1 \leq i \leq n$. There is a canonical homeomorphism*

$$\alpha : (A_1 \circ \cdots \circ A_r) \circ (A_{r+1} \circ \cdots \circ A_n) \to A_1 \circ \cdots \circ A_n.$$

Proof. The map is given by

$$\begin{array}{cccc} \alpha : & (A_1 \circ \cdots \circ A_r) \circ (A_{r+1} \circ \cdots \circ A_n) & \to & A_1 \circ \cdots \circ A_n \\ & s\left(t_1 a_1 \oplus \cdots \oplus t_r a_r\right) \oplus (1-s)\left(t_{r+1} a_{r+1}\right) & \mapsto & s t_1 a_1 \oplus \cdots s t_r a_r \oplus (1-s)\, t_{r+1} \oplus \cdots \oplus (1-s)\, t_n \end{array}$$

with inverse

$$\begin{array}{llll}\beta: & A_1 \circ \cdots \circ A_n & \to & (A_1 \circ \cdots \circ A_r) \circ (A_{r+1} \circ \cdots \circ A_n) \\ & t_1a_1 \oplus \cdots \oplus t_na_n & \mapsto & \left(\sum_{i=1}^{r} t_i\right) \left(\frac{t_1}{\sum_{i=1}^{r} t_i} a_1 \oplus \cdots \oplus \frac{t_r}{\sum_{i=1}^{r} t_i} a_r \right) \oplus \sum_{i=r}^{n} t_i \left(\frac{t_{r+1}}{\sum_{i=1}^{r} t_i} a_{r+1} \oplus \cdots \oplus \frac{t_n}{\sum_{i=r}^{n} t_i} a_n \right)\end{array}$$

which is well-defined because when $\sum_{i=1}^{r} t_i = 0$, the image is of the form $0 \oplus \cdots \oplus 0 \oplus t_{r+1}a_{r+1} \oplus \cdots \oplus t_na_n$, and $\sum_{i=1}^{r} t_i = 1$, the image is of the form $t_1a_1 \oplus \cdots \oplus t_ra_r \oplus 0 \oplus \cdots \oplus 0$. □

A subbasis of $A_1 \circ \cdots \circ A_n$ is given by two types of sets:

1. $\{t_1a_1 \oplus \cdots \oplus t_na_n | \alpha < t_i < \beta\}$, the set of all $t_1a_1 \oplus \cdots \oplus t_na_n$ such that $\alpha < t_i < \beta$.

2. $\{t_1a_1 \oplus \cdots \oplus t_na_n | t_i \neq 0, a_i \in U\}$, the set of all $t_1a_1 \oplus \cdots \oplus t_na_n$ suhc that $t_i \neq 0$ and $a_i \in U$ where U is an open set in A_i.

Lemma 3.1. *The reduced singular homology groups of the join $A \circ B$ with coefficient in a principal ideal domain are given by*

$$\tilde{H}_{r+1}(A \circ B) \cong \sum_{i+j=r} \tilde{H}_i(A) \otimes \tilde{H}_j(B) + \sum_{i+j=r-1} Tor\left(\tilde{H}_i(A), \tilde{H}_j(B)\right).$$

Proof. We have

$$A \circ B \cong (C(A) \times B) \bigcup_{A \times B} (A \times C(B))$$

so the triad $(A \circ B, C(A) \times B, A \times C(B))$ gives us the reduced Mayer-Vietoris sequence

$$\cdots \to \tilde{H}_{n+1}(A \circ B) \xrightarrow{\partial_*} \tilde{H}_n(A \times B) \xrightarrow{(i_*, j_*)} \tilde{H}_n(A) \oplus \tilde{H}_n(B) \xrightarrow{k_* - l_*} \tilde{H}_n(A \circ B) \xrightarrow{\partial_*} \tilde{H}_{n-1}(A \times B) \to \cdots \to \tilde{H}_0(A)$$

since the inclusion $k : A \times C(B) \to A \circ B$ and $l : C(A) \times B \to A \circ B$ are null-homotopic, this long exact sequence splits into short exact sequences

$$0 \to \tilde{H}_{n+1}(A \circ B) \xrightarrow{\partial_*} \tilde{H}_n(A \times B) \xrightarrow{(i_*, j_*)} \tilde{H}_n(A) \oplus \tilde{H}_n(B) \to 0.$$

Since the singular complex $S_*(A \times B)$ and $S_*(A) \otimes S_*(B)$ has the same homology group (theorem of Eilenberg and Zilber), and the later is given by the Künneth theorem (over PID),

$$0 \to \bigoplus_{i+j=k} H_i(S_*(A)) \otimes_R H_j(S_*(B)) \to H_k(S_*(A) \otimes S_*(B)) \to \bigoplus_{i+j=k-1} \mathrm{Tor}_1^R(H_i(S_*(A)), H_j(S_*(B))) \to 0.$$

By computing $\ker(i_*, j_*) \cong \tilde{H}_{n+1}(A \circ B)$ we get our desired result. □

Lemma 3.2. *If B is path connected and A is nonempty, then $A \circ B$ is simply connected.*

Proof. Note any $[f] \in \pi_1(A \circ B)$ is represented by a map

$$\begin{array}{rccc} f: & S^1 & \to & A \circ B \\ & s & \mapsto & t(s)\,a(s) \oplus (1 - t(s))\,b(s) \end{array}$$

where $t : S^1 \to [0,1]$, and $a(s)$ is defined when $t(s) \neq 0$ and $b(s)$ is defined when $t(s) \neq 1$. We will show that f is null-homotopic.

Since B is path-connected, we can define a map $p : S^1 \to B$ such that $p(s) = b(s)$ whenever $t(s) \leq \frac{1}{2}$. Fix $a_0 \in A$. Define a map

$$\begin{array}{rccc} t: & S^1 \times [0,2] & \to & [0,1] \\ & (s,t) & \mapsto & \begin{cases} \min\{1, (1+u)\,t(s)\} & 0 \leq u \leq 1 \\ (2-u)\,t(s,1) & 1 \leq u \leq 2 \end{cases} \end{array}$$

and our homotopy is given by

$$\begin{array}{rccc} F: & S^1 \times [0,3] & \to & A \circ B \\ & (s,t) & \mapsto & \begin{cases} t(s,u)\,a(s) \oplus (1 - t(s,u))\,b(s) & 0 \leq u \leq 1 \\ t(s,u)\,a(s) \oplus (1 - t(s,u))\,p(s) & 1 \leq u \leq 2 \\ (u-2)\,a_0 \oplus (3-u)\,p(s) & 2 \leq u \leq 3 \end{cases} \end{array}$$

where $F(s,0) = f(s)$ and $F(s,3) = (a_0 \oplus 0)$. □

We call a nonempty space (-1)-connected.

Lemma 3.3. *The join of $n+1$ nonempty spaces is always $(n-1)$-connected. In fact, if A_i is (c_i-1)-connected, then $A_0 \circ \cdots \circ A_n$ is $(c_0+\cdots+c_n+n-1)$-connected.*

Proof. Induction on n by applying Lemma 3.2. It suffices to prove for $n=1$.

When $c_0 = c_1 = 0$, there is a path between any point $t_0a_0 \oplus \cdots \oplus t_na_n$ with $t_i \neq 0$ and $x_i = 0 \oplus \cdots \oplus 0 \oplus a_i \oplus 0 \oplus \cdots \oplus 0$ given by

$$\begin{array}{rrcc} p: & I & \to & A_0 \circ \cdots \circ A_n \\ & t & \mapsto & (1-t)\,t_0a_0 \oplus \cdots \oplus (t_i+t-tt_i)\,a_i \oplus \cdots \oplus (1-t)\,t_na_n \end{array}$$

and each x_i is connected to x_j for $i \neq j$ via

$$\begin{array}{rrcc} p: & I & \to & A_0 \circ \cdots \circ A_n \\ & t & \mapsto & 0 \oplus \cdots \oplus ta_i \oplus \cdots \oplus (1-t)\,a_j \oplus \cdots \oplus 0 \end{array}$$

hence the join of any two nonempty spaces is path-connected, so the statement holds.

When $c_0 > 0$ or $c_1 > 0$, Lemma 3.2 implies $A_0 \circ A_1$ is connected, and Lemma 3.1 implies $H_r(A_0 \circ A_1) = 0, \forall r \leq c_0 + c_1$, so $A_0 \circ A_1$ is (c_0+c_1)-connected. □

Corollary 3.1. *The join of infinitely many nonempty spaces is always ∞-connected.*

Proof. Let $A = A_0 \circ A_1 \circ \cdots$ be a join of infinitely many nonempty spaces. We will prove by induction that $\pi_n(A) = 0, \forall n \geq 0$.

When $n = 0$, A is path-connected because $A = A_0 \circ (A_1 \circ \cdots)$ and by Lemma 3.3, A is path-connected.

Assume that A is $(n-1)$-connected, then $A = (A_0 \circ A_1 \circ \cdots \circ A_{n+2}) \circ A'$ where A' is a join of infinitely many nonempty spaces, so by induction hypothesis, A' is $n-1$-connected. Note $A_0 \circ \cdots \circ A_{n+2}$ is $(n+1)$-connected, so Lemma 3.3 implies A is $(2n)$-connected, and thus A is n-connected. □

3.3 Construction of BG

Definition 3.4. An *n-universal bundle* is a principal fibre bundle such that the bundle space is $(n-1)$-connected.

Given any topological group G, let $E_n = G \circ \cdots \circ G$ be the join of $n+1$ copies of G in the strong topology. Define the right translation

$$\begin{array}{rccc} R: & E_n \times G & \to & E_n \\ & (t_0 g_0 \oplus \cdots \oplus t_n g_n, g) & \mapsto & t_0 (g_0 g) \oplus \cdots \oplus t_n (g_n g) \end{array}$$

Let $X_n = E_n / G$ and $p : E_n \to X_n$ be tha canonical projection.

Theorem 3.2. *G is the group of the n-universal bundle $p : E_n \to X_n$.*

Proof. E_n is $(n-1)$-connected by 3.3. The bundle structure is defined as follows.

Let $V_j = \{p (t_0 g_0 \oplus \cdots \oplus t_n g_n) \in X_n | t_j \neq 0\}$ be an open cover X_n. Define a map

$$\begin{array}{rccc} \phi_j : & V_j \times G & \to & p^{-1} (V_j) \\ & (p (t_0 g_0 \oplus \cdots \oplus t_n g_n), g) & \mapsto & t_0 \left(g_0 g_j^{-1} g\right) \oplus \cdots \oplus t_n \left(g_n g_j^{-1} g\right) \end{array}$$

this is well-defined,

$$(p (t_0 (g_0 h) \oplus \cdots \oplus t_n (g_n h)), g) \mapsto t_0 (g_0 h) (g_j h)^{-1} g \oplus \cdots \oplus t_n (g_n h) (g_j h)^{-1} g = t_0 \left(g_0 g_j^{-1} g\right) \oplus \cdots \oplus t_n \left(g_n g_j^{-1} g\right)$$

with inverse

$$\begin{array}{rccc} \varphi_j = (p, p_j) : & p^{-1} (V_j) & \to & V_j \times G \\ & t_0 g_0 \oplus \cdots \oplus t_n g_n & \mapsto & (p (t_0 g_0 \oplus \cdots \oplus t_n g_n), g_j) \end{array}$$

The definition of the strong topology in the join shows that R and φ_j are continuous. Let $e \in G$ be identity, then

$$\phi_j (p (y), e) = R \left(e, p_j (y)^{-1}\right)$$

shows that $\phi_j (p (y), e)$ is continuous in y, so $\phi_j (x, e)$ is continuous in x, and hence $\phi_j (x, g) = R (\phi_j (x, e), g)$ is continuous.

Last, note that the right G-action commutes with the coordinate maps. The G-action on E_n is free and transitive. □

Remark 3.2. An ∞-universal bundle $p : E_\infty \to X_\infty$ can be constructed in the same way.

There is a natural embedding of $E_n \hookrightarrow E_{n+1}$ given by $x \mapsto x \oplus 0$, which gives a map betweem principal G-bundles and thus we have the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & E_{n-1} \arrow[r] \arrow[d] & E_n \arrow[r] \arrow[d] & E_{n+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & X_{n-1} \arrow[r] & X_n \arrow[r] & X_{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

and taking the colimit of this diagram we get $p : E_\infty \to X_\infty$. The topology on E_∞ is the colimit topology, i.e. the terminal topology with respect to the upper sequence of inclusions, equivalently, any $f : E_\infty \to Y$ is continuous if and only if the compositions $f \circ \varphi_n : E_n \to E_\infty \to Y$ is continuous. The topology on X_∞ is the quotient topology.

Claim 3.1. The space E_∞ is weakly contractible.

Proof. This is an immediate result of Corollary 3.1. □

3.4 Countable CW-groups

Theorem 3.3. *Every countable CW-group G is the group of an ∞-universal bundle for which the base space X_∞ is a countable CW complex.*

Remark 3.3. The continuity of $R : E_n \times G \to E_n$ requires the fact that E_n is a countable CW complex.

Proof. Let E_n be the join of $(n+1)$ copies of G in the weak topology, then we can construct $p : E_n \to X_n$ in the same way.

CW structure for E_n: induction on n. $E_0 = G$ same as G. For $E_n = (E_{n-1} \circ G)$, cells are of the form

$$\tau \circ \emptyset, \emptyset \circ \sigma, (\tau \circ e)\, \sigma = \{R\,(tx \oplus (1-t)\, e, g)\, |x \in \tau, g \in \sigma, t \in (0,1)\}$$

where τ is a generic cell in E_n and σ is a generic cell in G, and $\emptyset$ is the emptyset.

CW structure for X_n: for $X_n = p\,(E_{n-1} \circ G)$, cells are

1. cells in X_{n-1}, and
2. $p\,(\emptyset \circ e)$, and
3. $p\,(\tau \circ e)$ where τ is a cell in E_{n-1}.

Then X_n is a countable CW complex with respect to this subdivision.

The union E_∞, X_∞ of $\{E_n, X_n\}$ can be topologized as CW complexes and the map $p_\infty : E_\infty \to X_\infty$ is the projection map of an ∞-universal bundle. □

3.5 Universal Bundle of Base Space X

Theorem 3.4. *For any countable, connected simplicial complex X in the weak topology, there exists an ∞-universal bundle with base space X, and the bundle space and group being CW complexes.*

The standard construction of contractible fibre space over a space is based on X^I of paths in X. Our construction here is based on a similar space $\tilde{S}$ of simplicial paths in X.

Lemma 3.4. *[HA] The product of two countable CW complexes is a CW complex.*

Let $S_n = \left\{(x_n, x_{n-1}, \cdots, x_0) \in X^{n+1} || x_i, x_{i-1} \text{ lies in a common simplex of } X\right\} \subseteq X^{n+1}$.

Let $S = \bigsqcup_{n\geq 1} S_n$.

Let $\tilde{S} = S/\sim$ where $(x_n, \cdots, x_i, \cdots, x_0) \sim (x_n, \cdots, \hat{x}_i, \cdots, x_0)$ if either $x_i = x_{i-1}$ or $x_{i+1} = x_{i-1}$.

Fix $v_0 \in X$. Take $\tilde{E} = \left\{[x_n, \cdots, x_0] \in \tilde{S} | x_0 = v_0\right\}$ and

$$\begin{array}{rccc} p: & \tilde{E} & \to & X \\ & [x_n, \cdots, x_1, v_0] & \mapsto & x_n \end{array}$$

Take the fibre $\tilde{G} = p^{-1}(v_0) \left\{[x_n, \cdots, x_0] \in \tilde{S} | x_n = x_0 = v_0\right\} \subseteq \tilde{E}$.

A product on elements of $\tilde{S}$ is defined as follows: if $[x_n, \cdots, x_0]$ and $[y_m, \cdots, y_0]$ satisfy $x_0 = y_m$, then

$$[x_n, \cdots, x_0] \cdot [y_m, \cdots, y_0] = [x_n, \cdots, x_0, y_m, \cdots, y_0]$$

this multiplication is well-defined and associative.

Lemma 3.5. *The space $\tilde{S}$ can be given a structure of a CW-complex, with subcomplexes $\tilde{E}, \tilde{G}$.*

Let $D \subseteq S$ be the subset of all degenerate $(x_n, \cdots, x_0)$ in the sense that either $x_i = x_{i-1}$ or $x_{i-1} = x_{i+1}$.

Fact 3.1. *Every element of S is equivalent to a unique nondegenerate element in $S\backslash D$.*

Proof. Define a map $\mu : S \to S$ by

$$\mu\left(x_n, \cdots, x_0\right) = \begin{cases} \left(x_n, \cdots, x_0\right) & \text{nondegenerate} \\ \left(x_n, \cdots, \hat{x}_i, \cdots, x_0\right) & i = \max\left\{x_i = x_{-1} \vee x_{i+1} = x_{i-1}\right\} \end{cases}$$

Let $\nu : S \to S\backslash D$ be the map by applying μ until a nondegenerate point is obtained. Then $\nu(s) = \nu(s')$ whenever $s \sim s'$. □

Fact 3.2. *D is a subcomplex of S. In other words, if $x \in \sigma$ a simplex in S is a nondegenerate point, then $\forall y \in \sigma^o$, y is nondegenerate.*

Fact 3.3. *If σ is a simplex in S, then there is a unique map $\sigma \mapsto \sigma'$ where σ' is nondegenerate which maps points onto equivalent points.*

Lemma 3.6. *Let A be a simplicial complex and $\Delta_{ij} = \{(a_1, \cdots, a_n) \in A^n | a_i = a_j\}$. Then Δ_{ij} is a subcomplex of the first derived complex of A^n.*

Proposition 3.2. *Given a (not necessarily Hausdorff) space A and a collection of maps $f : \sigma^n \to A$ where each σ^n is a closed n-cell, and let $e^n = f(\sigma^n)^0$, $A^n = \bigcup_{i \leq n} e^i$. If the following conditions are satisfied, then A, together with cells $\{e^n\}$ form a CW complex.*

1. *$(\sigma^n)^0$ maps 1-1 onto correpsonding e^n, every point of A belongs to exactly one e^n.*
2. *$f(\partial\sigma^n) \subseteq A^{n-1}$.*
3. *A subset of $A^n, 0 \leq n \leq \infty$ is closed if and only if its inverse image in each cell σ^i is closed.*

Now we want to show that $\tilde{S}$ has a CW structure. The cells in $\tilde{S}$ are defined as the image of the interiors of the nondegenerate simplexes in S.

By applying the proposition, we see that condition (1) and (2) are satisfied, and for (3), consider any subset C of the n-skeleton of $\tilde{S}$ such that the inverse image in each nondegenerate simplex of dim $\leq n$ is closed, we need to show that its inverse image in any simplex is closed. This is true for any j-simplex $\sigma^j, j \leq n$, and for $j > n$, the inverse image contains no interior point. So we only need to consider the boundary and apply indcution hypothesis. Since S has the weak topology, the inverse image of C in S is a closed subset. Since $\tilde{S}$ is the quotient space, C is closed.

Lemma 3.7. *The projection map $p : \tilde{E} \to X$ is continuous.*

Proof. Let $\eta : S \to \tilde{S}$ be the quotient map and $\bar{p} : \tilde{S} \to X, [x_n, \cdots, x_0] \mapsto x_n$, since the compostion $\bar{p}\eta : S \to X$ is continuous, p is continuous. □

Lemma 3.8. *The product operation is continuous.*

Proof. Let Δ be the subcomplex of $\tilde{S}\times\tilde{S}$ consisting of $[x_n, \cdots, x_0], [y_m, \cdots, y_0]$ such that $x_0, y_m \in \sigma$ lies in the same simplex. Need to show $\Delta \to \tilde{S}$ is continuous, it suffices to verify on each closed cell of Δ. A cell in Δ is of the form $\eta(\sigma) \times \eta(\tau)$ with σ, τ nongenenerate, the composition $\sigma \times \tau \to \Delta \to \tilde{S}$ is the same as $\sigma \times \tau \to S \to \tilde{S}$ which is continuous. □

Remark 3.4. This follows from the fact that any CW complex is locally compact, and therefore the operations of taking quotients and products commute.

Every element $[x_n, \cdots, x_0]$ has an inverse $[x_n, \cdots, x_0]^{-1} = [x_0, \cdots, x_n]$ such that $[x_n, \cdots, x_0]^{-1} \cdot [x_n, \cdots, x_0] = [x_0, x_0]$.

Proposition 3.3. *$\tilde{G}$ is a topological group with identity $[v_0, v_0]$ with the above multiplication.*

The bundle structure is defined as follows. Let V_j be the star neighborhood of the j-th vertex $v_j \in X$. Fix $e_j = [v_j, x_{n-1}, \cdots, x_1, v_0] \in p^{-1}(v_j)$, we have coordinate map

$$\begin{array}{rccc} \phi_j: & V_j \times \tilde{G} & \to & p^{-1}(V_j) \\ & (x, g) & \mapsto & [x, v_j] \cdot e_j \cdot g \end{array}$$

which is continuous with inverse

$$\begin{array}{rccc} \varphi_j = (p, p_j): & p^{-1}(V_j) & \to & V_j \times \tilde{G} \\ & e & \mapsto & \left(p(e), e_j^{-1} \cdot [v_j, p(e)]\, e\right) \end{array}$$

which is also continuous. The transition map is

$$\begin{array}{rccc} g_{ij}: & V_i \cap V_j & \to & \tilde{G} \\ & x & \mapsto & e_i^{-1} \cdot [v_i, x, v_j] \cdot e_j \end{array}$$

which satisfies $\varphi_i \phi_j(x, g) = g_{ij}(x) \cdot g$.

Note that the right G-action commutes with the coordinate maps. Furthermore, the G-action on E_n is free and transitive. Thus $\left(\tilde{G} \hookrightarrow \tilde{E} \xrightarrow{p} X, \tilde{G}, \{V_j\}\right)$ is a principal fibre bundle.

Lemma 3.9. *$\tilde{E}$ is contractible.*

Proof. Let $E_n \subset S_n$ be the set of sequences $(x_n, \cdots, x_1, v_0)$ with endpoint v_0. $\tilde{E}_n$ is the image of E_n in $\tilde{S}_n$. Let $R_n = E_n \cap D$ the degenerate sequences.

We want to show that E_n, R_n are contractible. Then it follows from the below proposition that R_n is a strong deformation retract of E_n.

The map $\eta : E_n \to \tilde{E}_n$ takes R_n onto $\tilde{E}_{n-1}$ and carries $E_n \backslash R_n$ 1-1 onto $\tilde{E}_n \backslash \tilde{E}_{n-1}$ and maps simplexes of E_n onto cells of $\tilde{E}_n$, then $\tilde{E}_{n-1}$ is a strong deformation retract of $\tilde{E}_n$. (Let $r : E_n \times I \to E_n$ be the strong deformation retract of E_n to R_n, and define $\tilde{\eta} : E_n \times I \to \tilde{E}_n \times I, (e, t) \mapsto (\eta(e), t)$ then $\tilde{\eta} r \tilde{\eta}^{-1}$ strong deformation retract of $\tilde{E}_n$.) Thus we have $\tilde{E}_0 \subset \tilde{E}_1 \subset \cdots \subset \tilde{E}_n \subset \cdots$ where each $\tilde{E}_{n-1}$ is a strong deformation retract of $\tilde{E}_n$, hence $\tilde{E} = \bigcup \tilde{E}_n$ is contractible (each contraction of $\tilde{E}_{n-1}$ can be extended to a contraction of $\tilde{E}_n$ by extension theorem).

To show that E_n is contractible, let T_n be the linear graph with vertices $\{[0], [1], \cdots, [n]\}$ and edges $[0, 1], \cdots, [n-1, n]$, then we can identify E_n with the set of maps $(T_n, [0]) \to (X, v_0)$ which carry edges linearly into simplexes. es. A contraction of T_n is obtained by first deforming the edge $[n-1, n]$ into the vertex $[n-1]$; then deforming $[n-2, n-1]$ into $[n-2]$ etc. This induces a contraction of E_n.

To show R_n is contractible, note that $R_n = P_1 \cup P_2 \cup \cdots \cup P_s \cup Q_1 \cup \cdots \cup Q_t$ where $P_i = \{(x_n, \cdots, x_0) \in E_n | x_i = x_{i-1}\}$ and $Q_j = \{(x_n, \cdots, x_0) \in E_n | x_{j+1} = x_{j-1}\} \subset E_n$ it suffices to show that $R' = P_1 \cap \cdots \cap Q_t$ is contractible. Let T' be a linear graph obtained by identifying $[j_l + 1, j_l]$ and $[j_m - 1, j_m]$ of T_n for $1 \leq l \leq t$ and identifying all points in the edge $[i_l - 1, i_l]$ for $1 \leq l \leq s$ then R' can be considered as the set of maps $(T', [0]) \to (X, v_0)$. Since T' is a tree and contractible, R' is contractible. Thus R_n is contractible. □

Proposition 3.4. *If A is a contractible complex and B is contractible subcomplex, then B is a strong deformation retract of A.*

Theorem 3.5. *Let $p : \tilde{E} \to X$ be the universal bundle cosntrcuted with group $\tilde{G}$, then any principal G-bundle $E \to X$ is induced by a continuous map $h : \tilde{G} \to G$.*

4 The Bar and Cobar Construction

4.1 Simplicial Objects

The (Co)simplex Category Δ

Definition 4.1. Δ is the simplex category with

- objects are the finite totally ordered sets $[n] = \{0 \leq 1 \leq 2 \leq \cdots \leq n\}, n \geq 0$, and
- morphisms are order-preserving maps

$$\mathrm{Hom}_{\Delta}([m],[n]) \coloneqq \{f \in \mathrm{Hom}_{\mathbf{Set}}([m],[n]) \,|\, f(i) \leq f(j), \forall 0 \leq i \leq j \leq m\}$$

The morphisms in Δ are generated by the following two classes of morphisms:

- the *coface* maps $d^i : [n-1] \hookrightarrow [n], 0 \leq i \leq n, n \geq 1$ defined by the property that d^i is injective and contains no "i" in its image, i.e.

$$d^i(k) = \begin{cases} k & k < i \\ k-1 & k \geq i \end{cases}$$

- the *codegeneracy* maps $s^j : [n+1] \twoheadrightarrow [n], 0 \leq j \leq n, n \geq 0$ defined by the property that s^j is surjective and takes the value "j" twice, i.e.

$$s^j(k) = \begin{cases} k & k \leq i \\ k-1 & k > i \end{cases}$$

and they satisfies the following cosimplical conditions

$$\begin{aligned} d^j d^i &= d^{j+1} d^i && i \leq j \\ s^j d^i &= d^i s^{j+1} && i < j \\ s^j d^i &= \mathrm{Id} && i = j, j+1 \\ s^j d^i &= d^{i-1} s^j && i > j+1 \\ s^j s^i &= s^i s^{j+1} && i \leq j \end{aligned}$$

The dual category Δ^{op} has the presentation $Ob\left(\Delta^{op}\right) = Ob\left(\Delta\right) = \left\{\left[n\right], n \geq 0\right\}$, and morphisms are generated by the *face* maps $d_i : [n] \to [n-1]$ and the *degeneracy* maps $s_j : [n] \to [n+1]$ satisfying simplicial relations.

(Co)simplicial Objects

Definition 4.2. Let $\mathscr{C}$ be a category.

1. a *cosimplicial object* in $\mathcal{C}$ is a functor $X^* : \Delta \to \mathscr{C}, [n] \mapsto X^n \coloneqq X^*[n]$.
2. a *simplicial object* in $\mathcal{C}$ is a functor $X_* : \Delta^{op} \to \mathscr{C}, [n] \mapsto X_n \coloneqq X_*[n]$.

The category of simplicial ans cosimplicial objects in $\mathscr{C}$ are denoted $\mathbf{s}\mathscr{C} = \mathscr{C}^{\Delta^{op}} = \mathbf{Fun}\left(\Delta^{op}, \mathscr{C}\right) = \mathscr{C}_\Delta$ and $\mathbf{cs}\mathscr{C} = \mathscr{C}^\Delta = \mathbf{Fun}\left(\Delta, \mathscr{C}\right)$. In both cases, morphisms are natural transformations of functors.

We will discuss about simplical objects mostly, and dualize everything will give us the construction about cosimplicial objects.

Let $\mathcal{C}$ be a category. If $X_* \in Ob\left(\mathbf{sSet}\right)$, then $X_n \coloneqq X_*[n]$ is called the set of *n-simplices*. An n-simplex $x \in X_n$ is called *degenerate* if $x \in \operatorname{Im}\left(s_j\right)$ for some j.

Any simplicial object X_* can be written explicitly in the following manner.

$$X_* = \left[\begin{tikzcd} X_0 \arrow[r, dotted, "s_0" description] & X_1 \arrow[l, shift right=2, "d_0"'] \arrow[l, shift left=2, "d_1"] \arrow[r, dotted, shift left=1] \arrow[r, dotted, shift right=1] & \cdots \arrow[l, shift right=3] \arrow[l] \arrow[l, shift left=3] \end{tikzcd}\right]$$

where $d_i \coloneqq X_*\left(d^i\right), s_j \coloneqq X_*\left(s^j\right)$ satisfies the simplicial conditions

$$\begin{array}{rcll} d_i d_j & = & d_{j-1} d_i & i \leq j \\ d_i s_j & = & s_{j-1} d_i & i < j \\ d_i s_j & = & \mathrm{Id} & i = j.j+1 \\ d_i s_j & = & s_j d_{i-1} & i > j+1 \\ s_i s_j & = & s_{j+1} s_i & i \leq j \end{array}$$

Example 4.1. $\mathscr{C} = \mathbf{Set}, \mathbf{s}\mathscr{C} = \mathbf{sSet}$ is the category of simplicial sets. The *cartesian product* of simplicial sets is the categorical product in $\mathbf{sSet}$. Explicitly, given simplicial sets X and Y, the product $X \times Y$ is given by

$$\left(X \times Y\right)_n = X_n \times Y_n$$

with face and degenracy maps given by

$$\begin{aligned} d_i(x,y) &= (d_i x, d_i y) \\ s_j(x,y) &= (s_j x, s_j y) \end{aligned}$$

Definition 4.3. The *standard n-simplex*, denoted $\Delta[n]_*$, is a simplicial set defined as the functor $\mathrm{Hom}_\Delta(-,[n])$ where $[n]$ denotes the ordered set $\{0,1,\cdots n\}$.

By the Yoneda lemma, the n-simplices of a simplicial set X stand in 1-1 correspondence with the natural transformations from $\Delta[n]_*$ to X, i.e.

$$X_n = X([n]) \cong \mathrm{Hom}_{\mathbf{sSet}}(\Delta[n]*, X)$$

Remark 4.1. Here we abuse the notation s_j by taking it as $X_*(s_j)$ in the category $\mathcal{C}$. The proper elements should be considered when used.

4.2 Geometric Realization

Geometric n-simplex Δ^n

The *geometric n-simplex* Δ^n is the topological space defined by

$$\Delta^n = \left\{ (x_0,\cdots,x_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^n x_i = 1, x_i \geq 0, \forall 0 \leq i \leq n \right\},$$

i.e. it is the convex hull of unit vectors $\{e_i = (0,\cdots,1,\cdots,0)\}_{i=0}^n$ in $\mathbb{R}^{n+1}$.

Given a morphism $f:[n]\to[m]$ in Δ, we define

$$\begin{aligned} \Delta^*(f): \quad \Delta^n \subseteq \mathbb{R}^{n+1} &\longrightarrow \Delta^m \subseteq \mathbb{R}^{m+1} \\ e_i &\longmapsto e_{f(i)} \end{aligned}$$

which defines a functor

$$\begin{aligned} \Delta^*: \quad \Delta &\longrightarrow \mathbf{Top} \\ [n] &\longmapsto \Delta^n \end{aligned}$$

i.e. a cosimplicial space. Recall that

$$d^i(e_k) = \begin{cases} e_k, & k < i \\ e_{k+1}, & k \geq i \end{cases}$$

so it extends to $\sum_{k=0}^{n-1} x_k e_k \mapsto \sum_{k=0}^{i-1} x_k e_k + \sum_{k=i}^{n-1} x_k e_{k+1}$, and we have $d^i : \Delta^{n-1} \to \Delta^n, (x_0, \cdots, x_{n-1}) \mapsto (x_0, \cdots, x_{i-1}, 0, x_i, \cdots, x_n)$. Geometrically, in Δ^n we define i-th $(n-1)$-dimensional face to be the one opposite to e_i. Then $d^i : \Delta^{n-1} \hookrightarrow \Delta^n$ is the inclusion of i-th face into Δ^n.

Dually, $s^j : \Delta^{n+1} \to \Delta^n$ is given by $(x_0, \cdots, x_{n+1}) \mapsto (x_0, \cdots, x_{j-1}, x_j + x_{j+1}, \cdots, x_{n+1})$. Geometrically, s^j collapse the j-th and $(j+1)$-th vertices in Δ^{n+1} to a point.

Definition 4.4. The (classical) *geometric realization* of a simplicial set X is

$$|X| := \bigcup_{\sigma \in X} \Delta_\sigma = \coprod_{n \geq 0} X_n \times \Delta^n / \sim$$

where $(d_i x, \sigma) \sim (x, d_i \sigma)$ and $(s_j x, \sigma) \sim (x, s_j \sigma)$.

Theorem 4.1. *The natural map $|X \times Y| \to |X| \times |Y|$ is a bijection, and is a homeomorphism if the product is formed in the category of compactly generated spaces.*

Geometric Realization As Coends

Assume $\mathcal{D}$ is a cocomplete category with arbitrary coproducts and $\mathcal{C}$ is a small category.

Definition 4.5. Given a bifunctor $S : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$, define the coend of S by

$$\int^{c \in \mathrm{Ob}(\mathcal{C})} S(c,c) := \mathrm{Coeq} \left\{ \coprod_{\substack{f : c \to d \\ f \in \mathrm{Mor}(\mathcal{C})}} S(d,c) \underset{f_*}{\overset{f^*}{\rightrightarrows}} \coprod_{c \in \mathrm{Ob}(\mathcal{C})} S(c,c) \right\}$$

where $f^* = S(f, \mathrm{Id}) : S(d,c) \to S(c,c)$ and $f_* = S(\mathrm{Id}, f) : S(d,c) \to S(d,d)$. By UMP for colimits, a coend $X := \int^{c \in \mathrm{Ob}(\mathcal{C})} S(c,c)$ comes together with a family of morphisms $\{\varphi_c : S(c,c) \to X\}_{c \in \mathrm{Ob}(\mathcal{C})}$

making the diagram

$$\begin{tikzcd}
S(d,c) \arrow[r, "f^*"] \arrow[d, "f_*"'] & S(c,c) \arrow[d, "\varphi_c"] \\
S(d,d) \arrow[r, "\varphi_d"] & X
\end{tikzcd}$$

commute and is initial among all such pairs.

We can extend some natural constructions as coends of some bifunctors.

Functor Tensor Products

Definition 4.6. A category $\mathcal{S}$ is called *symmetric monoidal* if

1. there is a bifunctor $\otimes : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ called *tensor product*, and

2. there exists an object $\mathbf{1} \in Ob(\mathcal{S})$ called the *unit object* such that for any $a, b, c \in Ob(\mathcal{S})$, there are isomorphisms

$$a \otimes b \cong b \otimes a$$

$$(a \otimes b) \otimes c \xrightarrow[\cong]{\alpha_{a,b,c}} a \otimes (b \otimes c)$$

$$\mathbf{1} \otimes a \xrightarrow[\cong]{\lambda_a} a \xrightarrow[\cong]{\rho_a} a \otimes \mathbf{1}$$

which are natural in a, b, c and compatible in the sense that 2 axioms *triangle*

$$\begin{tikzcd}
(a \otimes \mathbf{1}) \otimes b \arrow[rr, "\alpha_{a,1,b}"] \arrow[dr, "\rho_a^{-1} \otimes \mathrm{Id}"'] & & a \otimes (\mathbf{1} \otimes b) \arrow[dl, "\mathrm{Id} \otimes \lambda_b"] \\
& a \otimes b &
\end{tikzcd}$$

and *pentagon*

$$\begin{tikzcd}
& (a \otimes (b \otimes c)) \otimes d \arrow[dr, "\alpha_{a,b\otimes c,d}"] & \\
((a \otimes b) \otimes c) \otimes d \arrow[ur, "\alpha_{a,b,c} \otimes \mathrm{Id}"] \arrow[d, "\alpha_{a\otimes b,c,d,}"'] & & a \otimes ((b \otimes c) \otimes d) \arrow[d, "\mathrm{Id} \otimes \alpha_{b,c,d}"] \\
(a \otimes b) \otimes (c \otimes d) \arrow[rr, "\alpha_{a,b,c\otimes d}"] & & a \otimes (b \otimes (c \otimes d))
\end{tikzcd}$$

holds.

Definition 4.7. $\mathcal{S}$ is called *closed* if there exists a bifunctor

$$\mathbf{Hom}_{\mathcal{S}}(-,-) : \mathcal{S}^{op} \times \mathcal{S} \to \mathcal{S}$$

such that

$$\mathbf{Hom}_{\mathcal{S}}(a \otimes b, c) \cong \mathbf{Hom}_{\mathcal{S}}(a, \mathbf{Hom}_{\mathcal{S}}(b, c)), \forall a, b, c \in Ob(\mathcal{S}).$$

Example 4.2. In $\mathcal{S} = \mathbf{sSet}$, the tensor product is given by

$$\begin{array}{rccc} \otimes = \times : & \mathbf{sSet} \times \mathbf{sSet} & \to & \mathbf{sSet} \\ & (X,Y) & \mapsto & X \times Y = \{X_n \times Y_n\}_{n \geq 0} \end{array}$$

and internal hom is given by

$$\begin{array}{rccc} \mathbf{Hom} : & \mathbf{sSet}^{op} \times \mathbf{sSet} & \to & \mathbf{sSet} \\ & (Y,Z) & \mapsto & \mathbf{Hom}(Y,Z) = \{\mathrm{Hom}_{\mathbf{sSet}}(Y \times \Delta[n], Z)\}_{n \geq 0} \end{array}$$

we have

$$\mathbf{Hom}(X \times Y, Z) \cong \mathbf{Hom}(X, \mathbf{Hom}(Y, Z)).$$

Definition 4.8. An $\mathcal{S}$-*category* is a category $\mathcal{M}$ enriched over $\mathcal{S}$, thus

(S1) For any objects X, Y in $\mathcal{M}$, there exists an object $\mathbf{Hom}_{\mathcal{M}}(X,Y)$ in $\mathcal{S}$

(S2) For any objects X, Y, Z in $\mathcal{M}$, there exists a morphism

$$c_{X,Y,Z} : \mathbf{Hom}_{\mathcal{M}}(Y,Z) \times \mathbf{Hom}_{\mathcal{M}}(X,Y) \to \mathbf{Hom}_{\mathcal{M}}(X,Z)$$

in $\mathcal{S}$ called *composition law.*

(S3) For any object X in $\mathcal{M}$, there is a morphism $i_X : \bullet \to \mathbf{Hom}_{\mathcal{M}}(X,X)$ called *unit.*

(S4) There is an isomorphism

$$\mathbf{Hom}_{\mathcal{S}}(\bullet, \mathbf{Hom}_{\mathcal{M}}(X,Y)) \cong \mathrm{Hom}_{\mathcal{M}}(X,Y).$$

Thus the usual hom can be recovered from internal hom.

These data satisfy the compatibility axioms (*triangle* and *pentagon*) similar to the ones in $\mathcal{S}$.

Definition 4.9. An $\mathcal{S}$-enriched category is called

1. *tensored over* $\mathcal{S}$ if there exists a bifunctor

$$\boxtimes : \mathcal{S} \times \mathcal{M} \to \mathcal{M}$$

viewed as an action of $\mathcal{S}$ on $\mathcal{M}$, such that

$$\mathbf{Hom}_{\mathcal{M}}(v \boxtimes x, y) \cong \mathbf{Hom}_{\mathcal{M}}(v, \mathbf{Hom}_{\mathcal{M}}(x, y)).$$

2. *cotensored over* $\mathcal{S}$ if there exists a bifunctor

$$(-)^{-} : \mathcal{S}^{op} \times \mathcal{M} \to \mathcal{M}$$

such that

$$\mathbf{Hom}_{\mathcal{M}}(v, \mathbf{Hom}_{\mathcal{M}}(x, y)) \cong \mathbf{Hom}_{\mathcal{M}}(x, y^{v}).$$

Let $\mathcal{S} = (\mathbf{sSet}, \times, \bullet)$ and $\mathcal{M} = \mathbf{s}\mathcal{C} = \mathbf{Fun}(\Delta^{op}, \mathcal{C})$, then $\mathcal{M}$ has a canonical structure of simplicial category tensored and cotensored over $\mathcal{S}$.

- Tensor

$$\begin{array}{rccc} \boxtimes : & \mathbf{sSet} \times \mathbf{s}\mathcal{C} & \to & \mathbf{s}\mathcal{C} \\ & (K, X) & \mapsto & K \boxtimes X = \left\{ \coprod_{K_n} X_n \right\}_{n \geq 0} \end{array}$$

- Internal hom

$$\mathbf{Hom}_{\mathbf{s}\mathcal{C}}(X, Y) = \left\{ \mathrm{Hom}_{\mathbf{s}\mathcal{C}}(\Delta[n] \boxtimes X, Y) \right\}_{n \geq 0}$$

Note that $(K \boxtimes L) \boxtimes X \cong K \boxtimes (L \boxtimes X)$ and $\Delta[0] \boxtimes X \cong X \cong * \boxtimes X$.

- Fix $K \in Ob(\mathbf{sSet})$, consider $K \boxtimes - : \mathbf{s}\mathcal{C} \to \mathbf{s}\mathcal{C}$. Since $\mathcal{C}$ is cocomplete, so is $\mathbf{s}\mathcal{C}$, hence $K \boxtimes -$

has right adjoint defined by left Kan extension

$$\begin{tikzcd} \mathbf{s}\mathcal{C} \arrow[r, "\mathrm{Id}"] \arrow[d, "K\boxtimes -"'] & \mathbf{s}\mathcal{C} \\ \mathbf{s}\mathcal{C} \arrow[ur, "L_{K\boxtimes -}(\mathrm{Id}_{\mathbf{s}\mathcal{C}})"'] & \end{tikzcd}$$

Denote

$$Y^K \coloneqq L_{K\boxtimes -}(\mathrm{Id}_{\mathbf{s}\mathcal{C}})(Y), \forall Y \in Ob(\mathbf{s}\mathcal{C}),$$

then by general properties of Kan extensions, we have

$$\mathrm{Hom}_{\mathbf{s}\mathcal{C}}(K \boxtimes X, Y) \cong \mathrm{Hom}_{\mathbf{s}\mathcal{C}}\left(X, Y^K\right).$$

This implies for any $n \geq 0$,

$$\begin{aligned} \mathbf{Hom}_{\mathbf{s}\mathcal{C}}(K \boxtimes X, Y)_n &= \mathrm{Hom}_{\mathbf{s}\mathcal{C}}(\Delta[n] \boxtimes K \boxtimes X, Y) \cong \mathrm{Hom}_{\mathbf{s}\mathcal{C}}(K \boxtimes \Delta[n] \boxtimes X, Y) \\ &\cong \mathrm{Hom}_{\mathbf{s}\mathcal{C}}\left(\Delta[n] \boxtimes X, Y^K\right) \eqqcolon \mathbf{Hom}_{\mathbf{s}\mathcal{C}}\left(X, Y^K\right)_n. \end{aligned}$$

Special cases

1. Let $\mathcal{C} = \mathbf{Mod}(R)$ where R is a unital (commutative) associative ring. Then $\coprod = \bigoplus$. Let $\mathcal{M} = \mathbf{sMod}(R) \cong \mathbf{Com}_{\geq 0}(R)$. The tensor product is given by

$$\begin{aligned} \bigoplus : \quad \mathbf{sSet} \times \mathbf{sMod}(R) &\to \mathbf{sMod}(R) \\ (K, X) &\mapsto \left\{\bigoplus_{K_n} X_n\right\}_{n\geq 0} = \{R[K_n] \otimes_R X_n\}_{n\geq 0} \end{aligned}$$

where $R[K_n]$ is the free bimodule based on K_n. We need to check that this agrees with simplicial operations. Internal hom is defined by

$$\mathbf{Hom}_{\mathbf{sMod}(R)}(X, Y) \coloneqq \left\{\mathrm{Hom}_{\mathbf{sMod}(R)}(R[\Delta[n]] \otimes_R X, Y)\right\}_{n\geq 0}.$$

And

$$Y^K \coloneqq \mathbf{Hom}_{\mathbf{sSet}}(X, Y)$$

where the R-module structure comes from the target.

2. Let $\mathcal{C} = \mathbf{CommAlg}_k$ with k a commutative ring. $\mathcal{M} = \mathbf{sCommAlg}_k$, then $\coprod = \bigotimes_k$. The tensor product is given by

$$K \boxtimes A \cong \left\{ \bigotimes_{K_n} A \right\}_{n \geq 0}.$$

And internal hom is defined as

$$\mathbf{Hom}_{\mathbf{sCommAlg}_k}(A, B) = \left\{ \mathrm{Hom}_{\mathbf{sCommAlg}_k}\left(\bigotimes_{\Delta[n]} A, B \right) \right\}_{n \geq 0}.$$

3. Let $\mathcal{M} = \mathbf{Top}$ the category of compactly generated weak Hausdorff spaces, then we have

$$\begin{aligned} \boxtimes : \quad \mathbf{sSet} \times \mathbf{Top} &\rightarrow \quad \mathbf{Top} \\ (K, X) &\mapsto \quad |K| \times X \end{aligned}$$

and

$$\begin{aligned} (-)^{-} : \quad \mathbf{sSet}^{op} \times \mathbf{Top} &\rightarrow \mathbf{Top} \\ (K, Y) &\mapsto Y^K := \mathbf{Map}(|K|, Y) \end{aligned}$$

Let $\mathcal{S} = (\mathcal{S}, \bigotimes, \mathbf{1})$ be a closed symmetric monoidal category and $\mathcal{C}$ be a small category. $\mathcal{M}$ is a cocomplete $\mathcal{S}$-category (tensored over $\mathcal{S}$).

$$\boxtimes : \mathcal{S} \times \mathcal{M} \rightarrow \mathcal{M}$$

Definition 4.10. Given two functors $G : \mathcal{C}^{op} \rightarrow \mathcal{S}$ and $F : \mathcal{C} \rightarrow \mathcal{M}$, define the *functor tensor product* $G \underset{\mathcal{C}}{\boxtimes} F$ as

$$G \underset{\mathcal{C}}{\boxtimes} F := \int^{c \in Ob(\mathcal{C})} G(c) \boxtimes F(c) = \mathrm{coeq} \left\{ \coprod_{f : c \rightarrow c'} G(c') \boxtimes F(c) \underset{f_*}{\overset{f^*}{\rightrightarrows}} \coprod_{c \in Ob(\mathcal{C})} G(c) \boxtimes F(c) \right\}$$

where

$$\begin{aligned} f^* : \quad G(c') \boxtimes F(c) &\xrightarrow{G(f)\boxtimes \mathrm{Id}} G(c) \boxtimes F(c) \hookrightarrow \coprod_{c\in Ob(\mathcal{C})} G(c) \boxtimes F(c) \\ f_* : \quad G(c') \boxtimes F(c) &\xrightarrow{\mathrm{Id}\boxtimes F(f)} G(c') \boxtimes F(c') \hookrightarrow \coprod_{c\in Ob(\mathcal{C})} G(c) \boxtimes F(c) \end{aligned}$$

Example 4.3. Let $\mathcal{S} = (\mathbf{Ab}, \bigotimes_{\mathbb{Z}}, \mathbb{Z})$ and $\mathcal{M} = \mathcal{S}$ with $\mathbf{Hom}_{\mathcal{S}} = \mathrm{Hom}_{\mathbf{Ab}} = \mathrm{Hom}_{\mathbb{Z}}$. $\mathcal{S}$ is enriched and tensored over itself with $\boxtimes = \bigotimes_{\mathbb{Z}}$.

Take a unital associative ring R, we can think of $\underline{R}$ as the category with one object $\{*\}$ enriched over $\mathcal{S}$.

A left module over R is an $\mathcal{S}$-functor

$$\begin{aligned} \underline{F} : \underline{R} &\to \mathbf{Ab} \\ * &\mapsto M \\ R &\mapsto \mathrm{End}(M) \end{aligned}$$

A right module over R is an $\mathcal{S}$-functor

$$\begin{aligned} \underline{G} : \underline{R}^{op} &\to \mathbf{Ab} \\ * &\mapsto N \\ R^{op} &\mapsto \mathrm{End}(N) \end{aligned}$$

Let's think of R as a monoid and take the underlying (unenriched) functors

$$\begin{aligned} F : R &\to \mathbf{Ab} \\ G : R^{op} &\to \mathbf{Ab} \end{aligned}$$

Then

$$G \boxtimes_R F = \int^R N \bigotimes_{\mathbb{Z}} M \cong \frac{N \bigotimes_{\mathbb{Z}} M}{\langle nr \otimes m - n \otimes rm \rangle} \cong N \bigotimes_R M$$

is the usual tensor product of left and right modules.

Example 4.4. (Kan extension) Given

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] \arrow[r, phantom, shift right=4, "\Downarrow\eta"] & \mathcal{D} \\ \mathcal{E} \arrow[ur, "\mathcal{L}_G(F)"'] & \end{tikzcd}$$

the left Kan extension can be interpreted as a tensor product

$$\mathcal{L}_G(F)(e) = (G \circ h_e) \underset{\mathcal{C}}{\boxtimes} F, e \in Ob(G).$$

where $G \circ h_e = \mathrm{Hom}_{\mathcal{D}}(G(-), e) : \mathcal{C}^{op} \to \mathbf{Set}$.

Proof. See later. We can also prove it by universal properties $\mathcal{L}_G(F)(e) \cong \mathrm{colim}_{G/e}(F)$.

$$(G \circ h_e)\underset{\mathcal{C}}{\boxtimes} F = \mathrm{coeq}\left\{ \coprod_{f:c\to c'} \mathrm{Hom}_{\mathcal{C}}(G(c'), e) \boxtimes F(c) \overset{f^*}{\underset{f_*}{\rightrightarrows}} \coprod_{c\in Ob(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(G(c), e) \boxtimes F(c) \right\} \cong \mathrm{colim}_{G/e}(F)$$

□

Example 4.5. (classical geometric realization) Given

$$\begin{tikzcd} \Delta \arrow[r, "\Delta^\bullet"] \arrow[d, "h"'] \arrow[r, phantom, shift right=4, "\Downarrow\eta"] & \mathbf{Top} \\ \mathbf{sSet} \arrow[ur, "|-|=\mathcal{L}_h(\Delta^\bullet)"'] & \end{tikzcd}$$

For $X \in Ob(\mathbf{sSet})$, $\mathcal{L}_h(\Delta^\bullet)(X) \cong \mathrm{colim}_{\Delta X}(\Delta^\bullet)$. Then

$$\mathcal{L}_h(\Delta^\bullet)(X) \cong \mathrm{Hom}_{\mathbf{sSet}}(\Delta[\cdot], X)\underset{\Delta}{\boxtimes}\Delta^\bullet \cong X\underset{\Delta}{\boxtimes}\Delta^\bullet = \coprod_{n\geq 0} X_n \times \Delta^n / \langle (x, f_*(u)) \sim (f^*(x), u) \rangle_{f\in Mor(\Delta)}$$

where X_n is given the discrete topology.

In particular, if $X : \Delta^{op} \to \mathbf{Space}$ is a simplicial space ($\mathbf{Top}$ or $\mathbf{sSet}$) then $|-| : \mathbf{Space}^{\Delta^{op}} \to \mathbf{Space}$ is given by

$$|X| \coloneqq X \underset{\Delta}{\boxtimes} \Delta^\bullet = \coprod_{n\geq 0} X_n \times \Delta^n / \sim$$

where $X_n \times \Delta^n$ is given the product topology.

4.3 Classifying Space of A Category

Nerve of a category

Recall a category $\mathscr{C}$ is small if its objects form a (proper) set. Associate to such a category a simplicial set $\mathcal{N}_*\mathscr{C}$ (or $\mathcal{B}_*\mathscr{C}$) defined by

$$
\begin{aligned}
\mathcal{N}_0\mathscr{C} &= \{\text{objects in } \mathscr{C}\} = Ob\,(\mathscr{C}) \\
\mathcal{N}_1\mathscr{C} &= \{\text{morphisms in } \mathscr{C}\} = Mor\,(\mathscr{C}) \\
\mathcal{N}_2\mathscr{C} &= \left\{\text{composable morphisms } c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} c_2 \text{ in } \mathscr{C}\right\} \\
&\cdots \\
\mathcal{N}_n\mathscr{C} &= \left\{n\text{-composable morphisms } c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} c_n \text{ in } \mathscr{C}\right\}
\end{aligned}
$$

$$
\mathcal{N}_2\mathscr{C}:\quad
\begin{tikzcd}
& c_1 \arrow[dr, "f_1"] & \\
\bullet \arrow[ur, "f_0"] \arrow[rr, dashed, "f_1 f_1"] & & c_2
\end{tikzcd}
$$

$$
\mathcal{N}_3\mathscr{C}:\quad
\begin{tikzcd}
& c_1 \arrow[drr, "f_2 f_1"] \arrow[ddr, "f_1"] & & \\
\bullet \arrow[ur, "f_0"] \arrow[rrr, dashed, "f_2 f_1 f_0"] \arrow[dr, "f_1 f_0"'] & & & c_3 \\
& & c_2 \arrow[ur, "f_2"'] &
\end{tikzcd}
$$

This suggests that

$$
\begin{array}{llcl}
d_i: & \mathcal{N}_n\mathscr{C} & \longrightarrow & \mathcal{N}_{n-1}\mathscr{C} \\
& \left(c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} c_n\right) & \longmapsto & \left(c_0 \xrightarrow{f_0} \cdots \xrightarrow{f_{i-2}} c_{i-1} \xrightarrow{f_{i-1}} \hat{c}_i \xrightarrow{f_i} c_{i+1} \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_{n-1}} c_n\right) \\
& (f_{n-1}, \cdots, f_0) & \longmapsto & \begin{cases} (f_{n-1}, \cdots, f_1) & i = 0 \\ (f_{n-1}, \cdots, f_i f_{i-1}, \cdots, f_0) & 1 \le i \le n-1 \\ (f_{n-2}, \cdots, f_0) & i = n \end{cases}
\end{array}
$$

$$\begin{array}{cccc} s_j : & \mathcal{N}_n\mathscr{C} & \longrightarrow & \mathcal{N}_{n+1}\mathscr{C} \\ & \left(c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} c_n\right) & \longmapsto & \left(c_0 \xrightarrow{f_0} \cdots \xrightarrow{f_{i-1}} c_i \xrightarrow{\mathrm{Id}} c_i \xrightarrow{f_i} c_{i+1} \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_{n-1}} c_n\right) \\ & (f_{n-1}, \cdots, f_0) & \longmapsto & (f_{n-1}, \cdots, f_i, f_i, \cdots, f_0) \end{array}$$

Another way to view this construction is the following. Let

- $s, t : Mor\,(\mathscr{C}) \to Ob\,(\mathscr{C})$ the source and target maps.
- $i : Ob\,(\mathscr{C}) \to Mor\,(\mathscr{C})$ the identity morphism map.
- $\circ : Mor\,(\mathscr{C}) \times_{Ob(\mathscr{C})} Mor\,(\mathscr{C}) \to Mor\,(\mathscr{C})$ where $Mor\,(\mathscr{C}) \times_{Ob(\mathscr{C})} Mor\,(\mathscr{C})$ is the fibred product

$$\begin{tikzcd} Mor\,(\mathscr{C}) \times_{Ob(\mathscr{C})} Mor\,(\mathscr{C}) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"'] & Mor\,(\mathscr{C}) \arrow[d, "t"] \\ Mor\,(\mathscr{C}) \arrow[r, "s"] & Ob\,(\mathscr{C}) \end{tikzcd}$$

Notice that the structure of $\mathscr{C}$ gives us

$$\begin{tikzcd} Ob\,(\mathscr{C}) \arrow[r, dashed, "i" description] & Mor\,(\mathscr{C}) \arrow[l, shift right=2, "s"'] \arrow[l, shift left=2, "t"] \arrow[r, dashed, shift left=1] \arrow[r, dashed, shift right=1] & Mor\,(\mathscr{C}) \times_{Ob(\mathscr{C})} Mor\,(\mathscr{C}) \arrow[l, shift right=3] \arrow[l] \arrow[l, shift left=3] \end{tikzcd}.$$

The new construction can be viewed as an "extension" of a category to a "full" simplicial set

$$\begin{tikzcd} \mathcal{N}_0\,(\mathscr{C}) \arrow[r, dashed, "i" description] & \mathcal{N}_1\,(\mathscr{C}) \arrow[l, shift right=2, "s"'] \arrow[l, shift left=2, "t"] \arrow[r, dashed, shift left=1] \arrow[r, dashed, shift right=1] & \mathcal{N}_2\,(\mathscr{C}) \cdots \arrow[l, shift right=3] \arrow[l] \arrow[l, shift left=3] \end{tikzcd}$$

where

$$\mathcal{N}_n\,(\mathscr{C}) = \underbrace{Mor\,(\mathscr{C}) \times_{Ob(\mathscr{C})} Mor\,(\mathscr{C}) \times_{Ob(\mathscr{C})} \cdots \times_{Ob(\mathscr{C})} Mor\,(\mathscr{C})}_{n} =: Mor_n\,(\mathscr{C})$$

The maps $d_i's$ and s_j's are the structure maps of iterated fibred products.

$\mathcal{N}_*\mathscr{C}$ (as a simplicial set up to isomorphism) determines $\mathscr{C}$ uniquely up to isomorphism.

Definition 4.11. The *classifying space* $B\mathcal{C}$ of a category $\mathcal{C}$ is the geometric realization of $\mathcal{N}\mathcal{C}$.

Proposition 4.1. *If $\mathcal{C}, \mathcal{C}'$ are topological categories and $F_0, F_1 : \mathcal{C} \to \mathcal{C}'$ are continuous functors, and $\alpha : F_0 \to F_1$ is a morphism of functors, then the induced map $BF_0, BF_1 : B\mathcal{C} \to B\mathcal{C}'$ are homotopic.*

Proof. α can be regarded as a functor $\mathcal{C} \times \mathbf{1} \to \mathcal{C}'$ where $\mathbf{1} = \{0 < 1\}$ is a category, then

$$B\alpha : B(\mathcal{C} \times \mathbf{1}) \cong B\mathcal{C} \times B\mathbf{1} = B\mathcal{C} \times I \to B\mathcal{C}'$$

is a homotopy between BF_0 and BF_1. □

4.4 The Classifying Space of a Topological Group

Let G be a topological group. G can be viewed as a topological category with a single object $*$ and $Mor(G) = G$, then $\mathcal{N}_* G = \left\{\mathcal{N}_k G = G^k\right\}_{k \geq 0}$.

The space $BG = |\mathcal{N}_* G|$ is the classifying space of G in the usual sense as one can see in the follows.

Consider the category $\overline{G}$ with $Ob\left(\overline{G}\right) = G$ and $Mor\left(\overline{G}\right) = G \times G$ where there is a unique isomorphism (g_1, g_2) between any two objects $g_1 \to g_2$, then $\overline{G} \simeq *$ as category via

$$p : \overline{G} \rightleftarrows \bullet : i$$

where $i(\bullet) = e \in G$ and $i(\mathrm{Id}_\bullet) = (e, e) \in G$ and $p(g) = \bullet$ and $p(g_1, g_2) = \mathrm{Id}_\bullet$, then we observe that

$$pi = \mathrm{Id}_\bullet$$

and

$$ip \cong \mathrm{Id}_{\overline{G}}$$

via the natural isomorphism $\eta : \mathrm{Id}_{\overline{G}} \Rightarrow ip$ where $\eta_{(g)} = \left(g, g^{-1}\right)$, and we have

$$\begin{tikzcd}
g_1 \arrow[r, "\left(g_1, g_1^{-1}\right)"] \arrow[d, "\left(g_1, g_1^{-1} g_2\right)"'] & ip(g_1) = e \arrow[d, "(e,e)"] \\
g_2 \arrow[r, "\left(g_2, g_2^{-1}\right)"'] & e
\end{tikzcd}$$

so $EG \coloneqq B\overline{G}$ is contractible.

.

Note G acts on $\mathcal{N}_* \overline{G}$ freely, G acts on EG freely,

Relation between $\mathcal{N}_*\bar{G}$ and $\mathcal{N}_*G$

1. There is a functor $\overline{G} \to G, (g_1, g_2) \mapsto g_2 g_1^{-1}$ and it induces a map $\mathcal{N}_*\overline{G} \to \mathcal{N}_*G$ given by

$$\begin{array}{ccc} \mathcal{N}_n\overline{G} & \longrightarrow & \mathcal{N}_nG \\ (g_0, g_1, \cdots, g_n) & \mapsto & \left(g_1 g_0^{-1}, g_2 g_1^{-1}, \cdots, g_n g_{n-1}^{-1}\right) \end{array}$$

2. There is a right G-action of $\mathcal{N}_*\bar{G}$

$$\begin{array}{ccc} \mathcal{N}_n\overline{G} \times G & \longrightarrow & \mathcal{N}_nG \\ (g_0, \cdots, g_n) \times g & \longmapsto & (g_0 g, \cdots, g_n g) \end{array}$$

such that

$$\begin{tikzcd} \mathcal{N}_*\overline{G} \arrow[rr, "p"] \arrow[dr] & & \mathcal{N}_*G \\ & \mathcal{N}_*\overline{G}/G \arrow[ur, "\cong"'] & \end{tikzcd}$$

This gives an example of simplicial principal G-bundle. In particular, this gives us

$$EG/G \cong BG.$$

3. Note that $\mathcal{N}_*\bar{G}$ is a simplicial group which acts on the left on $\mathcal{N}_*G$

$$\begin{array}{ccc} \mathcal{N}_*\bar{G} \times \mathcal{N}_*G & \longrightarrow & \mathcal{N}_*G \\ (g_0, \cdots, g_n) \times (h_1, \cdots, h_n) & \longmapsto & \left(g_0 h_1 g_1^{-1}, g_1 h_2 g_2^{-1}, \cdots, g_{n-1} h_{n-1} g_n^{-1}\right) \end{array}$$

Relation between $B\underline{G}$ and BG

Recall Milnor's construction of a classifying space $X^\infty G$ for a topological group G in last chapter, and we will compare this with our construction here when view G as a topological category G with single object and morphisms are elements in G. Since $X^\infty G$ and and BG can both be identified with the quotient of a contractible space, it suffices compare $E^\infty G$ and EG.

If G is a countable CW group, then $E^\infty G \cong EG$ since we can take the normal join topology on $E^n G$. Otherwise the topology on $E^\infty G$ is stronger than EG.

4.5 Categorical Bar Construction

(Co)Monad

Definition 4.12. A *monad* (*triple*) on a category $\mathcal{C}$ is given by an endofunctor $T : \mathcal{C} \Rightarrow \mathcal{C}$ with two morphisms $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow T$ and $\mu : T \circ T \Rightarrow T$ satisfying

1. associativity

$$\begin{tikzcd} T \circ T \circ T \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"'] & T \circ T \arrow[d, Rightarrow, "\mu"] \\ T \circ T \arrow[r, Rightarrow, "\mu"] & T \end{tikzcd}$$

2. unitality

$$\begin{tikzcd} T \arrow[r, Rightarrow, "\eta T"] \arrow[d, Rightarrow, "T\eta"'] & T \circ T \arrow[d, Rightarrow, "\mu"] \\ T \circ T \arrow[r, Rightarrow, "\mu"] & T \end{tikzcd}$$

Remark 4.2. This can be regarded as a "generalized" associative, unital algebras.

Definition 4.13. A *comonad* (*cotriple*) on a category $\mathcal{C}$ is given by an endofunctor $\perp : \mathcal{C} \Rightarrow \mathcal{C}$ with two morphisms $\varepsilon : \perp \Rightarrow \mathrm{Id}_{\mathcal{C}}$ and $\delta : \perp \Rightarrow \perp \circ \perp$ satisfying coassociative and counital diagrams.

Main application

Given a pair of adjoint functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ with unit $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow UF$ and counit $\varepsilon : FU \Rightarrow \mathrm{Id}_{\mathcal{D}}$, we can define

$$\begin{aligned} T = UF : \mathcal{C} \to \mathcal{C} \quad & \mu : T \circ T \Rightarrow T, UFUF \overset{U\varepsilon F}{\Longrightarrow} UF \\ \perp = FU : \mathcal{D} \to \mathcal{D} \quad & \delta : \perp \to \perp \circ \perp, FU \overset{F\eta U}{\Longrightarrow} FUFU \end{aligned}$$

Claim 4.1. $(T = UF, \eta, \mu)$ is a monad on $\mathcal{C}$ and $(\perp = FU, \varepsilon, \delta)$ is a comonad in $\mathcal{D}$.

Proof. Use identities **??** for adjunction morphisms. Then we have

$$\left(FU \overset{F\eta U}{\Rightarrow} FUFU \overset{\varepsilon FU}{\Rightarrow} FU \right) = \mathrm{Id}_{FU}$$
$$\left(UF \overset{\eta UF}{\Rightarrow} UFUF \overset{U\varepsilon F}{\Rightarrow} UF \right) = \mathrm{Id}_{UF}$$

which gives the unitality diagrams. The associativity diagram follows from naturality of the unit and counit functors, i.e. we have

$$\begin{tikzcd}
UFUFUFx \arrow[r, "UFU\varepsilon_{Fx}"] \arrow[d, "U\varepsilon_{FUFx}"'] & UFUFx \arrow[d, "U\varepsilon_{Fx}"] \\
UFUFx \arrow[r, "U\varepsilon_{Fx}"] & UFx
\end{tikzcd}$$

and

$$\begin{tikzcd}
FUy \arrow[r, "F\eta_{Uy}"] \arrow[d, "F\eta_{Uy}"'] & FUFUy \arrow[d, "FUF\eta_{Uy}"] \\
FUFUy \arrow[r, "F\eta_{UFUy}"] & FUFUFUy
\end{tikzcd}$$

□

Claim 4.2. Every monad in $\mathcal{C}$ gives a functor $\mathcal{C} \to \mathbf{c}\mathcal{C}$ and every comonad gives a functor $\mathcal{D} \to \mathbf{s}\mathcal{D}$.

Proof. Given $(\perp, \varepsilon, \delta)$ on $\mathcal{D}$, and $A \in Ob(\mathcal{D})$, we define

$$\begin{aligned}
\perp_* : \quad \mathcal{D} &\to \mathbf{s}\mathcal{D} \\
A &\mapsto \perp_* A = \{\perp_n A\}_{n \geq 0}
\end{aligned}$$

where $\perp_n A = \perp^{n+1} A$ and

$$\begin{aligned}
d_i = \perp^i \cdot \varepsilon \cdot \perp^{n-i} : \quad \perp^{n+1} A &\to \perp^n A \\
s_j = \perp^j \cdot \delta \cdot \perp^{n-j} : \quad \perp^{n+1} A &\to \perp^{n+2} A
\end{aligned}$$

which is similar to the bar construction. Explicitly, $\perp_* A$ can be expressed as follows

$$\begin{tikzcd}
\perp A & \perp^2 A \arrow[l, shift left] \arrow[l, shift right] & \perp^3 A \cdots \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2]
\end{tikzcd}$$

The simplicial identities are satisfied because of the functoriality of units and counits and the identity for adjunction morphisms. In particular, $d_i s_j = \mathrm{Id}, i = j, j+1$ follows from

$$\left(FU \overset{F\eta U}{\Rightarrow} FUFU \overset{\varepsilon FU}{\Rightarrow} FU\right) = \mathrm{Id}_{FU}$$
$$\left(FU \overset{F\eta U}{\Rightarrow} FUFU \overset{FU\varepsilon}{\Rightarrow} FU\right) = \mathrm{Id}_{FU}$$

□

Definition 4.14. Let $\mathcal{C}$ be a category and (T, η, μ) be a monad on $\mathcal{C}$. A T-*algebra* is a pair (A, ρ) where A is an object in $\mathcal{C}$ and $\rho : TA \to A$ called the structure map of the algebra such that the diagrams

$$\begin{tikzcd} T^2A \arrow[r, "T\rho"] \arrow[d, "\mu_A"'] & TA \arrow[d, "\rho"] \\ TA \arrow[r, "\rho"] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, equal] & TA \arrow[d, "\rho"] \\ & A \end{tikzcd}$$

commute. A morphism $f : (A, \rho_A) \to (B, \rho_B)$ of T-algebras is an morphism $f : A \to B$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\rho_A"'] & TB \arrow[d, "\rho_B"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

commutes.

The category $\mathcal{C}^T$ of T-algebras and their morphisms is called the *Eilenberg–Moore category* or category of (Eilenberg–Moore) algebras of the monad T. The forgetful functor $U : \mathcal{C}^T \to \mathcal{C}$ has a left adjoint $F : \mathcal{C} \to \mathcal{C}^T$ which takes A to (TA, μ_A).

Classifying Space of a Category

Let $\mathcal{C}$ be a category and $(T, \eta, \mu))$ be a monad on $\mathcal{C}$. Let $\mathcal{C}^T$ be the category of T-algebras in $\mathcal{C}$, there is a natural fogetful functor $U : \mathcal{C}^T \to \mathcal{C}$ which has a left adjoint $F : \mathcal{C} \to \mathcal{C}^T$. This pair of adjoint functors gives a comonad

$$\perp = FU : \mathcal{C}^T \to \mathcal{C}^T, \eta : \perp \Rightarrow \mathrm{Id}_{\mathcal{C}^T}, \delta : \perp \Rightarrow \perp \circ \perp, FU \overset{F\eta U}{\Longrightarrow} FUFU$$

on $\mathcal{C}^T$, and it induces a functor

$$\begin{array}{rccl} \perp_* : & \mathcal{C}^T & \to & \mathbf{s}\mathcal{C}^T \\ & A & \mapsto & \perp_* A = \left\{ \perp_n A = \perp^{n+1} A \right\}_{n \geq 0} \end{array}$$

where $\perp_n A = \perp^{n+1} A$.

Recall that the (augmented) simplex category Δ_+ is defined as follows. The category Δ has terminal object [0] but no initial object. Define the augmented simplex categor Δ_+ as

$Ob\,(\Delta_+) = Ob\,(\Delta) \cup \{[-1]\}$.

$$\mathrm{Hom}_{\Delta_+}([n],[m]) = \begin{cases} \mathrm{Hom}_{\Delta}([n],[m]) & n,m \geq 0 \\ [-1] \to [m] & n = -1 \\ \emptyset & m = -1, n \geq 0 \end{cases}$$

Definition 4.15. For any category $\mathcal{C}$ we define the *augmented simplicial object* as a functor $X : \Delta_+ \to \mathcal{C}$. Denote $\mathbf{s}_+\mathcal{C} = \mathbf{Fun}\,(\Delta_+, \mathcal{C})$.

Explicitly, each $X \in Ob\,(\mathbf{s}_+\mathcal{C})$ is given by $X \in Ob\,(\mathbf{s}\mathcal{C})$ together with $X_{-1} \in Ob\,(\mathcal{C})$ and $\varepsilon : X_0 \to X_{-1}$ in $Mor\,(\mathcal{C})$ such that

$$X_{-1} \xleftarrow{\varepsilon} X_0 \underset{d_0}{\overset{d_1}{\leftleftarrows}} X_1 \cdots$$

agrees in the sense that $\varepsilon d_1 = \varepsilon d_0$. We can denote $\varepsilon = d_0$ and extend $d_i d_j = d_{j-1} d_i, i < j$ for $n = 0$.

Therefore the above functor $\perp_*$ is in fact a functor $\perp_* : \mathcal{C}^T \to \mathbf{s}_+\mathcal{C}^T$.

Definition 4.16. The *bar construction* $B\,(T, A)$ is the simplicial T-algebra given by $\perp_*(A)$. If we forget the T-algebra structure on A, $\perp_*(A)$ is an (aumented) simplicial object in $\mathcal{C}$ which is called the bar resolution of A.

4.6 Special cases (Algebraic Bar Construction)

Modules over Commutative Algebras

Let A be a commutative associative algebras over some ring k. Write $\mathrm{Mod}\,(A)$ for the category of connective chain complexes of modules over A.

For N a right module, $A \otimes_k N$ is canonically a module. This construction extends to a functor

$$(-) \otimes_k A : \mathrm{Mod}\,(A) \to \mathrm{Mod}\,(A)\,.$$

The monoid-structure on A makes this a monad in **Cat**: the monad product and unit are given by the product and unit in A.

For N a module its right action $\rho : N \otimes_k A \to N$ makes the module an algebra over this monad.

The bar construction $B(A,N)$ is then the simplicial module

$$\cdots \longrightarrow N \otimes_k A \otimes_k A \underset{\rho\otimes 1_A}{\overset{1_N\otimes\mu_A}{\rightrightarrows}} N \otimes_k A$$

Under the Moore complex functor of the Dold-Kan correspondence this is identified with a chain complex whose differential is given by the alternating sums of the face maps indicated above.

This chain complex provides a resolution that computes the Tor. This gives the Hochschild homology of A.

Remark 4.3. This chain complex is what originally was called the bar complex in homological algebra. Because the first authors denoted its elements using a notation involving vertical bars (Ginzburg).

Bar and cobar constructions of differential graded Hopf algebras

Definition 4.17. A $\mathbb{Z}$-graded Hopf algebra is a $\mathbb{Z}$-graded vector space, which, for that grading, is both a $\mathbb{Z}$-graded algebra, (A,μ,ε), with unity $\varepsilon : k \to A$, and a Z-graded coalgebra (A,Δ,η) such that:

- $\varepsilon : k \to A$ is a morphism of $\mathbb{Z}$-graded coalgebras;
- $\eta : A \to k$ is a morphism of $\mathbb{Z}$-graded algebras;
- $\mu : A \otimes A \to A$ is a morphism of $\mathbb{Z}$-graded coalgebras.

Bar Construction Let (A,d,ε) be a commutative, augmented differential $\mathbb{Z}$-graded algebra, $d(A_n) \subseteq A_{n-1}$ and $\overline{A} = \mathrm{Ker}(\varepsilon)$.

The *bar construction* BA is given by

$$BA = \left(T\left(s\overline{A}\right), D\right)$$

where

- $T\left(s\overline{A}\right)$ is the commutative differential graded Hopf algebra generated by $s\overline{A}$ where $s : A \to A$ is the suspension operator, i.e. $(sA)_n = A_{n-1}$.

- $D = d_I + d_E$, where

$$\begin{aligned} d_I\left(sa_1 \otimes \cdots \otimes sa_n\right) &= -\sum_{i=1}^{n} \eta\left(i-1\right) sa_1 \otimes \cdots \otimes sa_{i-1} \otimes sda_i \otimes \cdots sa_n \\ d_E\left(sa_1 \otimes \cdots \otimes sa_n\right) &= -\sum_{i=1}^{n} \eta\left(i-1\right) sa_1 \otimes \cdots \otimes sa_{i-1}a_i \otimes \cdots sa_n \end{aligned}$$

with $\eta\left(i\right) = \left(-1\right)^{\sum_{k=1}^{i}|sa_k|}$.

BA has a tensor algebra construction. This from one point of view handles the formal concatenation aspect, but has also a structure of a coalgebraic structure with reduced diagonal, given by

$$\overline{\Delta}\left(v_1 \otimes \cdots \otimes v_n\right) = \sum_{i=1}^{n-1} \left(v_1 \otimes \cdots \otimes v_p\right) \otimes \left(v_{p+1} \otimes \cdots \otimes v_n\right).$$

Cobar Construction Let (C, ∂, η) be a cocommutative differential Z-graded coaugmented coalgeba, $\partial\left(C_n\right) \subseteq C_{n-1}$, $\overline{C} = C/\eta\left(k\right)$, $\overline{\Delta} : \overline{C} \to \overline{C} \otimes \overline{C}$.

The cobar construction ΩC is the cocommutative pre-dgha defined by

$$\Omega C = \left(T\left(s^{-1}\overline{C}\right), \delta\right)$$

where

- $T\left(s^{-1}\overline{C}\right)$ is the commutative differential graded Hopf algebra generated by $s^{-1}\overline{C}$.

- $\delta = \partial_I + \partial_E$, where

$$\begin{aligned} \partial_I\left(s^{-1}c_1 \otimes \cdots \otimes s^{-1}c_n\right) &= -\sum_{i=1}^{n} \eta\left(i-1\right) s^{-1}c_1 \otimes \cdots \otimes s^{-1}c_{i-1} \otimes s^{-1}\partial c_i \otimes \cdots s^{-1}c_n \\ \partial_E\left(s^{-1}c_1 \otimes \cdots \otimes s^{-1}c_n\right) &= -\sum_{i=1}^{n} \eta\left(i-1\right) \sum_{\mu} \left(-1\right)^{|c'_{i\mu}|+1} s^{-1}c_1 \otimes \cdots \otimes s^{-1}c'_{i\mu} \otimes s^{-1}c''_{i\mu} \otimes \cdots s^{-1}c_n \end{aligned}$$

with $\eta\left(i\right) = \left(-1\right)^{\sum_{k=1}^{i}|s^{-1}c_k|}$ and $\overline{\Delta}\left(c_i\right) = \sum_{\mu} c'_{i\mu} \otimes c''_{i\mu}$.

Remark 4.4. If A is not (graded) commutative, the differential d_E of BA does not respect the shuffle product on $T\left(s\overline{A}\right)$; BA thus becomes merely a differential $\mathbb{Z}$-graded coalgebra. Similarly if C is not (graded) cocommutative ΩC is merely a differential $\mathbb{Z}$-graded algebra.

Twisting cochains

Definition 4.18. Let (C, d_C) be a differential graded coalgebra with comultiplication Δ and (A, d_A) a dg-algebra with multiplication μ. A *twisting cochain* is a morphism $\tau : C \to A[1]$ such that the following Maurer-Cartan equation holds:

$$d_A \circ \tau + \tau \circ d_C + \mu \circ (\tau \otimes \tau) \circ \Delta = 0.$$

Let $\mathbf{DGC}_0$ be the category of connected dg-coalgebras and $\mathbf{DGA}_a$ the category of augmented dg-algebras. Then the barconstruction functor $B : \mathbf{DGA}_a \to \mathbf{DGC}_0$ is a right adjoint to the cobar construction functor $\Omega : \mathbf{DGC}_0 \to \mathbf{DGA}_a$.

Starting from a dg-coalgebra map $f : C \to BA$, one constructs a twisting cochain τf by post-composing f by the natural projection $BA \to A[1]$, the Maurer-Cartan equation for τf translates to saying that f is a chain map, $d_{BA} \circ f = f \circ d_C$. One then replaces τf by the composition of the evident canonical map $\tau_0 : \Omega C \to C[-1]$ (called the canonical twisting cochain) and $\tau f[-1] : C[-1] \to A$ to obtain a morphism $f' : \Omega C \to A$. The Maurer–Cartan equation for τ is equivalent also to saying that f' is a chain map, i.e. $d_A \circ f = f' \circ d_{\Omega C}$.

4.7 *Dold-Kan Correspondence

4.8 Pointwise Kan Extensions

Definition 4.19. A right Kan extension is called *pointwise* if it is preserved by all (covariant) representable functors $h^d = \mathrm{Hom}(d, -) : \mathcal{D} \to \mathbf{Set}, d \in Ob(\mathcal{D})$.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\Uparrow"] & \mathcal{D} \arrow[r, "h^d"] & \mathbf{Set} \\
\mathcal{D} \arrow[ur, "L_G(F)"'] \arrow[urr, bend right, "L_G(h^d \circ F)"'] & &
\end{tikzcd}$$

Definition 4.20. A left Kan extension is called *pointwise* if it is mapped to a right Kan extension by all (contravariant) representable functors $h_d = \mathrm{Hom}\,(-,d) : \mathcal{D}^{op} \to \mathbf{Set}, d \in Ob\,(\mathcal{D})$.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\Downarrow"] & \mathcal{D} \arrow[r, "h_d"] & \mathbf{Set}^{op} \\
\mathcal{D} \arrow[ur, "R_G(F)"'] \arrow[urr, bend right, "L_G(h_d \circ F)"'] & &
\end{tikzcd}$$

Remark 4.5. This is very similar to the property of limits and colimits.

We have the following:

$$\text{Absolute Kan extensions} \subsetneqq \text{pointwise Kan extensions} \subsetneqq \text{Kan extensions.}$$

We will give a characterization (formula) for pointwise Kan extensions.

Comma Category

Definition 4.21. Given a functor $F : \mathcal{C} \to \mathcal{D}$ and an object $d \in Ob\,(\mathcal{D})$, we can define the *comma category* F/d (or $F \downarrow d$) as follows.

Objects: $Ob\,(F/d) = \{(c, f)\,|c \in Ob\,(\mathcal{C})\,, f \in \mathrm{Hom}_{\mathcal{D}}\,(Fc, d)\}$.

Morphisms: $\mathrm{Hom}_{F/d}\,((c, f)\,, (c', f')) = \{\varphi \in \mathrm{Hom}_{\mathcal{C}}\,(c, c')\,|f' \circ F\varphi = f\}$, i.e. morphisms are those $\varphi : c \to c'$ such that the following diagram

$$\begin{tikzcd}
Fc \arrow[rr, "F\varphi"] \arrow[dr, "f"'] & & Fc' \arrow[dl, "f'"] \\
& d &
\end{tikzcd}$$

commutes.

Dually we can define cocomma category $d\backslash F$ (or $d \downarrow F$).

Note that there is a forgetful functor $U : F/d \to \mathcal{C}$ which can be thought as a fibre functor.

Example 4.6. Let $F = \mathrm{Id}_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$, then F/d is the category over d and $d\backslash F$ is the category under d.

Example 4.7. (Category of simplicial sets) Take

$$\begin{array}{rccc} F = h_* : & \Delta & \to & \mathbf{sSet} \\ & [n] & \mapsto & \Delta[n]_* \end{array}$$

For any $X \in Ob(\mathbf{sSet})$ we call the category $\Delta X \coloneqq h_*/X$ the category of simplices of X, which is given by

Objects: $Ob(\Delta X) = \{([n], x) : [n] \in \Delta, x \in \mathrm{Hom}_{\mathbf{sSet}}(\Delta[n]_*, X) = X_n\} = \coprod_{n \geq 0} X_n$.

Morphisms: $\mathrm{Hom}(([n], x), ([m], y)) = \{f : [n] \to [m] \,|X(f)\, y = y \circ h_*(f) = x\}$.

Another way to define ΔX is to consider $X : \Delta^{op} \to \mathbf{Set}$ and take the Grothendieck construction $\Delta^{op}_X = \Delta^{op} \int X$ where

Objects: $Ob\left(\Delta^{op}_X\right) = \{([m], y) \,|\, [m] \in Ob(\Delta^{op}) = Ob(\Delta), y \in X([m]) = X_m\}$.

Morphisms: $\mathrm{Hom}(([n], x)([m], y)) = \{f \in \mathrm{Hom}_{\Delta^{op}}([n], [m]) \,|X(f)\, x = y\}$.

Hence $\Delta^{op}_X \cong (\Delta X)^{op}$.

Example 4.8. Let $\mathcal{C}$ be a small category and take

$$\begin{array}{rccc} \Delta^* : & \Delta & \hookrightarrow & \mathbf{Cat} \\ & [n] & \mapsto & \overrightarrow{n} = \{0 \to 1 \to \cdots \to n\} \end{array}$$

This is a fully faithful functor. Then $\Delta^*/\mathcal{C} \coloneqq$ the simplicial complex over $\mathcal{C}$ defined by

Objects: $Ob(\Delta^*/\mathcal{C}) = \{([n], f) \,|\, [n] \in \Delta, f : \overrightarrow{n} \to \mathcal{C}\} \cong \coprod_{n \geq 0} \mathcal{N}_n \mathcal{C}$.

Morphisms: $\mathrm{Hom}(([n], f), ([m], g)) = \{\varphi : \overrightarrow{n} \to \overrightarrow{m} | g \circ \Delta^*(\varphi) = f\}$.

Therefore $\Delta^*/\mathcal{C} \cong \Delta\mathcal{N}\mathcal{C}$.

Remark 4.6. ΔX and $(\Delta X)^{op}$ are examples of Reedy categories (with fibrant or cofibrant, respectively).

Computing Kan extension via (co)limits

Theorem 4.2. *A left Kan extension is pointwise if and only if it can be computed by the formula*

$$L_G(F)(e) = colim_{G/e}\left(G/e \xrightarrow{U} \mathcal{C} \xrightarrow{F} \mathcal{D}\right).$$

Theorem 4.3. *(Dual version) A right Kan extension is pointwise if and only if it can be computed by the formula*

$$R_G(F)(e) = lim_{G/e}\left(e\backslash G \xrightarrow{U} \mathcal{C} \xrightarrow{F} \mathcal{D}\right).$$

Proof. It suffices (and more convenient) to prove the dual version. Indeed, $L_G(F)$ is characterized by

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Fun}(\mathcal{E},\mathcal{D})}(L_G(F), H) & \cong & \mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathcal{D})}(F, G_*H) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{Hom}_{\mathbf{Fun}(\mathcal{E},\mathcal{D})^{op}}(H, L_G(F)) & & \mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathcal{D})^{op}}(G_*H, F) \end{array}$$

Note that $\mathbf{Fun}(\mathcal{C},\mathcal{D})^{op} = \mathbf{Fun}(\mathcal{C}^{op},\mathcal{D})$, so $R_{G^0}\left(F^0\right) \cong L_G(F)$.

Since limits commutes with representable functors, i.e. given $F : \mathcal{J} \to \mathcal{D}$,

$$\mathrm{Hom}_{\mathcal{D}}\left(d, \lim_{\mathcal{J}} F\right) \cong \lim_{\mathcal{J}}\left(\mathrm{Hom}_{\mathcal{D}}(d, F(-))\right)$$

so if $R_G(F)$ is given by such a formula, then it automatically commutes with representable functors $h^d = \mathrm{Hom}_{\mathcal{D}}(d, -), \forall d \in \mathcal{D}$.

Assume that $R_G(F)$ is pointwise, then for any $d \in Ob(\mathcal{D})$ and any $e \in Ob(\mathcal{E})$,

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}}(d, R_GF(e)) &= h^d(R_GF(e)) = \left(h^d \circ R_GF\right)(e) \\ &= \mathrm{Hom}_{\mathbf{Fun}(\mathcal{E},\mathbf{Set})}\left(h^e, h^d \circ R_GF\right) \\ &\cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{E},\mathbf{Set})}\left(h^e, R_G\left(h^d \circ F\right)\right) \\ &\cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathbf{Set})}\left(h^e \circ G, h^d \circ F\right) \\ &\cong \mathrm{Hom}_{\mathbf{Fun}(\mathcal{C},\mathbf{Set})}\left(\mathrm{Hom}_{\mathcal{E}}(e, G(-)), \mathrm{Hom}_{\mathbf{Fun}(\mathcal{D},\mathbf{Set})}(d, F(-))\right) \\ &\cong \text{the set of cones under } d \text{ of the functor } FU \\ &= \mathrm{Hom}_{\mathbf{Fun}(e\backslash G,\mathcal{D})}(\mathrm{const}_d, FU) \\ &\cong \mathrm{Hom}_{\mathcal{D}}\left(d, \lim\nolimits_{e\backslash G} FU\right) \end{aligned}$$

By Yoneda lemma, $R_G F(e) \cong \lim_{e\backslash G}(FU)$. $\square$

Corollary 4.1. *If $\mathcal{D}$ is cocomplete then every left Kan extension of $F : \mathcal{C} \to \mathcal{D}$ exists and is pointwise. If $\mathcal{D}$ is complete then every right Kan extension of $F : \mathcal{C} \to \mathcal{D}$ exists and is pointwise.*

Corollary 4.2. *If $\mathcal{D}$ is cocomplete and G is fully faithful, then η_{un} is an isomorphism of functors.*

Proof. Take $c \in Ob(\mathcal{C})$ and consider $G/G(c)$, then G is fully faithful implies that $G/G(c)$ has terminal object. Indeed, $Ob(G/G(c)) = \{(c', f') \mid c' \in Ob(\mathcal{C}), f' : Gc' \to Gc\}$. Then $\left(c, \mathrm{Id}_{G(c)}\right)$ is terminal in $G/G(c)$ because

$$\mathrm{Hom}\left(\left(c', f'\right), \left(c, \mathrm{Id}_{G(c)}\right)\right) = \left\{h : c' \to c \middle| G(h) \circ \mathrm{Id}_{G(e)} = f'\right\} = G^{-1}\left(f'\right)$$

contains only one element.

Recall, by UMP of colimits, if $\mathcal{J}$ has terminal object $*$, then for any $F : \mathcal{J} \to \mathcal{D}$, $\mathrm{colim}_{\mathcal{J}}(F) = F(*)$. Now for any $c \in Ob(\mathcal{C})$, take $e = G(c)$ and apply formula

$$L_G(F)(e) = \mathrm{colim}_{G/G(e)}(FU) \cong FU\left(c, \mathrm{Id}_{G(c)}\right) = F(c)$$

So $L_G F \circ G \cong F$. $\square$

Example 4.9. (Co-Yoneda lemma) Simplest version. Consider

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "\mathrm{Id}_{\mathcal{C}}"'] & \mathcal{D} \\
\mathcal{C} \arrow[ur, "L_{\mathrm{Id}_{\mathcal{C}}} F \cong F"']
\end{tikzcd}$$

then we have $L_{\mathrm{Id}_{\mathcal{C}}} F \cong F$ and $F(c) \cong \mathrm{colim}_{\mathcal{C}/c}\left(\mathcal{C}/c \xrightarrow{U} \mathcal{C} \xrightarrow{F} \mathcal{D}\right)$.

Example 4.10. Take $\hat{\mathcal{C}} = \mathbf{Fun}(\mathcal{C}^{op}, \mathbf{Set})$ and Yoneda functor $h_* : \mathcal{C} \hookrightarrow \hat{\mathcal{C}}$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook, "h_*"] \arrow[d, hook, "h_*"'] & \hat{\mathcal{C}} \\
\hat{\mathcal{C}} \arrow[ur, "L_h(h) \cong \mathrm{Id}_{\hat{\mathcal{C}}}"']
\end{tikzcd}$$

Every presheaf on a small category $\mathcal{C}$ is canonically a colimit of representable presheaf. For any $X \in Ob\left(\hat{\mathcal{C}}\right)$,

$$X \cong \operatorname{colim}_{h_*/X}\left(h_*/X \xrightarrow{U} \mathcal{C} \xrightarrow{h_*} \hat{\mathcal{C}}\right).$$

Take $\mathcal{C} = \Delta$, then $\hat{\mathcal{C}} = \mathbf{sSet}$ and $h/X = \Delta X$ is the category of simplices over X.

$Ob\left(\Delta X\right) = \coprod_{n \geq 0} X_n$. Note $X_n \cong \operatorname{Hom}_{\mathbf{sSet}}\left(\Delta\left[n\right]_*, X\right)$.

$\operatorname{Hom}_{\Delta X}\left(\left(\left[n\right], x\right), \left(\left[m\right], y\right)\right) = \left\{f : \left[n\right] \to \left[m\right] \middle| X\left(f\right) y = x\right\}$.

$$\begin{tikzcd}
\Delta \arrow[r, hook, "h_*"] \arrow[d, "h_*"'] & \hat{\Delta} = \mathbf{sSet} \\
\mathbf{sSet} \arrow[ur, "L_h(h) \cong \mathrm{Id}_{\mathbf{sSet}}"'] &
\end{tikzcd}$$

So we have $X = \operatorname{colim}_{\Delta X}\left(\Delta X \xrightarrow{U} \Delta \xrightarrow{h} \mathbf{sSet}\right) = \operatorname{colim}_{\Delta[n]_* \to X} \Delta\left[n\right]_*$.

Equivalence of Categories

Given two locally small categories $\mathcal{C}$ and $\mathcal{D}$, define $\mathbf{Adj}\left(\mathcal{C}, \mathcal{D}\right)$ to be the category of adjunctions as

Objects: $\left\{\left(L, R, \varphi\right) \mid L : \mathcal{C} \rightleftarrows \mathcal{D} : R, \varphi : \operatorname{Hom}_{\mathcal{D}}\left(F\left(-\right), -\right) \xrightarrow{\cong} \operatorname{Hom}_{\mathcal{C}}\left(-, G\left(-\right)\right)\right\}$ where φ is an isomorphism of bifunctors $\mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$.

Morphisms: $\operatorname{Hom}_{\mathbf{Adj}(\mathcal{C},\mathcal{D})}\left(\left(L, R, \varphi\right), \left(L', R', \varphi'\right)\right) = \left\{\left(\alpha, \beta\right) \middle| \alpha : L \Rightarrow L', \beta : R' \Rightarrow R, \varphi' = \beta_* \circ \varphi \circ \alpha^*\right\}$.

Explicitly, for each $c \in Ob\left(\mathcal{C}\right), d \in Ob\left(\mathcal{D}\right)$, we have a commutative (factorization) diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{D}}\left(L'\left(c\right), d\right) \arrow[r, "(\alpha_c)^*"] \arrow[d, "\varphi'_{c,d}"'] & \operatorname{Hom}_{\mathcal{D}}\left(L\left(c\right), d\right) \arrow[d, "\varphi_{c,d}"] \\
\operatorname{Hom}_{\mathcal{C}}\left(c, R'\left(d\right)\right) & \operatorname{Hom}_{\mathcal{C}}\left(c, R\left(d\right)\right) \arrow[l, "(\beta_d)_*"']
\end{tikzcd}$$

Proposition 4.2. *Let $\mathcal{C}$ be a small category, and $\mathcal{D}$ a locally small, cocomplete, then there exists a natural equivalence of categories*

$$\Phi : \mathcal{D}^{\mathcal{C}} \underset{}{\overset{\cong}{\rightleftarrows}} \mathbf{Adj}\left(\hat{\mathcal{C}}, \mathcal{D}\right) : \Psi$$

where Ψ is defined by restriction

$$\Psi\left(\ L : \hat{\mathcal{C}} \rightleftarrows \mathcal{D} : R\ , \varphi\right) = \left(h^*(L) = L \circ h : \mathcal{C} \overset{h}{\hookrightarrow} \hat{\mathcal{C}} \xrightarrow{L} \mathcal{D}\right)$$
$$\Psi(\alpha, \beta) = h^*(\alpha) = \alpha \circ h$$

Proof. Construction of Φ. Given $F \in Ob\left(\mathcal{D}^{\mathcal{C}}\right)$, define

$$\Phi(F) : \left(\ L(F) : \hat{\mathcal{C}} \rightleftarrows \mathcal{D} : R(F)\ , \varphi\right)$$

where $L(F) = L_h(F)$ and

$$\begin{array}{rccl} R(F): & \mathcal{D} & \to & \hat{\mathcal{C}} \\ & d & \mapsto & \mathrm{Hom}_{\mathcal{D}}(F(-), d) = h_d \circ F : \mathcal{C}^{op} \to \mathbf{Set} \end{array}$$

Take any $c \in Ob(\mathcal{C}), d \in Ob(\mathcal{D})$ and consider $h_c \in Ob\left(\hat{\mathcal{C}}\right)$,

$$\mathrm{Hom}_{\hat{\mathcal{C}}}(h_c, RF(d)) \overset{\text{Yoneda}}{\cong} RF(d)(c) = \mathrm{Hom}_{\mathcal{D}}(Fc, d) = \mathrm{Hom}_{\mathcal{D}}(L(F))h_c, d)$$

where the last equality follows from lemma on left Kan extension along fully faithful functors.

Hence $R(F)$ is right adjoint to $L(F)$ on the representable functors.

To extend it to all presheaves $X : \mathcal{C}^{op} \to \mathbf{Set}$, we need the following facts

1. Co-Yonada lemma.

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook, "h"] \arrow[d, hook, "h"'] & \hat{\mathcal{C}} \\ \hat{\mathcal{C}} \arrow[ur, "\mathrm{Id}_{\mathcal{C}}"'] & \end{tikzcd}$$

   Every X is canonically a colimit to h_c's.

$$X \cong \mathrm{colim}_{h/X}\left(h/X \xrightarrow{U} \mathcal{C} \overset{h}{\hookrightarrow} \hat{\mathcal{C}}\right) \cong \mathrm{colim}_{h/X}(h).$$

2. Left Kan extension is given by

$$L(F)X = L_hF(X) := \mathrm{colim}_{h/X}\left(h/X \xrightarrow{U} \mathcal{C} \xrightarrow{F} \mathcal{D}\right) = \mathrm{colim}_{h/X}(F).$$

Since colimit commutes with hom,

$$\begin{aligned}
\mathrm{Hom}_{\hat{\mathcal{C}}}(X, RF(d)) &\cong \mathrm{Hom}_{\hat{\mathcal{C}}}\left(\mathrm{colim}_{h/X}(h), R(F)(d)\right) \\
&\cong \lim_{h/X} \mathrm{Hom}_{\hat{\mathcal{C}}}(hU(-), R(F)(d)) \\
&\cong \lim_{h/X} (\mathrm{Hom}_{\mathcal{D}}(LF(hU(-)), d)) \\
&\cong \mathrm{Hom}_{\mathcal{D}}\left(\mathrm{colim}_{h/X} LF(hU(-)), d\right) \\
&\cong \mathrm{Hom}_{\mathcal{D}}\left(LF\left(\mathrm{colim}_{h/X} h\right), d\right) \\
&\cong \mathrm{Hom}_{\mathcal{D}}(LFX, d)
\end{aligned}$$

We have $\Phi \circ \Psi\left(L : \hat{\mathcal{C}} \rightleftarrows \mathcal{D} : R\, , \varphi\right) = \left(L : \hat{\mathcal{C}} \rightleftarrows \mathcal{D} : R\, , \varphi\right)$ and a natural transformation $F \Rightarrow \Psi \circ \Phi(F) = h \circ L_h F$ by the universal property of left Kan extension.

Check that Φ and Ψ are inverse to each other. $\square$

Example 4.11. [Dold-Kan] The adjoint pair

$$\mathcal{N} : \mathbf{Ab}^{\Delta^{op}} \rightleftarrows \mathbf{Ch}_{\geq 0}(\mathbf{Ab}) : \Gamma$$

comes from a cosimplicial object

$$\begin{array}{ccccccc}
\Delta & \xrightarrow{\Delta^*} & \mathbf{sSet} & \xrightarrow{\mathbb{Z}[-]} & \mathbf{sAb} & \xrightarrow{\mathcal{N}} & \mathbf{Ch}_{\geq 0}(\mathbf{Ab}) \\
[n] & \mapsto & \Delta[n] & \mapsto & \mathbb{Z}[\Delta[n]] & \mapsto & \mathcal{N}_*(\mathbb{Z}[\Delta[n]])
\end{array}$$

4.9 *Simplicial groups and spaces

Twisted Cartesian products and principal bundles

Let $\mathbf{sGr} = \mathbf{Fun}(\Delta^{op}, \mathbf{Gr})$ be the category of simplicial groups.

Let $G_* = \{G_n\}_{n\geq 0} \in Ob(\mathbf{sGr})$ and $X_* \in Ob(\mathbf{sSet})$.

Definition 4.22. A *twisting function* $\tau : X_* \to G_{*-1}$ is a family of maps $\{\tau_n : X_n \to G_{n-1}\}_{n\geq 1}$

such that

$$\begin{array}{rcll} d_0\left(\tau\left(x\right)\right) & = & \tau\left(d_0 x\right)^{-1}\tau\left(d_1 x\right) & \\ d_i\left(\tau\left(x\right)\right) & = & \tau\left(d_{i+1}x\right) & i\geq 1 \\ s_j\left(\tau\left(x\right)\right) & = & \tau\left(s_{j+1}x\right) & j\geq 0 \\ \tau\left(s_0\left(x\right)\right) & = & 1_{G_n} & \forall x\in G_n \end{array}$$

Definition 4.23. A *(principal) twisted Cartesian product* with fibre G_* and base X_*, and twisting function $\tau : X_* \to G_{*-1}$ is a simplicial set $E_* = G_* \times_\tau X_*$ with

$$E_n := G_n \times X_n, n \geq 0$$

and

$$\begin{array}{rcll} d_i\left(g,x\right) & = & \begin{cases} \left(\tau\left(x\right)\cdot d_0 g, d_0 x\right) & i=0, \\ \left(d_i g, d_i x\right) & i>0. \end{cases} \\ s_j\left(g,x\right) & = & \left(s_j g, s_j x\right) & j\geq 0. \end{array}$$

Proposition 4.3. *Any principal G_*-fibration $p_* : E_* \to X_*$ with right G_* action on E_* with local cross section $\sigma_* : X_* \to E_*$ (i.e. $\sigma_n : X_n \to E_n$ such that $p_n\sigma_n = Id_{X_n}$ and $d_i\sigma = \sigma d_i, \forall i > 0$, $s_j\sigma = \sigma s_j, \forall j \geq 0$) can be identified with $G \times_\tau X \to X$ where $\tau : X_* \to G_{*-1}$ is determined by $d_0\sigma\left(x\right) = \sigma\left(d_0 x\right)\cdot\tau\left(x\right)$.*

The classifying space of a simplicial group $\overline{\mathcal{W}}$

Given $G_* \in Ob\left(\mathbf{sGr}\right)$, define a reduced simplicial set $\overline{\mathcal{W}}\left(G_*\right)$ by

$$\overline{\mathcal{W}}_0\left(G\right) := \left\{*\right\}, \overline{\mathcal{W}}_n\left(G\right) := G_{n-1}\times G_{n-2}\times\cdots\times G_n, n\geq 0$$

with

$$\begin{array}{rrcl} s_0 : & \overline{\mathcal{W}}_0\left(G\right) & \to & \overline{\mathcal{W}}_1\left(G\right) \\ & * & \mapsto & 1_{G_0} \\ d_0 = d_1 : & \overline{\mathcal{W}}_1\left(G\right) & \to & \overline{\mathcal{W}}_0\left(G\right) \\ & g & \mapsto & * \end{array}$$

and for $n \geq 1$,

$$\begin{aligned}
d_0 (g_{n-1}, \cdots, g_0) &= (g_{n-2}, \cdots, g_0) \\
d_{i+1} (g_{n-1}, \cdots, g_0) &= (d_i g_{n-1}, \cdots, d_1 g_{n-i}, g_{n-i-2} \cdot d_0 g_{n-i-1}, g_{n-i-3}, \cdots, g_0) \\
s_0 (g_{n-1}, \cdots, g_0) &= (1, g_{n-1}, \cdots, g_0) \\
s_{j+1} (g_{n-1}, \cdots, g_0) &= (s_j g_{n-2}, \cdots s_0 g_{n-j-1}, 1, g_{n-i-2}, \cdots, g_0)
\end{aligned}$$

This is a simplicial set with a twisting function

$$\begin{aligned}
\tau_n (G): \quad \overline{\mathcal{W}}_n (G) &\to G_{n-1} \\
(g_{n-1}, \cdots, g_0) &\mapsto g_{n-1}
\end{aligned}$$

Lemma 4.1. *$\tau (G)$ is a universal twisting function in the sense that any principal twisted product $G \times_\tau X$ can be induced from $G_* \times_{\tau(G)} X_*$ by a unique classifying map $X_* \mapsto \overline{\mathcal{W}} (G_*)$ given by*

$$x \in X_n \mapsto \left(\tau (x), \tau (d_0 x), \cdots, \tau \left(d_0^{n-1} x\right)\right) \in \overline{\mathcal{W}}_n (G_*).$$

Example 4.12. If $G_* = \{G_n\}_{n \geq 0}$ is a discrete simplicial group, $\overline{\mathcal{W}} (G) = B_* G$ the simplicial nerve of G, then

$$\begin{aligned}
G \times_{\tau(G)} \overline{\mathcal{W}} (G) &\cong E_* G \\
G^{n+1} &\longleftrightarrow E_n (G) \\
(g_0, g_0 g_1, \cdots, g_0 \cdots g_n) &\longleftrightarrow (g_0, \cdots, g_n)
\end{aligned}$$

The Kan loop group of simplicial sets

Conversely, given $X_* \in Ob (\mathbf{sSet}_0)$ a reduced simplicial set, define the Kan loop group of X, $\mathbb{G} (X)_* \in Ob (\mathbf{sGr})$ by

$$\mathbb{G}_n (X) \coloneqq \mathbb{F} \langle X_{n+1} \rangle / (s_0 (x) = 1, \forall x \in X_n)$$

induced by

$$B_n = X_{n+1} \backslash s_0 (X_n) \hookrightarrow X_{n+1}$$

(but $\{B_n\}$ do not form a simplicial set), with

$$\begin{aligned} d_i^{\mathbb{G}}(x) &= \begin{cases} d_1(x)\, d_0(x)^{-1} & i = 0, \\ d_{i+1}x & i > 0. \end{cases} \\ s_j^{\mathbb{G}}(x) &= s_{j+1}x \qquad\qquad j \geq 0. \end{aligned}$$

Define

$$\tau(X) : X_* \to \mathbb{G}(X)_{*-1}$$

by

$$\tau_n(X) : X_n \hookrightarrow \mathbb{F}\langle X_n \rangle \twoheadrightarrow \mathbb{G}X_{n-1}.$$

Given $X_* \in Ob(\mathbf{sSet})$ and $G_* \in Ob(\mathbf{sGr})$ define

$$\mathrm{Tw}(X_*, G_*) \coloneqq \{\text{twisting functions } \tau : X_* \to G_*\}.$$

Theorem 4.4. *There are natural bijections*

$$\begin{array}{ccccc} Hom_{\mathbf{sGr}}(\mathbb{G}X_*, G_*) & \xrightarrow{\sim} & Tw(X_*, G_*) & \xleftarrow{\sim} & Hom_{\mathbf{sSet}}\left(X_*, \overline{\mathcal{W}}G_*\right) \\ f & \mapsto & f \circ \tau(X) & & \\ & & \tau(G) \circ g & \leftarrow\!\shortmid & g \end{array}$$

Hence we have adjunction

$$\mathbb{G} : \mathbf{sSet}_0 \rightleftarrows \mathbf{sGr} : \overline{\mathcal{W}}.$$

Theorem 4.4, proposition 4.3 and lemma 4.1 implies

Corollary 4.3. *For fixed $G_* \in Ob(\mathbf{sGr}), X \in Ob(\mathbf{sSet})$, there is a natural bijection between the set of twisting function $Tw(X_*, G_*)$ and the isomorphism classes of pairs (E_*, G_*) where E_* is a principal G_*-bundle over X_* with local section $\sigma : X_* \to E_*$. The bijection is given by*

$$\tau \mapsto (G \times_\tau X, \sigma)$$

where $d_0\sigma_n(x) = \sigma_{n-1}(d_0x)\,\tau(x)$.

Our main theorem is the following.

Theorem 4.5. *(Kan)*

1. *For $X_* \in Ob(\mathbf{sSet}_0)$ a reduced simplicial set and $G_* \in Ob(\mathbf{sGr})$, there are weak homotopy equivalences of spaces*

$$\begin{aligned} |\mathbb{G}(X_*)| &\simeq \Omega|X_*| \\ \left|\overline{\mathcal{W}}(G_*)\right| &\simeq B|G_*| \end{aligned}$$

This shows that the homotopy type of $|\mathbb{G}(X_*)|$ is the loop space of $|X|$, and the homotopy type of $\left|\overline{\mathcal{W}}(G_*)\right|$ is the classifying space of $|G_*|$, which is the reason for the name of the two functors.

2. *The adjoint functors $\left(\mathbb{G}, \overline{\mathcal{W}}\right)$ give Quillen pair of model categories*

$$\mathbb{G} : \mathbf{sSet}_0 \rightleftarrows \mathbf{sGr} : \overline{\mathcal{W}}$$

and

$$Ho(\mathbf{sSet}_0) \cong Ho(\mathbf{sGr}).$$

4.10 Geometric Bar Construction

Let $\mathcal{U}$ be the category of compactly generated weak Hausdorff spaces. Let $\mathcal{W}$ be the full subcategory of $\mathcal{U}$ of spaces having the same homotopy type of CW complexes.

Neighborhood Deformation Retract

The object is to define a category of pairs (X, A) where $X \in Ob(\mathcal{U})$ and A is a closed subset in X having certain useful properties. Most important is that each (X, A) should have the homotopy extension property (i.e. the inclusion $A \subset X$ is a cofibration). Also we require that the category should be closed under the operations of forming products and adjunction spaces.

Definition 4.24. [M1] A closed subset A of a space $X \in Ob(\mathcal{U})$ is called a *neighborhood deformation retract* (NDR) in X if there exists a map $u : X \to I$ such that $A = u^{-1}(0)$ and a homotopy $h : I \times X \to X$ such that $h(0, x) = x$ for all $x \in X$, $h(t, a) = a$ for all $(t, a) \in I \times A$, and $h(1, x) \in A$ for all $x \in u^{-1}[0, 1)$. The pair (X, A) is called an *NDR-pair*. The pair (h, u) is said to be a *representation* of (X, A) as an NDR-pair. If in addition, $h(1, x) \in A$ for all $x \in X$, then A is a *deformation retract* of X and (X, A) is a *DR-pair*. An NDR-pair (X, A) is a *strong NDR-pair* if $u \circ h(t, x) < 1$ whenever $u(x) < 1$; thus if $B = u^{-1}[0, 1)$, it is required that (h, u) restrict to a representation of (B, A) as a DR-pair.

The pair $(X, \emptyset)$ is always an NDR pair. (X, X) and $(I, \{0\})$ is always a DR pair.

Theorem 4.6. *If (X, A) and (Y, B) are NDR pairs, then so is their product*

$$(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y).$$

If in addition, one of them is a DR pair, then their product is a DR pair.

Proof. Let (h, u) and (j, v) be the representations of (X, A) and (Y, B) respectively. Define

$$\begin{aligned} w: \quad X \times Y & \to \quad I \\ (x, y) & \mapsto u(x) v(y) \end{aligned}$$

then $X \times B \cup A \times Y = w^{-1}(0)$. Define the homotopy

$$\begin{aligned} q: \quad I \times X \times Y & \to \quad X \times Y \\ (t, x, y) & \mapsto \begin{cases} (x, y) & x \in A, y \in B \\ \left(h(t, x), j\left(\frac{u(x)}{v(y)} t, y\right)\right) & u(x) \leq v(y), v(y) > 0 \\ \left(h\left(\frac{v(y)}{u(x)} t, x\right), j(t, y)\right) & v(y) \leq u(x), u(x) > 0 \end{cases} \end{aligned}$$

The domain of the last two lanes intersect in the relatively closed set where $u(x) = v(y) > 0$ which both reduce to $(h(t, x), j(t, y))$, thus q is continuous on $I \times (X \times Y - A \times B)$. Thus we just need to verify that q is continuous at a point (t, x, y) in $I \times A \times B$. Let $U \ni x, V \ni y$ be open sets in X and Y respectively, then we have $I \times \{x\} \in h^{-1}(U)$ and $I \times \{y\} \in j^{-1}(V)$. Since I is compact and $h^{-1}(U)$ is open, there is an open neighborhood $S \ni x$ such that $I \times S \subset h^{-1}(U)$. Similarly, there is an open neighborhood $T \ni y$ such that $I \times T \subset j^{-1}(V)$, so $q(I \times S \times T) \subset U \times V$. Thus q is continuous.

We also see that $q(t, x, y) = (x, y)$ whenever $x \in A$ (equivalently, $u(x) = 0$)or $y \in B$ (equivalently, $v(y) = 0$).

When $t = 1$, suppose that $0 < w(x, y) < 1$.

1. If $u(x) < 1$, we have
   (a) $u(x) \le v(y)$, then $q(1, x, y) = \left(h(1, x), j\left(\frac{u(x)}{v(y)}, y\right)\right) \in A \times Y$.
   (b) $v(y) \le u(x)$, then $q(1, x, y) = \left(h\left(\frac{v(y)}{u(x)}, x\right), j(1, y)\right) \in X \times B$.
2. If $v(y) < 1$, the proof is similar.

Thus $(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y)$ is a NDR pair.

In addition, if (h, u) represents (X, A) as a DR pair, replace u by $u' = \frac{1}{2}u$, then (h, u') also represents (X, A) as a DR pair, then the product is a DR pair. □

Theorem 4.7. *If X is CGWH and A is closed in X, then TFAE:*

1. *(X, A) is an NDR,*
2. *$\{0\} \times X \cup I \times A$ is a DR of $I \times X$,*
3. *$\{0\} \times X \cup I \times A$ is a retract of $I \times X$,*
4. *(X, A) has the homotopy extension property, i.e. the inclusion $A \subset X$ is a cofibration.*

Proof. $1 \implies 2$: By previous theorem 4.6.

$2 \implies 3$: trivial.

$3 \implies 4$: trivial.

$4 \implies 1$: □

Geometric Realization

Let Δ^n denote the standar topological n-simplex

$$\Delta^n = \left\{(x_0, \cdots, x_n) \in \mathbf{R}^{n+1} | \sum x_i = 1, x_i \ge 0\right\}.$$

Define $\delta_i : \Delta^{n-1} \to \Delta^n$ and $\sigma_j : \Delta^{n+1} \to \Delta^n$ for $0 \leq i, j \leq n$ by

$$\begin{aligned} \delta_i (x_0, \cdots, x_{n-1}) &= (x_0, \cdots, x_{i-1}, 0, x_i, \cdots, x_{n-1}) \\ \sigma_j (x_0, \cdots, x_{n+1}) &= (x_0, \cdots, x_{i-1}, x_i + x_{i+1}, \cdots, x_{n+1}) \end{aligned}$$

Definition 4.25. Given a simplicial object X in $\mathcal{U}$, $X \in Ob(s\mathcal{U})$, the *geometric realization* $|X|$ of X is a topological space defined as

$$|X| = \coprod_{n \geq 0} X_n \times \Delta^n / \sim$$

where $(\partial_i x, u) \sim (x, \delta_i u)$ for $x \in X_n, u \in \Delta^{n-1}$ and $(s_j x, u) \sim (x, \sigma_j)$ for $x \in X_n, u \in \Delta^{n+1}$. The topology on X is given by the coproduct topology of the quotient topology of the product topology.

If X is a simplicial set, then the classical geometric realization of X coincides with the geometric realization of X as a discrete simplicial space.

Definition 4.26. Let X be a simplicial space, $X \in Ob(s\mathcal{U})$, define $sX_n = \bigcup_{j=0}^{n} s_j X_n \subset X_{n+1}$. We say X is *proper* if each (X_{n+1}, sX_n) is a strong NDR pair and X is *strictly proper* if in addition, each $(X_{n+1}, s_k X_n), 0 \leq k \leq n$ is a strong NDR pair via a homotopy $h : I \times X_{n+1} \to X_{n+1}$ such that

$$h\left(I \times \bigcup_{j=0}^{k-1} s_j X_n\right) \subset \bigcup_{j=1}^{k-1} s_j X_n$$

Theorem 4.8. *Fix $n \geq 0$. If X is a (strictly) proper simplicial space scuh that each X_i is $(n-i)$-connected for all $i \leq n$, then $|X|$ is n-connected.*

Proof. □

Definition 4.27. An object $X \in Ob(s\mathcal{U})$ is *cellular* if each X_n is a CW complex and each ∂_i, s_j is a cellular map.

Theorem 4.9. *For $X, Y \in Ob(s\mathcal{U})$, the map $|\pi_1| \times |\pi_2| : |X \times Y| \to |X| \times |Y|$ is a natural homeomorphism whose inverse ζ is commutative and associative and is cellular if X and Y are cellular.*

Classifying Spaces of Topological Monoid

Let G be a topological monoid such that its identity element e is a strongly nondegenerated basepoint (in the sense that (G,e) is a strong NDR-pair).

Let X and Y be left and right G-spaces. Define a simplicical topological space $B_*(Y,G,X)$ by letting the space of j-simplices be $Y\times G^j\times X$ with elements $y[g_1,\cdots,g_j]x$ and the face and degeneracy maps are

$$\partial_i\left(y[g_1,\cdots,g_j]x\right)=\begin{cases} yg_1[g_2,\cdots,g_j]x & i=0\\ y[g_1,\cdots,g_{i-1},g_ig_{i+1},g_{i+2},\cdots,g_j]x & 1\le i<j\\ y[g_1,\cdots,g_{j-1}]g_jx & i=j\end{cases}$$

and

$$s_j\left(y[g_1,\cdots,g_j]x\right)=y[g_1,\cdots,g_i,e,g_{i+1},\cdots,g_j]x.$$

Let $B(Y,G,X)$ be the geometric realization of $B_*(Y,G,X)$. Then B is a functor

$$\begin{array}{rccc} B: & \mathcal{A}(\mathcal{U}) & \to & \mathcal{U}\\ & (Y,G,X) & \mapsto & |B_*(Y,G,X)|\end{array}$$

where $\mathcal{A}(\mathcal{U})$ is the category where objects are triples (Y,G,X) and morphisms are (k,f,j) : $(Y,G,X)\to(Y',G',X')$ where $f:G\to G'$ is a map of topological monoids, and $k:Y\to Y'$ and $j:X\to X'$ are f-equivariant maps, i.e. $j(gx)=f(g)j(x)$ and $k(yg)=k(y)f(g)$.

Let $*$ be the one-point G-space and define

$$\begin{aligned} BG &= B(*,G,*)\\ EG &= B(*,G,G)\end{aligned}$$

then BG is the standard classifying space of G.

Now we state basic facts about the topological behavior of the functor B.

Proposition 4.4. *$B_*(Y,G,X)$ is a proper simplicial space. $B(Y,G,X)$ is n-connected if G is $(n-1)$-connected and X and Y are n-connected.*

Proof. The first statement is equivalent to say that

$$(Y, \emptyset) \times (G, e)^j \times (X, \emptyset)$$

is a strong NDR pair, which follows from Theorem 4.6.

The second statement follows from Theorem 4.8. □

Proposition 4.5. *If Y, G, X are in $\mathcal{W}$ then so is $B(Y, G, X)$.*

Proposition 4.6. *Let $(k, f, j) : (Y, G, X) \to (Y', G', X')$ be a morphism in $\mathcal{A}(\mathcal{U})$.*

1. *If k, f, j induces isomorphsims on integral homology, then so does $B(k, f, j)$.*

2. *If k, f, j are homotopy equivalences, then so is $B(k, f, j)$.*

Proposition 4.7. *B preserves products.*

Definition 4.28. A map $p : E \to B$ is a *quasifibration* if p is onto and

$$p_* : \pi_i\left(E, p^{-1}(x), y\right) \to \pi_i(B, x)$$

is an isomorphism for all $x \in B$, $y \in p^{-1}(x)$ and $i \geq 0$. A subset $U \subseteq B$ is *distinguished* if $p : p^{-1}(U) \to U$ is a quasi-fibration.

Lemma 4.2. *Let $p : E \to B$ be a map onto a filtered space B, then each $F_j B$ is distinguished and p is a quasifibration provided that*

1. *$F_0 B$ and each open subset of $F_j B - F_{j-1} B$ for $j > 0$ is distinguished.*

2. *For each $j > 0$, there is an open subset U of $F_j B$ which contains $F_{j-1} B$ and there are homotopies $h_t : U \to U$ and $H_t : p^{-1}(U) \to p^{-1}(U)$ such that*

   (a) $h_0 = 1, h_t(F_{j-1}B) \subset F_{j-1}B$ and $h_1(U) \subset F_{i-1}B$.

   (b) $H_0 = 1$ and H covers h, $pH_t = h_t p$; and

   (c) $H_1 : p^{-1}(x) \to p^{-1}(h_1(x))$ is a weak homotopy equivalence for all $x \in U$.

Let $p : B(Y, G, X) \to B(Y, G, *)$ and $q : (Y, G, X) \to B(*, G, X)$ be the maps induced by the trivial G-maps $X \to *$ and $Y \to *$.

Theorem 4.10. *If G is grouplike, then p and q are quasi-fibrations.*

Definition 4.29. A cover $\mathscr{C}$ of a space B is *numerable* if it is locally finite and if for each $U \in \mathscr{C}$, there is a map $\lambda_U : B \to I$ such that $U = \lambda_U^{-1}(0, 1]$.

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