MATH 7350
Homotopical Algebra, Homotopy Colimits and ∞-Categories

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Spring, 2021

Last updated April 25, 2024.

Contents

Contents by Lecture

[1]

[1] These notes will not follow exactly the order in class for the sake of completeness and readability, so the Content by Lecture may not be precise.

1 Tentative Topics

1. Quick introduction to ∞-categories.

   (a) Motivation.

   - categorical motivation: nerves of categories.
   - topological motivation: singular complex of spaces.

   Each perspective has its own problems, and they are modified such that nerves are replaced by quasi-categories and singular complexes are replaced by simplicial spaces. It turns our that the modified versions give two models of ∞-categories.

   (b) Five models (examples) of ∞-categories.

   - simplicial categories by Dywer and Kan [DK2].
   - complete Segal spaces by Rezk [R1].
   - Segal categories [B4].
   - Quasi-categories by Boardman and Vogt [BV], Joyal [J2] [J3], and Lurie [L].
   - Relative categories by Barwick and Kan [BK1].

   (c) Comparision theorems.

2. Model categories.

   (a) Review of Quillen's theory of model categories.

   i. Bousfield Localization of model categories [H2].

   ii. Universal model categories [D3] and presentation of model categories [D2]. Examples includes $\mathbb{A}^1$-homotopy theory of smooth schemes, see [D3] section 8.

   (b) Homotopy (co)limits in model categories.

   (c) Examples.

   i. Homology decomposition of classifying spaces (for compact Lie groups).

   ii. Homotopy colomits as tools of quantum deformation. For instance, we can think of $\mathbb{S}^3$ as a homotopy colimit

   $$\mathbb{S}^3 \cong \mathbb{S}^1 * \mathbb{S}^1 = \operatorname{hocolim}\left\{ \mathbb{S}^1 \xleftarrow{\pi_1} \mathbb{S}^1 \times \mathbb{S}^1 \xrightarrow{\pi_2} \mathbb{S}^1 \right\}$$

   so we can deform it via

   $$\gamma = \begin{bmatrix} p & a \\ q & b \end{bmatrix} \in \mathrm{MCG}(\mathbb{S}^1 \times \mathbb{S}^1) = \mathrm{SL}_2(\mathbb{Z})$$

as follows

$$\operatorname{hocolim}\left\{ \mathbb{S}^1 \xleftarrow{\pi_1} \mathbb{S}^1 \times \mathbb{S}^1 \xrightarrow{\pi_2 \circ \gamma} \mathbb{S}^1 \right\} =: L(p, q)$$

which gives us the Lens space $L(p, q)$.

iii. Khovanov homology as homotopy limit.

3. Homotopy theory of ∞-categories with a view towards homotopy colimits.

Remark. Given a small category $\mathcal{C}$ and a model catefgory $\mathcal{M}$, the functor category $\mathcal{M}^{\mathcal{C}} = \text{Fun}(\mathcal{C}, \mathcal{M})$ is not always a model category.

When $\mathcal{C}$ is good enough, $\mathcal{M}^{\mathcal{C}}$ is a model category, so we can define homotopy colimit (resp. limits) as the left (resp. right) derived functor of colimit (resp. limit) functor, which is left (resp. right) adjoint to the constant functor const : $\mathcal{M} \to \mathcal{M}^{\mathcal{C}}$ which assigns to each object $X \in Ob(\mathcal{M})$ the constant functor $\underline{X} : \mathcal{C} \to \mathcal{M}$ which takes every object in $\mathcal{C}$ to X.

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{C}} \arrow[d, bend right=60, "\text{colim}"'] \arrow[d, bend left=60, "\lim"] \\
\mathcal{M} \arrow[u, "\text{const}"]
\end{tikzcd}$$

For any arbitrary $\mathcal{C}$, the existence of homotopy (co)limits are studied via the following several methods.

1. Model approximation [CCS].

2. Homotopical categories [DHKS].

3. Derivators [G2].

2 Quick Introduction to ∞-categories

2.1 Motivation

Notation. Let $\mathcal{C}$ be a small category and $\mathcal{D}$ be any category.

We will write $\mathcal{D}^{\mathcal{C}} = \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ for the category of covariant functors from $\mathcal{C}$ to $\mathcal{D}$ (where morphisms are natural transformations between functors) and $\mathcal{D}_{\mathcal{C}} = \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{D})$ for the category of contravariant functors from $\mathcal{C}$ to $\mathcal{D}$.

In particular, $\widehat{\mathcal{C}} = \mathrm{Pre}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{op}, \mathbf{Set})$ is called the category of presheaves of sets on $\mathcal{C}$.

We begin with a general categorical principle.

Proposition 2.1. *Let $\mathcal{C}$ be a small category and $\mathcal{D}$ be a locally small cocomplete category. There is an equivalence of categories*

$$\Phi : \mathcal{D}^{\mathcal{C}} \xrightarrow{\simeq} \boldsymbol{Adj}(\widehat{\mathcal{C}}, \mathcal{D}) \tag{2.1}$$ {{eq:cosimplicial-adj}}

where $\boldsymbol{Adj}(\widehat{\mathcal{C}}, \mathcal{D})$ is the category of triples

$$\left(L : \widehat{\mathcal{C}} \rightleftarrows \mathcal{D} : R,\ \psi : \mathrm{Hom}(L(-), *) \xrightarrow{\cong} \mathrm{Hom}(-, R(*)) \right)$$

and ψ is a binatural isomorphism.

Explicitly, given any $F : \mathcal{C} \to \mathcal{D}$, $\Phi(F) = (L_F, R_F, \psi_F)$ is defined as follows.

- $R_F : \mathcal{D} \to \widehat{\mathcal{C}}$ is given by

$$d \longmapsto R_d : c \mapsto \mathrm{Hom}(Fc, d)$$

- $L_F : \widehat{\mathcal{C}} \to \mathcal{D}$ is the left Kan extension (please refer to classical category theory references if you're not familiar with this construction) of F along the Yoneda functor $h : \mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, hook, "h"'] \arrow[dr, phantom, "\eta\Downarrow"] & \mathcal{D} \\ \widehat{\mathcal{C}} \arrow[ur, bend right, "L_h(F)=:L_F"'] & \end{tikzcd}$$

{simp-adj}

Corollary 2.2. *Constructing a simplicial adjunction with values in $\mathcal{D}$*

$$L : \mathbf{sSet} \rightleftarrows \mathcal{D} : R$$

is equivalent to specifying a cosimplicial object $\Delta \to \mathcal{D}$.

The main index category $\mathcal{C}$ that we are interested in is the simplicial category Δ and it gives us the corollary above. Let's start with a brief review of this category.

2.1.1 The Simplex Category

Δ is the simplex category defined with

- objects are finite ordered sets $[n] = \{0 < 1 < \cdots < n\}$ for $n \geq 0$, and]item morphisms are order-preserving maps $[n] \to [m]$.

Δ is generated by the coface maps

$$d^i : [n-1] \longrightarrow [n]$$

which is the unique map that skips i and codegeneracy maps

$$s^j : [n+1] \longrightarrow [n]$$

which is the unique map that repeats i twice.

Notation. Let Δ^- be the category with the same objects and morphisms are generated by d^i's. Let Δ^+ be the category with the same objects and morphisms are generated by s^j's.

Exercise 2.3. Δ is a crossed category.

$$\Delta = \Delta^- \bowtie \Delta^+.$$

For every morphism $f : [n] \to [m]$ in Δ, there is a unique object $[k]$ in Δ and morphisms $s : [n] \to [k]$ in Δ^+ and $d : [k] \to [m]$ in Δ^- such that f can be decomposed uniquely as $f = d \circ s$. [2]

Given a simplicial set $K_\bullet$, we may present it as a diagram

$$K_\bullet = \left\{ \begin{tikzcd} K_0 & K_1 \arrow[l, shift left] \arrow[l, shift right] & K_2 \cdots \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \end{tikzcd} \right\}$$

we are omitting the degeneracy maps here so simplicity.

Notation. We will write $\mathbf{sSet} = \mathrm{Fun}(\Delta^{op}, \mathbf{Set}) = \mathbf{Set}_\Delta$ the category of simplicial sets and $\mathbf{s}\mathcal{D} = \mathrm{Fun}(\Delta^{op}, \mathcal{D}) = \mathcal{D}_\Delta$ the category of simplicial objects in $\mathcal{D}$.

Dually, we will write $\mathbf{c}\mathcal{D} = \mathrm{Fun}(\Delta, \mathcal{D}) = \mathcal{D}^\Delta$ the category of cosimplicial objects in $\mathcal{D}$.

There are two examples that are especially important for our discussion.

[2] This is called the PBW property.

2.1.2 Geometric Realization.

Let $\mathcal{D} = \mathbf{Top}$ be the category of compactly generated weakly Hausdorff spaces. There is a natural cosimplicial space

$$\begin{aligned} \Delta^\bullet : \Delta &\longrightarrow \mathbf{Top} \\ [n] &\longmapsto \Delta^n = \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} | \sum x_i = 1, x_i \geq 0\} \\ f : [n] \to [m] &\longmapsto \hat{f} : \mathbf{e}_i \mapsto \mathbf{e}_{f(i)} \end{aligned}$$

which by Corollary 2.2 gives us the following adjunction

$$|-| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : \mathcal{S}ing$$

where $|-|$ is the geometric realization functor

$$\begin{aligned} |K_\bullet| &= \int^{[n] \in Ob(\Delta)} K_n \times \Delta^n \\ &= \coprod_{n \geq 0} K_n \times \Delta^n / \sim \end{aligned}$$

where the equivalence relation is

$$(\varphi_* x, \sigma) \sim (x, \varphi^* \sigma),$$

for $x \in K_m, \sigma \in \Delta^n, \varphi : [n] \to [m] \in Mor(\Delta)$, and $\mathcal{S}ing$ is the singular complex functor which is given by

$$\mathcal{S}ing(X)_n = \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, X)$$

and the corresponding face and degeneracy maps are induced by the coface and codegeneracy maps on $\Delta^\bullet$.

2.1.3 Nerves of Categories.

Let $\mathcal{D} = \mathbf{Cat}$ be the category of small categories. We have a natural functor

$$\begin{aligned} i : \Delta &\longrightarrow \mathbf{Cat} \\ [n] &\mapsto \vec{n} = \{0 \to \cdots \to n\} \end{aligned}$$

where $\vec{n}$ is the category that represents the quiver of type A_n. This induces an adjunction

$$c : \mathbf{sSet} \rightleftarrows \mathbf{Cat} : \mathcal{N}.$$

The nerve of a category $\mathcal{C}$ is given by

$$\mathcal{N}_n\mathcal{C} = \mathrm{Hom}_{\mathbf{Cat}}(\vec{n}, \mathcal{C}) = \{X_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} X_n, n\text{-composable morphisms in } \mathcal{C}\}.$$

In other words,

$$\begin{aligned}
\mathcal{N}_0\mathcal{C} &= \mathrm{Ob}(\mathcal{C}) \\
\mathcal{N}_1\mathcal{C} &= \mathrm{Mor}(\mathcal{C}) \\
\mathcal{N}_n\mathcal{C} &= \underbrace{\mathrm{Mor}(\mathcal{C}) \times_{\mathrm{Ob}(\mathcal{C})} \cdots \times_{\mathrm{Ob}(\mathcal{C})} \mathrm{Mor}(\mathcal{C})}_{n}.
\end{aligned}$$

And for $\varphi : [m] \to [n]$ in Δ,

$$\mathcal{N}\varphi : \mathcal{N}_n\mathcal{C} \longrightarrow \mathcal{N}_m\mathcal{C}$$

$$(X_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} X_n) \mapsto (X_{\varphi(0)} \xrightarrow{g_1} \cdots \xrightarrow{g_m} X_{\varphi(m)}).$$

where $g_i : X_{\varphi(i-1)} \to X_{\varphi(i)}$ is defined as $g_i = f_{\varphi(i)} \circ \cdots \circ f_{\varphi(i-1)+1}$, and when $\varphi(i-1) = \varphi(i)$, $g_i = \mathrm{id}$.

The category $c(K)$ is defined as follows.

- objects are elements in K_0, and
- morphisms are generate by elements in K_1 in the following sense: given $f \in K_1$, it gives a morphism

  $$f : d_0 f \longrightarrow d_1 f$$

  subject to the composition law

  $$d_1\alpha = d_0\alpha \circ d_2\alpha$$

  for any $\alpha \in K_2$.

  $$\begin{tikzcd} & x_1 \arrow[dr, "d_0\alpha"] \arrow[d, phantom, "\Downarrow\alpha"] & \\ x_0 \arrow[ur, "d_2\alpha"] \arrow[rr, "d_1\alpha"'] & & x_2 \end{tikzcd}$$

The counit $c\mathcal{N} \Rightarrow \mathrm{id}$ is a natural isomorphism.

Example 2.4 (Bar construction of groups)**.** Let **Gr** be the category of discrete groups. wW can view groups as categories with a single object with objects in the groups as morphisms in this category, in this way we get a functor

$$\underline{B_*} : \mathbf{Gr} \longrightarrow \mathbf{Cat}$$

Composing this functor with the nerve functor, we get the **simplicial classifying space** or the **bar construction** $B_*G = \mathcal{N}(\underline{B_*G})$ of a group G, and the **classifying space** of G is defined to be the geometric realization $BG := |B_*G|$. Note the functor

$$B : \mathbf{Gr} \hookrightarrow \mathrm{Ho}(\mathbf{Top}_{0,*})$$

is fully faithful, and we can view groups as homotopy types of aspherical spaces (whose higher homotopy groups vanishes) by taking their fundamental groups.

From this point of view, we can generalize many operations on groups to space. For instance, abelianization of groups generalizes to homology of spaces (i.e. the homotopy type pf infinity symmetryc products of spaces [H1]), and nilpotent completion of groups generalizes to Bousfield-Kan completion of spaces [BK2].

A useful way to view bar construction is as following. By Yoneda lemma we have a fully faithful embedding

$$\mathbf{Gr} \hookrightarrow \mathrm{Fun}(\mathbf{Gr}^{op}, \mathbf{Set}) = \widehat{\mathbf{Gr}}$$

how can we characterize the essential images? Of course we can answer representable functors, but in fact we can do better. The idea is as follows. We want to restrict to a subcategory $\mathcal{G} \subset \mathbf{Gr}$ which is as small as possible, and without lose of information of the group structures. This indeed works, when we take $\mathcal{G}$ to be the full subcategory of finitely generated free groups, where objects are $\langle n \rangle = \mathbb{F}_n (n \geq 0)$ the free group with n generators. The composite

$$\mathbf{Gr} \overset{h}{\hookrightarrow} \mathrm{Fun}(\mathbf{Gr}^{op}, \mathbf{Set}) \xrightarrow{res} \mathrm{Fun}(\mathcal{G}^{op}, \mathbf{Set})$$

$$G \longmapsto \mathrm{Hom}_{\mathbf{Gr}}(-, G) \mapsto \underline{G} : \mathcal{G}^{op} \to \mathbf{Set}, \langle n \rangle \mapsto G^n$$

remains fully faithful.

Furthermore, $\mathcal{G}$ has (strict) monoidal structure given by coproduct (i.e. free product of groups)

$$\langle n \rangle * \langle m \rangle = \langle n + m \rangle$$

and the above functor $\underline{G} : \mathcal{G}^{op} \to \mathbf{Set}$ is strict monoidal preserving (i.e. it takes products in $\mathcal{G}^{op}$, or equivalently coproducts in $\mathcal{G}$, to products in **Set**). Let $\mathbf{Set}_{\mathcal{G}}^{\otimes} = \mathrm{Fun}^{\otimes}(\mathcal{G}^{op}, \mathbf{Set})$ be the category of strict monoidal functors, we see the image of **Gr** under the above composite indeed lies in $\mathbf{Set}_{\mathcal{G}}^{\otimes}$.

Exercise 2.5 (Lawvere)**.** Prove that the the above construction induces an equivalence of categories

$$\mathbf{Gr} \simeq \mathbf{Set}_{\mathcal{G}}^{\otimes}$$

Thus, we can view the bar construction in terms of functor categories

$$B_* : \mathbf{Set}_{\mathcal{G}}^{\otimes} \simeq \mathbf{Gr} \hookrightarrow \mathbf{Cat} \overset{\mathcal{N}}{\hookrightarrow} \mathbf{sSet} = \mathrm{Fun}(\Delta^{op}, \mathbf{Set})$$

$$G \longmapsto \underline{B_* G} \longmapsto B_* G$$

Exercise 2.6. Show that B_* is induced by a functor $b : \Delta \to \mathcal{G}$ in **Cat**, i.e. we have

$$B_* \cong b^* : \mathbf{Set}_{\mathcal{G}}^{\otimes} \simeq \mathbf{Gr} \to \mathbf{sSet} = \mathrm{Fun}(\Delta^{op}, \mathbf{Set})$$

$$(\mathcal{G}^{op} \xrightarrow{G} \mathbf{Set}) \longmapsto (\Delta^{op} \xrightarrow{b^{op}} \mathcal{G}^{op} \xrightarrow{G} \mathbf{Set})$$

Remark. Let $\mathcal{M}$ be a monoidal category. Then the group object in $\mathcal{M}$ can be defined as $\mathcal{M}_{\mathcal{G}}^{\otimes}$. For example, cocommutative Hopf algebras are group objects in the category of commutative coalgebras. [3] In this way, we can define internal nerves given be the functor

$$B_* : \mathcal{M}_{\mathcal{G}}^{\otimes} \longrightarrow \mathcal{M}_{\Delta}$$

$$(\mathcal{G}^{op} \xrightarrow{G} \mathcal{M}) \mapsto (\Delta^{op} \xrightarrow{b^{op}} \mathcal{G}^{op} \xrightarrow{G} \mathcal{M})$$

In fact, we can replace $\mathcal{G}$ by the algebraic theory of monoids and give similar construction.

General property of nerve construction. The nerve functor is fully faithful, in fact $\mathcal{N}$ is the left inverse of c, $c \circ \mathcal{N} \simeq \mathrm{id}$, so we have an embedding

$$\mathcal{N} : \mathbf{Cat} \hookrightarrow \mathbf{sSet}$$

whose essential images are 2-coskeletal simplicial sets.

Proposition 2.7. *If $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories, $\mathcal{N}_* F : \mathcal{N}_* \mathcal{C} \to \mathcal{N}_* \mathcal{D}$ is a weak equivalences of simplicial sets.*

The converse is not true. For example, take $\mathcal{C} = \overrightarrow{0} = \{*\}$ and $\mathcal{D} = \overrightarrow{1} = \{0 \to 1\}$, we have a functor $F : \overrightarrow{0} \hookrightarrow \overrightarrow{1}$ which induces a map

$$\mathcal{N}_* F : \mathcal{N}_* \overrightarrow{0} \xrightarrow{\simeq} \mathcal{N}_* \overrightarrow{1}$$

which is a weak homotopy equivalence of simplicial sets, but F is not a category equivalence.

Proposition 2.8. *If both $\mathcal{C}$ and $\mathcal{D}$ are groupoids, then $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if $\mathcal{N}_* F : \mathcal{N}_* \mathcal{C} \to \mathcal{N}_* \mathcal{D}$ is a weak equivalences of simplicial sets.*

We want some refinement so that the nerve functor reflects weak equivalences of categories. There are mainly two approach.

[3] It seems that not all cocommutative Hopf algebras are group objects in the category of cocommutative coalgebras, since the antipode in the definition of Hopf algebra is not required to be a coalgebra map. See here.

1. We can change the weak equivalences in simplicial sets (fewer) so that if $\mathcal{C} \not\simeq \mathcal{D}$ as categories, we have $\mathcal{N}_*\mathcal{C} \not\simeq \mathcal{D}$ as simplicial sets. This new notion of weak equivalences will force us to modify the model structure on **sSet**. This is Joyal's model structure on **sSet**, with fibrations being quasi-categories. [4]

2. We can replace the target of the functor $\mathcal{N}$ to be **sSp** the category of simplicial spaces (or equivalently, bisimplicial sets). This leads to the construction of complete Segal spaces [R1].

2.1.4 Basic Examples of Simplicial Sets

Standard n-simplex. Recall we have the Yoneda functor $h : \Delta \to \mathbf{sSet}$, and we will denote $\Delta[n]_* = h_{[n]}$ the **standard n-simplex**. Explicitly,

$$\Delta[n]_k = \{(i_0, \cdots, i_k) | 0 \le i_0 \le \cdots \le i_k \le n\}$$

By Yoneda embedding, we see that

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta[n]_*, X) \cong X_n, n \ge 0$$

for any simplicial set X. Thus $\Delta[n]_*$ represents the functor

$$\begin{aligned} (-)_n : \mathbf{sSet} &\to \mathbf{Set} \\ X_* &\longmapsto X_n \end{aligned}$$

Properties

1. There is an embedding

$$\begin{aligned} \Delta &\hookrightarrow \mathbf{Cat} \\ [n] &\mapsto \overrightarrow{n} \end{aligned}$$

Since this is a fully faithful functor, we have

$$\mathrm{Hom}_{\mathbf{Cat}}(\overrightarrow{m}, \overrightarrow{n}) \cong \mathrm{Hom}_{\Delta}([m], [n])$$

which induces a natural isomorphism of functors $\Delta^{op} \to \mathbf{Set}$

$$\mathcal{N}_*(\overrightarrow{n}) = \mathrm{Hom}_{\mathbf{Cat}}(*, \overrightarrow{n}) \cong \mathrm{Hom}_{\Delta}(*, [n]) = \Delta[n]_*$$

[4] The general principle is, whenever you have a model category structure, the class of fibrant/cofibrant objects usually have many interesting properties. More precisely, if you are working with (topologically) model categories, many interesting algebraic conditions that you want to impose on objects in order to do homotopy theory can be interpreted as being in the class of cofibrant/cofibrant-fibrant objects.

2. We have a functor $\Delta \to \mathbf{Top}$ (which is faithful but not full) and the following diagram of left Kan extensions

$$\begin{tikzcd} \Delta \arrow[r] \arrow[d, hook, "h"'] & \mathbf{Top} \\ \mathbf{sSet} \arrow[ur, "|-|=\mathrm{Lan}_h(\Delta^*"'] & \end{tikzcd}$$

which is in fact commutative since h is fully faithful. The left Kan extension is exactly the geometric realization of simplicial sets.

The above two properties imply $B(\vec{n}) \cong |\Delta[n]_*| = \Delta^n$.

Notice the assignment $[n] \mapsto \Delta[n]_*$ gives a (Yoneda) functor $\Delta \to \mathbf{sSet}$, which is a cosimplicial simplicial set, called the **standard cosimplicial simplicial set**.

$$\mathrm{Fun}(\Delta, \mathbf{sSet}) = \mathrm{Fun}(\Delta, \mathbf{Fun}(\Delta^{op}, \mathbf{Set})) \cong \mathrm{Fun}(\Delta \times \Delta^{op}, \mathrm{Set})$$

We call a functor $\Delta \times \Delta^{op} \to \mathrm{Set}$ a Δ-bimodule. In particular, the standard cosimplicial simplicial set is given by the following Δ-bimodule

$$\begin{aligned} \Delta \times \Delta^{op} &\longrightarrow \mathbf{Set} \\ ([n],[k]) &\mapsto \mathrm{Hom}_\Delta([n],[k]) \end{aligned}$$

The left and right Δ-actions commutes, in particular, applying this to $\Delta[\bullet]_*$, we get coface and codegeneracy maps

$$\begin{aligned} d^i : \Delta[n-1] &\to \Delta[n] \qquad n \geq 1, 0 \leq i \leq n \\ s^j : \Delta[n+1] &\to \Delta[n] \qquad n \geq 0, 0 \leq j \leq n \end{aligned}$$

We can write the standard cosimplicial simplicial set explicitly as follows (omitting codegeneracy maps as usual)

$$\begin{tikzcd} \Delta[0]_* \arrow[r, shift left] \arrow[r, shift right] & \Delta[1]_* \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \Delta[2]_* \cdots \end{tikzcd}$$

Definition 2.9. The n**-th boundary** simplex of $\Delta[n]_*$ is the simplicial subset of $\Delta[n]_*$

$$\partial\Delta[n] := \bigcup_{0 \leq i \leq n} d^i(\Delta[n-1]_*) \subset \Delta[n]_*.$$

Example 2.10. 1. For $n = 0$, $\Delta[0]_* = \{*\}$ with

$$\Delta[0]_k = \{\underbrace{(0, \cdots, 0)}_{k}\}$$

is a discrete simplicial set which is the terminal object in $\mathbf{sSet}$, and $\partial\Delta[0]_* = \emptyset$.

2. For $n = 1$,
$$\Delta[1]_k = \{(\underbrace{0, \cdots, 0}_{i}, \underbrace{1, \cdots, 1}_{k+1-i}), 0 \leq i \leq k+1\}$$

   and
$$\partial\Delta[1]_k = d^0(\Delta[0]_k) \bigcup d^1(\Delta[0]_*) = \{\underbrace{(0, \cdots, 0)}_{k}, \underbrace{(1, \cdots, 1)}_{k}\}$$

   is the discrete simplicial set $\{0, 1\}$ (i.e. sets viewed as constant simplicial set with identity as face and degeneracy maps).

Definition 2.11. For $0 \leq k \leq n$, the k**-th horn** of $\Delta[n]_*$ is the simplicial subset
$$\Lambda_k[n]_* := \bigcup_{i \neq k, 0 \leq i \leq n} d^i(\Delta[n-1]_*) \subset \Delta[n]_*$$
which is the cone centered at vertex k.

To illustrate the definition, we will work the following example.

Example 2.12. We will use topological simplex for demonstration. Take $n = 2$, consider the following picture of Δ^2 in Figure 1. Then boundary is given by

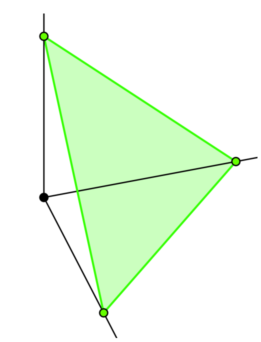

Figure 1: 2 simplex in $\mathbb{R}^3$ {fig:2simplex}

$$\begin{tikzcd}
 & 1 \arrow[dr, dash] & \\
0 \arrow[ur, dash] \arrow[rr, dash] & & 2
\end{tikzcd}$$

and the 0-th horn is

$$\begin{tikzcd}
 & 1 & \\
0 \arrow[ur, dash] \arrow[rr, dash] & & 2
\end{tikzcd}$$

and the 1-th horn is

$$\begin{tikzcd} & 1 \arrow[dl, dash] \arrow[dr, dash] & \\ 0 & & 2 \end{tikzcd}$$

and the 2-th horn is

$$\begin{tikzcd} & 1 \arrow[dr, dash] & \\ 0 \arrow[rr, dash] & & 2 \end{tikzcd}$$

Kan Complexes. For $0 \leq k \leq n$, we have the inclusion functor

$$\Lambda_k[n] \hookrightarrow \Delta[n]_*$$

which induces a map between sets

$$X_n = \mathrm{Hom}_{\mathbf{sSet}}(\Delta[n]_*, X) \longrightarrow \mathrm{Hom}_{\mathbf{sSet}}(\Lambda_k[n], X) =: \Lambda_k(X). \tag{2.2}$$ {{eq:horn-lifting}}

Exercise 2.13 ([GJ], Corollary 3.2)**.** Show that

$$\Lambda_k(X) \cong \{(x_0, \cdots, \widehat{x_k}, \cdots, x_n) \in X_{n-1} \times \cdots \times X_{n-1} | \\ d_i x_j = d_{j-1} x_i, \forall 0 \leq i < j \leq n, i, j \neq k\} \tag{2.3}$$

Definition 2.14. A **Kan complex** is a simplicial set X such that the above maps 2.2 are surjective for all $0 \leq k \leq n, n \geq 0$, i.e.

$$\begin{tikzcd} \Lambda_k[n]_* \arrow[r] \arrow[d, hook] & X \\ \Delta[n]_* \arrow[ur, dashed, "\exists"'] & \end{tikzcd} \tag{2.4}$$ {{eq:horn-filling}}

every horn can be filled in to an n-simplex.

Example 2.15.

1. Let X be a topological space, then $\mathrm{Sing}(X)$ is a Kan complex.
2. (Moore) Every simplicial group G (by forgetting the group structure) is a Kan complex.
3. The nerve of every groupoid $\mathcal{C}$ is a Kan complex.
   (*Sketch of proof.*) The lifting condition is roughly speaking the requirement of left and right inverses for every morphism in $\mathcal{C}$, and every morphism

in groupoid has an inverse. For instance, when $n = 2$, we have following lifting diagrams

$$\begin{tikzcd} & 1 \arrow[dr, "g"] & \\ 0 \arrow[ur, dashed] \arrow[rr, "1"] & & 2 \end{tikzcd} \quad \begin{tikzcd} & 1 \arrow[dr, "g"] & \\ 0 \arrow[ur, "f"] \arrow[rr, dashed] & & 2 \end{tikzcd} \quad \begin{tikzcd} & 1 \arrow[dr, dashed] & \\ 0 \arrow[ur, "f"] \arrow[rr, "1"] & & 2 \end{tikzcd}$$

In the first diagram, dashed arrow represents the left inverse of g. The second diagram represents composition of morphisms and the last diagram represents the right inverse of f.

Remark (Moral). Kan complexes are groupoids up to homotopy. In fact, in the ∞-category language, they are ∞-groupoids.

Theorem 2.16 ([L], Propostion 1.1.2.2)**.** *A simplicial set X is the nerve of a small category $\mathcal{C}$ if and only there are unique inner horn (i.e. $\Lambda_k[n]_*$ for $0 < k < n$) filling.*

Quasi-categories

Definition 2.17. A simplicial set X is called a **quasi-category** if for every inner horn $\Lambda_k[n]_*(0 < k < n)$, the lifting 2.4 always exists. [5]

2.2 Homotopy Category

In a general sense, to define a "homotopy theory" on a category $\mathcal{C}$ is to specify a class of morphisms $\mathcal{W}$ (called weak equivalences) which we want to treat as "isomorphisms" in $\mathcal{C}$.

Example 2.18. Take $\mathcal{C} = \textbf{Top}$ and $\mathcal{W} =$ homotopy equialence/weak homotopy equivalences.

One can always define a category $\mathcal{C}[\mathcal{W}^{-1}]$ together with a functor $\mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ satisfying the following universal property.

For any $F : \mathcal{C} \to \mathcal{D}$ such that $F(\mathcal{W}) \subseteq \mathrm{Iso}(\mathcal{D})$, there is a functor $\overline{F} : \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}$ together with a natural isomorphism $\alpha : F \cong \overline{F} \circ \gamma$

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "\gamma"'] & & \mathcal{D} \\ & \mathcal{C}[\mathcal{W}^{-1}] \arrow[ur, "\overline{F}"'] & \end{tikzcd}$$

$\alpha : F \cong \overline{F} \circ \gamma$. where $\overline{F}$ is unique up to unique isomorphisms. The category of all factorization $(\overline{F}, \alpha)$ forms a contractible groupoid.

However, there are some problems with $\mathcal{C}[\mathcal{W}^{-1}]$.

[5] Sometimes, they are also called weak Kan complexes or inner Kan complexes.

1. The category $\mathcal{C}[\mathcal{W}^{-1}]$ ([GZ]) is not locally small.

2. We can not control maps in $\mathcal{C}$ that become isomorphisms in $\mathcal{C}[\mathcal{W}^{-1}]$.

Problem 2 is detected by the following classes of morphisms. Define $\tilde{\mathcal{W}} := \{f \in \mathrm{Mor}(\mathcal{C}), \gamma(f) \in \mathrm{Iso}(\mathcal{C}[\mathcal{W}^{-1}]), $ then $\mathcal{W} \subset \tilde{\mathcal{W}}$ but this is not always equal. We say $\mathcal{W}$ is **saturated** if equality holds.

Moral. One has to impose additional conditions on $\mathcal{W}$.

Here is a list of possible conditions.

1. 2-of-3 property: for any composable pair f, g in $\mathcal{C}$, if any two of $f, g, g \circ f$ are in $\mathcal{W}$, so is the third.

2. 2-of-6 property [DHKS]: for any triple of composable maps

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, dashed] \arrow[drr, dashed] & B \arrow[d, "g"] \arrow[dr, dashed] & \\ & C \arrow[r, "h"'] & D \end{tikzcd}$$

   $hg, gf \in \mathcal{W}$ implies $f, g, h, hgf \in \mathcal{W}$.

3. Retract axiom: given

$$\begin{tikzcd} X \arrow[r] \arrow[d, "r"] & \tilde{X} \arrow[r] \arrow[d, "f"] & X \arrow[d, "r"] \\ A \arrow[r] & \tilde{A} \arrow[r] & A \end{tikzcd}$$

   where horizontal compositions are identities, r is called a **retract** of f. Then $f \in \mathcal{W} \implies r \in \mathcal{W}$.

4. Wide subcategory: $\mathcal{W}$ is closed under composition and contains all identity maps. In particular, $\mathrm{Ob}(\mathcal{W}) = \mathrm{Ob}(\mathcal{C})$.

5. $\mathcal{W}$ contains all all isomorphisms.

Remark. 1. 2-of-6 property implies 2-3 property, but not vice versa.

2. The class of all isomorphisms in any category $\mathcal{C}$ satisfies 2-6 property. Indeed,

   - $f(gf)^{-1}$ is right inverse of g: $g \circ f(gf)^{-1} = \mathrm{id}$.
   - $hg \in \mathrm{Iso}(\mathcal{C})$ implies g is monic, so $gf(gf)^{-1}g = g \circ \mathrm{id}$ implies $f(gf)^{-1}g = \mathrm{id}$, so $f(gf)^{-1}$ is also left inverse of g.

Remark. Check this last fact is used to prove that if $f : X \rightleftarrows Y : g$ are homotopy equivalent such that $fg \sim \mathrm{id}$ and $gf \sim \mathrm{id}$ then f, g are weak homotopy equivalences.

Corollary 2.19. *If $\mathcal{W}$ is saturated, then 2-6 property holds because $\mathcal{W}$ is characterized by isomorphisms in $\mathcal{C}[\mathcal{W}^{-1}]$.*

Remark. In any model category, the class of weak equivalences satisfies 2-of-6 property.

We will use three types of weak equivalences

1. Quillen's weak equivalences in model categories: $\mathcal{W}$ satisfies all 5 properties.
2. In homotopical categories [DHKS], $\mathcal{W}$ satisfies (2) and (5).
3. In relative categories $(\mathcal{C}, \mathcal{W})$ (category with equivalences [BK1]), $\mathcal{W}$ satisfies (4). [6]

Exercise 2.20. Let $\mathcal{W}$ be a class of morphisms in a category $\mathcal{C}$ such that there is a functor $F : \mathcal{C} \to \mathcal{D}$ with $F(\mathcal{W}) \subset \mathrm{Iso}(\mathcal{D})$, then $\mathcal{W}$ must satisfy all the five properties.

Not sure if you want to require $\mathcal{W}$ to be saturated?

(2-categorical) universal mapping property for localization. [GZ] Let $\mathcal{C}$ be a category and $\mathcal{W}$ a class of morphisms in $\mathcal{C}$ together with a localization

$$\gamma : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}].$$

For any category $\mathcal{M}$, γ induces a fully faithful embedding

$$\mathrm{Fun}(\mathcal{C}\left[\mathcal{W}^{-1}\right], \mathcal{M}) \xhookrightarrow{\gamma^*} \mathrm{Fun}(\mathcal{C}, \mathcal{M})$$

with essential images are isomorphic to the full subcategory $\mathrm{Fun}_{\mathcal{W}}(\mathcal{C}, \mathcal{M})$ that inverts $\mathcal{W}$.

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{C}\left[\mathcal{W}^{-1}\right], \mathcal{M}) \arrow[rr, hook, "\gamma"] \arrow[dr, "\cong"'] & & \mathrm{Fun}(\mathcal{C}, \mathcal{M}) \\
& \mathrm{Fun}_{\mathcal{W}}(\mathcal{C}, \mathcal{M}) \arrow[ur, hook] &
\end{tikzcd}$$

[6] On small relative categories, there is a nontrivial model structure that is Quillen equivalent to Joyal model category on **sSet**.

2.3 Model Categories

Definition 2.21 (Quillen, [Q1] [Q2]). A **model category** $\mathcal{C}$ is a category with 3 distinguished classes of morphisms

- We: Weak equivalences, denoted by $\xrightarrow{\sim}$,
- Fib: Fibrations, denoted by $\twoheadrightarrow$,
- Cof: Cofibrations, denoted by $\hookrightarrow$,

all of which form wide subcategories of $\mathcal{C}$ satisfying

MC1 $\mathcal{C}$ has finite limits and colimits with initial object $\varnothing$ and terminal object $*$.

MC2 2-of-3 property for weak equivalences.

MC3 We, Fib, Cof satisfy retract axiom.

MC4 Cofibrations satisfy left lifting property with respect to acyclic fibrations (i.e. fibrations which are also weak equivalences) and fibrations satisfy right lifting property with respect to acyclic cofibrations (cofibrations which are also weak equivalences).

MC5 Factorization property: for every morphism in $\mathcal{C}$ there are two ways of factorizing morphisms

$$A \overset{\sim}{\hookrightarrow} B \twoheadrightarrow X$$

$$A \hookrightarrow C \overset{\sim}{\twoheadrightarrow} X$$

where the first is an acyclic cofibration followed by a fibration, and the second is a cofibration followed by an acyclic fibration.

Remark. There are some modifications that we usually take.

1. in MC1, we require existence of all small limits and colimits.
2. In MC5, the two factorization are functorial.
3. In MC4, the following notation is used. Given two morphisms $i : A \to B$ and $p : X \to Y$, consider commutative diagrams

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

we say $i \in \mathrm{LLP}(p)$ and $p \in \mathrm{RLP})i)$ if there exists $h : B \to X$ such that then MC4 can be restated as $\mathrm{Cof} \in \mathrm{LLP}(\mathrm{Fib} \cap We)$ and $\mathrm{Fib} \in \mathrm{RLP}(\mathrm{Cof} \cap We)$.

Lemma 2.22. [7] *In any model category,*

$$\mathtt{Cof} = \mathrm{LLP}(\mathtt{Fib} \cap We), \mathtt{Fib} = \mathrm{RLP}(\mathtt{Fib} \cap We),$$

Moreover,

$$\mathtt{Cof} \cap We = \mathrm{LLP}(\mathtt{Fib}), \mathtt{Fib} \cap We = \mathrm{RLP}(\mathtt{Cof}).$$

Remark. A model category is closed in the sense that 2 of 3 classes of WE,Fib,Cof determines the third. Except that Fib, Cob cannot determine We.

Definition 2.23. A model category is called **cofibrantly generated** if there is a class (usually countable set) of morphisms $I \subset \mathtt{Cof}$ generating cofibrations and a class (usually countable set) $J \subset \mathtt{Cof} \cap We$ of acyclic cofibrations such that

$$\mathtt{Fib} = \mathrm{RLP}(J), \mathtt{Fib} \cap We = \mathrm{LLP}(I)$$

which satisfy small object argument.

Definition 2.24. 1. An object Q in $\mathcal{C}$ is cofibrant if $\varnothing \hookrightarrow Q$ is a cofibration.

2. An object R in $\mathcal{C}$ is fibrant if $Q \twoheadrightarrow R$ is a fibration.

We can get cofibrant replacement

$$\begin{tikzcd} \varnothing \arrow[rr] \arrow[dr, hook] & & A \\ & QA \arrow[ur, two heads, "\sim"] & \end{tikzcd}$$

and fibrant replacement

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, hook, "\sim"'] & & * \\ & RX \arrow[ur] & \end{tikzcd}$$

2.3.1 Quillen's Model Structure

Theorem 2.25 (Quillen)**.** *There is a proper, combinatorial, cofibrantly generated model structure on* **sSet** *with*

- *Weak equivalences are Quillen's weak equivalence of simplicial sets, i.e.* $f : X_* \to Y_*$ *is a weak equivalence if the induced map* $|f| : |X| \to |Y|$ *on their geometric realizations is a weak homotopy equivalence.*
- *Cofibrations are monomorphisms of simplicial sets.*

[7] See old notes.

- *Fibrations are Kan fibrations, i.e. morphisms which have right lifting property with respect to all horn lifting maps.*

In this model category, all simplicial sets are cofibrant, and Kan complexes are exactly the fibrant objects.

The class of fibration in this model category can be described as maps having RLP with respect to horn lifting maps

$$\begin{tikzcd} \Lambda_k[n]_* \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ \Delta[n]_* \arrow[r] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

for all $n \geq 1, 0 \leq k \leq n$.

Remark. This model structure is cofibrantly generated. More precisely, fibrations can be characterized as

$$\mathrm{Fib} = \mathrm{RLP}(J)$$

where $J = \{\Lambda_k[n]_* \hookrightarrow \Delta[n]_*, n \geq 1, 0 \leq k \leq n\}$ is the set of all horn inclusions, and

$$We \cap \mathrm{Fib} = \mathrm{RLP}(I)$$

where $I = \{\partial\Delta[n]_* \hookrightarrow \Delta[n]_*, n \geq 0\}$ is the set of all boundary inclusions.

2.3.2 Joyal Model Structure

We want a model structure on **sSet** such that quasi-categories are the fibrant objects, this is Joyal's model structure [DS] [L].

We can apply the categorical principle 2.1 in the following way. Fix a simplicial set Y, we can construct a cosimplicial simplicial set

$$\begin{aligned} (-) \times Y : \Delta &\longrightarrow \mathbf{sSet} \\ [n] &\longmapsto \Delta[n]_* \times Y \end{aligned}$$

which induces an adjunction

$$(-) \times Y : \mathbf{sSet} \rightleftarrows \mathbf{sSet} : \mathrm{Map}(Y, -).$$

where

$$\mathrm{Map}(Y, Z)_n = \mathrm{Hom}_{\mathbf{sSet}}(\Delta[n]_* \times Y, Z)$$

gives the internal hom of **sSet**.

Consider the functor

$$\begin{aligned} \tau_0 : \mathbf{sSet} \xrightarrow{c} \mathbf{Cat} &\xrightarrow{\pi_0} \mathbf{Set} \\ X \longmapsto c(X) &\mapsto \pi_0 c(X) \end{aligned}$$

where $\pi_0\mathcal{C}$ is the iso-classes of obejcts in $\mathcal{C}$. Using this to define

$$\tau_0\,[\mathrm{Map}(-,-)] : \mathbf{sSet}^{op} \times \mathbf{sSet} \to \mathbf{Set}.$$

Definition 2.26 (Joyal)**.** A map $f : A \to B$ in **sSet** is called a **Joyal equivalence** if for any quasi-category K,

$$f^* : \tau_0(\mathrm{Map}(B,K)) \xrightarrow{\sim} \tau_0(\mathrm{Map}(A,K))$$

is an isomorphism of sets.

There is another equivalent description of Joyal equivalences.

Definition 2.27 ([DS])**.** Consider the category $\overleftrightarrow{I} = \{\ \bullet \rightleftarrows \bullet\ \}$ the groupoid with two objects and two isomorphisms. We identity $\mathcal{N}\overrightarrow{I} = \overrightarrow{I}$.

Given $f, g : A \to K$ with K a quasi-category, an $\overrightarrow{I}$-homotopy is a map $A \times \overleftrightarrow{I} \xrightarrow{h} K$

$$\begin{tikzcd} A \arrow[r, hook] \arrow[dr, "f"'] & A \times \overleftrightarrow{I} \arrow[d, "h"] & A \arrow[l, hook'] \arrow[dl, "g"] \\ & K & \end{tikzcd}$$

When K is a quasi-category, this defines an equivalence relation $\sim_I$ on $\mathrm{Hom}(A,K)$ and we define

$$[A,K] \coloneqq \mathrm{Hom}(A,K)/\sim_I$$

Lemma 2.28 ([DS])**.** *f is a Joyal equivalence if and only if for any quasi-category K,*

$$f^* : [B,K] \to [A,K]$$

is an isomorphism of simplicial sets.

Theorem 2.29 (Joyal)**.** *The Joyal's model structure is given by*

- *Weak equivalences are Joyal weak equivalences.*
- *Cofibrations are monomoprhisms.*
- *Fibrations are weak Kan fibrations, i.e. morphisms which have right lifting property with respect to all inner horn lifting maps.*

Remark. This model category is a cofibrant model category and the fibrant objects are quasi-categories.

Remark (Nerves)*.* 1. The adjunction

$$c : \mathbf{sSet} \rightleftarrows \mathbf{Cat} : \mathcal{N}.$$

(in particular the categorification functor c) has poor homotopical properties. For instance, this is not a Quillen adjunction.

2. The nerve has nice homotopy inverses; this implies in particular that the nerve functor induces an equivalence of homotopy categories.

Exercise 2.30. Take $\mathbb{S}^n_* := \Delta[n]_*/\partial\Delta[n]_*$ the simplicial n-sphere, $|\mathbb{S}^n_*| \cong S^n$. $\mathcal{N}c\mathbb{S}^n_* \cong \Delta[n]_*$ for any $n \geq 2$, thus $c(\mathbb{S}^n_*)$ is contractible.

To see that the nerve construction has nice homotopy inverse, we need to introduce the simplex category.

Simplex category and category of simplices

Definition 2.31 (Slice category)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Fix an object $d \in \mathrm{Ob}(\mathcal{D})$ and define a category $F \downarrow d$ with

- objects are pairs (c, f) where $c \in \mathrm{Ob}(\mathcal{C})$ and $f : F(c) \to d$ is a morphism in $\mathcal{D}$, and
- morphims between (c, f) and (c', f') are morhpisms $\varphi : c \to c'$ in $\mathcal{C}$ such that $f = f' \circ F(\varphi)$.

$$\begin{tikzcd} F(c) \arrow[rr, "F(\varphi)"] \arrow[dr, "f"'] & & F(c') \arrow[dl, "f'"] \\ & d & \end{tikzcd}$$

Example 2.32. 1. The simplx category of a small category $\mathcal{C}$ is

$$\Delta\mathcal{C} := \Delta^* \downarrow \mathcal{C}$$

with

- objects are $([n], f)$, where $[n] \in \mathrm{Ob}(\Delta)$ and $f : \vec{n} \to \mathcal{C}$ is a functor between categories. Explicitly, objects are n-composable morphisms in $\mathcal{C}$.
- morphisms are functor $\varphi : [n] \to [m]$ in Δ such that

$$\begin{tikzcd} \vec{n} \arrow[rr, "\Delta^*(\varphi)"] \arrow[dr, "f"'] & & \vec{m} \arrow[dl, "f'"] \\ & \mathcal{C} & \end{tikzcd}$$

commutes.

2. Given a simplicial set X, the Yoneda functor

$$h : \Delta \hookrightarrow \mathbf{sSet}$$

defines the category $\Delta X := \Delta \downarrow X$ of simplices of X where

- objects are $([n], x : \Delta[n]_* \to X)$, i.e. elements in $\coprod_{n \in \mathbb{N}} X_n$, and
- morphisms are $f : [n] \to [m]$ in Δ such that $x = f_*(y)$.

$$\begin{tikzcd} \Delta[n]_* \arrow[rr, "f_*"] \arrow[dr, "x"'] & & \Delta[m]_* \arrow[dl, "y"] \\ & X & \end{tikzcd}$$

Exercise 2.33. $\Delta\mathcal{C} \cong \Delta(\mathcal{N}\mathcal{C})$ is an isomorphism of categories.

Theorem 2.34 ([I], Theorem 3.2)**.** *The functor*

$$\Delta \downarrow (-) : \mathbf{sSet} \to \mathbf{Cat}$$

is the homotopy inverse to

$$\mathcal{N} : \mathbf{Cat} \to \mathbf{sSet}$$

Test Categories The notion of a test category was introduced by Grothendieck in [S4]. A test category $\mathcal{A}$ is meant to axiomatize common features of categories of shapes used to model homotopy types in homotopy theory,

$$\mathrm{Ho}(\mathrm{Pr}(\mathcal{A})) \cong \mathrm{Ho}(\mathbf{Top})$$

such as the categories of simplicial sets, or cubical sets. The natural question is, what are the possible test categories $\mathcal{A}$? Various conjectures made in [S4] are proven in [C1] which moreover develops the main toolset and establishes the model structure on presheaves over a test category.

3 Complete Segal Spaces

The main reference for this section is [R1].

One of the motivation for this section is, we've seen quasi-category as a model for ∞-categories, and there are many (at least four) other models, each of which is a nice model for ∞-categories, why do people still care about other models given the fact that quasi-categories are quite nice, accessible combinatorial models? One potential answer/motivation is, can we generalize everything to other algebraic constructions, namely, can we consider groups or more complicated algebraic structures in a similar manner. And instead of working with quasi-categories, we can also work with other models. It turns out that for groups for example, structures like monoid and group actions, while they can be considered up to homotopy in algebraic topology, are studied in the other models if you want to develop a theory in the same direction as ∞-categories. Namely, Complete Segal spaces and Segal categories, which we will discuss in this section, are suitable models for this purpose.

Let's start with a motivating problem: We observe the nerve functor $\mathcal{N}$: $\mathbf{Cat} \to \mathbf{sSet}$ is fully faithful, and two categories $\mathcal{C}_1 \cong \mathcal{C}_2$ are isomorphic if and only if their nerves $\mathcal{N}_*\mathcal{C}_1 \cong \mathcal{N}_*\mathcal{C}_2$ are isomorphic. However, $\mathcal{N}_*$ does not capture equivalences: Kan weak equivalences $\mathcal{N}_*\mathcal{C}_1 \simeq \mathcal{N}_*\mathcal{C}_2$ between nerves of categories does not imply category equivalences $C_1 \simeq \mathcal{C}_2$. Thus the functor

$$\begin{tikzcd}
\mathbf{Cat} \arrow[r, "\mathcal{N}"] \arrow[d] & \mathbf{sSet} \arrow[d] \\
\mathbf{Cat}[Eq^{-1}] \arrow[r, "\overline{\mathcal{N}}"] & \mathrm{Ho}(\mathbf{sSet}) = \mathbf{sSet}[WE^{-1}]
\end{tikzcd}$$

$\overline{\mathcal{N}}$ is not fully faithful.

Notation. We will call objects in $\mathbf{sSet}$ spaces, and objects in $\mathbf{sSet}_\Delta$ simplicial spaces. we will write $X_* \in \mathrm{Fun}(\Delta^{op}, \mathbf{sSet})$ as

$$X_\bullet = \left\{ X_0 \leftleftarrows X_1 \; \substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow} \; X_2 \cdots \right\}$$

where each X_n is a vertical simplicial set.

There are many interesting model structures on bisimplicial sets.

1. Bousfield-Kan (BK) model structure (projective)
2. Reedy model structures
3. Bergner model structure (left Bousfield localization of BK model structure on Segal maps).
4. Rezk model structure (left Bousfield localization of Reedy model structure on Segal maps).

3.1 Classical Homotopy Theory of Categories

In order to answer the following question, we start with some classical results. [8]

Question 3.1. *When exactly does a functor* $F : \mathcal{C} \to \mathcal{D}$ *induce a Kan weak equivalences* $\mathcal{N}_* F : \mathcal{N}_*\mathcal{C} \to \mathcal{N}_*\mathcal{D}$*?*

We start with the following observation: given two functors $F, G : \mathcal{C} \to \mathcal{D}$ and a natural transformation $\alpha : F \Rightarrow G$, α induces a functor

$$\begin{aligned} H_\alpha : \mathcal{C} \times \mathcal{I} &\longrightarrow \mathcal{D} \\ (c, 0) &\longmapsto F(c) \\ (c, 1) &\longmapsto G(c) \end{aligned}$$

where $\mathcal{I} = \{0 \to 1\}$. And we get a commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "i_0"] \arrow[dr, "F"'] & \mathcal{C} \times \mathcal{I} \arrow[d, "H_\alpha"] & \mathcal{C} \arrow[l, "i_1"'] \arrow[dl, "G"] \\ & \mathcal{D} & \end{tikzcd}$$

Note $\mathcal{N}_*\mathcal{I} = \Delta[1]_*$, so it induces a simplicial homotopy

$$\begin{tikzcd} \mathcal{N}_*\mathcal{C} \arrow[r, "\mathcal{N}_* i_0"] \arrow[dr, "\mathcal{N}_* F"'] & \mathcal{N}_*\mathcal{C} \times \Delta[1]_* \arrow[d, "\mathcal{N}_* H_\alpha"] & \mathcal{N}_*\mathcal{C} \arrow[l, "\mathcal{N}_* i_1"'] \arrow[dl, "\mathcal{N}_* G"] \\ & \mathcal{D} & \end{tikzcd}$$

To sum up, any natural transformation $\alpha : F \to G$ gives a homotopy of simplicial set maps. As a consequence, if $F : \mathcal{C} \to \mathcal{D}$ is an adjoint functor, then the unit and counit gives us

$$\begin{aligned} \mathcal{N}_* F \circ \mathcal{N}_* G &\simeq \mathrm{id}_{\mathcal{N}_*\mathcal{C}} \\ \mathcal{N}_* G \circ \mathcal{N}_* F &\simeq \mathrm{id}_{\mathcal{N}_*\mathcal{D}} \end{aligned}$$

after applying the nerve functor.

One of the most important application of slice category is the following theorem.

Theorem 3.2 (Quillen, Theorem A). *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor. If* $\mathcal{N}_*(F/d) \simeq \{*\}$ *for all* $d \in \mathrm{Ob}(\mathcal{D})$*, then* $\mathcal{N}_* F : \mathcal{N}_*\mathcal{C} \to \mathcal{N}_*\mathcal{D}$ *is a weak homotopy equivalence, i.e.* $BF : B\mathcal{C} \xrightarrow{\sim} B\mathcal{D}$.

[8] Proofs are in old notes.

Notation. For any $F : \mathcal{C} \to \mathcal{D}$, we write $F/d = F \downarrow d$ for any object d in $\mathcal{D}$.

Example 3.3. Let $F : \mathcal{C} \to \mathcal{D}$ be a left adjoint with righr adjoint G, then F/d is contractible.

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

The counit natural transformation gives $\varepsilon_d : FGd \xrightarrow{\sim} d$ for any $d \in \mathrm{Ob}(\mathcal{D})$, so (Gd, ε) is a terminal object in F/d, which implies that F/d is weakly contractible.

Quillen's Theorem A can be deduced from Theorem B.
Notice,

1. $d \mapsto F/d$ defines a functor

$$\begin{aligned} F/(-) : \mathcal{D} &\longrightarrow \mathbf{Cat} \\ d &\longmapsto F/d \\ (\beta : d \to d') &\mapsto (F/\beta : F/d \to F/d') \end{aligned} \tag{3.1}$$ {{eq:slice-functor}}

2. There is a forgetful functor

$$j_d : F/d \to \mathcal{C}$$

which composes

$$F/d \xrightarrow{j_d} \mathcal{C} \xrightarrow{F} \mathcal{D}$$

and on nerves

$$\mathcal{N}_*(F/d) \xrightarrow{\mathcal{N}_* j_d} \mathcal{N}\mathcal{C} \xrightarrow{\mathcal{N}_* F} \mathcal{N}_*\mathcal{D}.$$

Quillen's theorem B detects when this sequence is a fibration.

Theorem 3.4 (Quillen, Theorem B)**.** *Assume that for any morphism $\beta : d \to d'$ in $\mathcal{D}$,*

$$\mathcal{N}_*(F/\beta) : \mathcal{N}_*(F/d) \to \mathcal{N}_*(F/d')$$

is a weak equivalence, then

$$\mathcal{N}_*(F/d) \longrightarrow \mathcal{N}_*\mathcal{C} \longrightarrow \mathcal{N}_*\mathcal{D}$$

is a homotopy fibration sequence for any d, i.e.

$$\begin{tikzcd} \mathcal{N}_*(F/d) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{N}_*\mathcal{C} \arrow[d] \\ \{*\} \arrow[r] & \mathcal{N}_*\mathcal{D} \end{tikzcd}$$

is homotopy Cartesian.

Note $\mathcal{N}(F/d) \simeq \mathrm{hocofib}(\mathcal{N}F)$, so by applying Theorem B, we see that trivial homotopy fibers implies that $\mathcal{N}_*F : \mathcal{N}_*\mathcal{C} \to \mathcal{N}_*\mathcal{D}$ is a weak homotopy equivalence.

There is another proof of Theorem A based on the ideal of homotopy decompositions.

The functor 3.1 gives a diagram of small categories of shape $\mathcal{D}$. Applying nerve functor we get

$$\mathcal{N}_*(F/(-)) : \mathcal{D} \to \mathbf{Cat} \to \mathbf{sSet}$$

and the aasignment

$$j_d : F/d \to \mathcal{C}$$

gives a functor

$$j = \mathrm{const}_{\mathcal{C}} : F/(-) \to \mathcal{C}$$

which induces

$$\mathcal{N}_*j : \mathcal{N}_*[F/(-)] \to \mathcal{N}_*\mathcal{C}$$

and a map in $\mathrm{Ho}(\mathbf{sSet})$

$$\mathcal{N}j_* : \mathrm{hocolim}_{\mathcal{D}}\, \mathcal{N}_*[F/(-)] \to \mathcal{N}_*\mathcal{C}$$

{lem:whe}

Lemma 3.5 (Quillen). *For any $F : \mathcal{C} \to \mathcal{D}$, the map*

$$\mathcal{N}j_* : \mathrm{hocolim}_{\mathcal{D}}\, \mathcal{N}_*[F/(-)] \xrightarrow{\sim} \mathcal{N}_*\mathcal{C}$$

is a weak homotopy equivalence, i.e.

$$\mathrm{hocolim}_{\mathcal{D}}[B(F/-)] \xrightarrow{\sim} B\mathcal{C}. \tag{3.2}$$ {{eq:whe}}

is a weak homotopy equivalence of topological spaces.

Question 3.6. *How does lemma implies Theorem A?*

Given any diagram $\Gamma : \mathcal{D} \to \mathbf{sSet}$, there is a canonical map $\Gamma \to * = \mathrm{const}_{\Delta[0]_*}$ and therefore a map between homotopy colomits

$$\mathrm{can} : \mathrm{hocolim}_{\mathcal{D}}\, \Gamma \to \mathrm{hocolim}_{\mathcal{D}} * \cong \mathcal{N}_*\mathcal{D} \tag{3.3}$$ {{eq:can}}

Combining these together we get a commutative diagram

$$\begin{tikzcd}
 & \mathrm{hocolim}_{\mathcal{D}}\,[\mathcal{N}_*F/(-)] \arrow[dl, "\mathcal{N}j_*"', "\sim"] \arrow[dr, "\mathrm{can}", "\sim"'] & \\
\mathcal{N}_*\mathcal{C} \arrow[rr, "\mathcal{N}_*F", "\sim"'] & & \mathcal{N}_*\mathcal{D}
\end{tikzcd}$$

where both can and $\mathcal{N}_*F$ are weak equivalences, so by 2-of-3 property of weak equivalences, $\mathcal{N}j_*$ is also a weak equivalence.

Lemma 3.5 is a consequence of a beautiful formula of Thomason.

Theorem 3.7 ([T]). *Let* $F : \mathcal{D} \to \mathbf{Cat}$ *be a functor, then*

$$\mathrm{hocolim}_{\mathcal{D}}(\mathcal{N}_* F) \simeq \mathcal{N}_*(\mathcal{D} \int F)$$

where $\mathcal{D} \int F$ *is the Grothendieck construction.*

Definition 3.8. Let $F : \mathcal{D} \to \mathbf{Cat}$ be a functor, the Grothendieck construction $\mathcal{D} \int F$ is a category with

- objects are pairs (d, x) where $d \in \mathrm{Ob}(\mathcal{D})$ and $x \in \mathrm{Ob}(F(d))$.
- morphisms are

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D} \int F}((d,x),(d',x')) = \{(f, \varphi) : f : d \to d', \\ (\varphi : F(f)x \to x') \in \mathrm{Hom}_{F(f)d}(F(f)x, x')\}.\end{aligned}$$

Definition 3.9. $F/\mathcal{D} = \mathcal{D} \int (F/-)$ is the **global slice category** with objects $\bigcup_{d \in \mathcal{D}} \mathrm{Ob}(F/d)$. Explicitly, objects are triples $(c, d, f : Fc \to d)$ where $c \in \mathrm{Ob}(\mathcal{C})$, $d \in \mathrm{Ob}(\mathcal{D})$, $f : Fc \to d \in \mathrm{Mor}(\mathcal{D})$.

Then we can replace the canonical map 3.3 by the map $\mathcal{N}(F/\mathcal{D}) \xrightarrow{\sim} \mathcal{N}\mathcal{C}$.

Remark. We can think of 3.2 as a kind of homotopy decomposition.

There are important similar decomposition for other spaces.

Example 3.10. Let G be a compact Lie group (e.g. finite group). Let $BG := |\mathcal{N}_* G|$ be the geometric realization of simplicial spaces.

Theorem 3.11 ([N]). *Let* $G \cong G'$ *be compact connected Lie groups if and only if* $BG \simeq BG'$.

Therefore if you are interested in Lie groups from homotopical point of view, you can replace Lie groups by their classifying spaces, and there are many interesting Lie theory constructions, e.g. notions of closed subgroups, centralizers, maximal torus, that can be translated in the language of classifying spaces. In particular, there are many tools of studying classifying spaces which rely on this idea of homotopy decomposition.

Definition 3.12. Let G be a compact connected Lie group and p be a prime. A **homology decomposition** ($\mod p$) of BG is given by a functor $F : \mathcal{D} \to \mathbf{Top}$ together with a map $f : \mathrm{hocolim}_{\mathcal{D}} \mathcal{F} \to BG$ such that

$$f_* : H_*(\mathrm{hocolim}_{\mathcal{D}}(F), \mathbb{F}_p) \simeq H_*(BG, \mathbb{F}_p)$$

and for any $d \in \mathrm{Ob}(\mathcal{D})$, there is a closed subgroup $H_d \subseteq G$ such that $F(d) \simeq BH_d$, and

$$BH_d \simeq F(d) \to \mathrm{hocolim}_{\mathcal{D}}(F) \xrightarrow{f} BG$$

is induced by the inclusion $H_d \hookrightarrow G$.

Notation. We shall denote $\mathbf{Sp} = \mathbf{sSet}$ and $\mathbf{sSp} = \mathbf{sSet}_{\Delta}$.

Complete Segal Spaces The concept of complete Segal spaces is after Rezk [R1]. This is one the model for ∞-category. It requires a lot of background in order to introduce the definition, but to deal with them it is actually easier than the other models. And they come very natural in practice. The main constructions are

- Classifying diagram of a category (Rezk nerve).
- Reedy model structure on simplicial spaces.
- Localization of model structures (Bousfield localization).

3.2 Classifying Diagram of a Category

Observation. Let **Gpd** be the category of (small) groupoids, then there is a natural inclusion

$$i : \mathbf{Gpd} \hookrightarrow \mathbf{Cat}$$

which has a left adjoint

$$\mathrm{loc} : \mathbf{Cat} \longrightarrow \mathbf{Gpd}$$

$$\mathcal{C} \longmapsto \mathcal{C}[\mathrm{Mor}(\mathcal{C})^{1}]$$

and a right adjoint

$$\mathrm{Iso} : \mathbf{Cat} \hookrightarrow \mathbf{Gpd}$$

$$\mathcal{C} \longmapsto \mathrm{Iso}(\mathcal{C})$$

where $\mathrm{Iso}(\mathcal{C})$ is the maximal subgroupoid of $\mathcal{C}$. Explicitly,

$$\mathrm{Hom}_{\mathrm{Iso}(\mathcal{C})}(x, y) = \{\text{invertible arrows } x \to y \text{ in } \mathcal{C}\}.$$

$$\begin{tikzcd} \mathbf{Gpd} \arrow[d, hook, "i"] \\ \mathbf{Cat} \arrow[u, bend left=60, "\mathrm{loc}"] \arrow[u, bend right=60, "\mathrm{Iso}"'] \end{tikzcd}$$

Notation. Let $\mathcal{C}$ be a small category. For any $n \geq 0$, define $\mathcal{C}^{[n]} = \mathrm{Fun}(\vec{n}, \mathcal{C})$ the functor category with objects

$$(f_0, \cdots, f_{n-1}) = (c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} c_n)$$

n-tuples of composable morphisms in $\mathcal{C}$.

Definition 3.13. The **classifying diagram** (Rezk nerve) of $\mathcal{C}$ is defined to be the simplciial space

$$\begin{aligned} \mathcal{N}^R(\mathcal{C})_* : \Delta^{op} &\longrightarrow \mathbf{sSet} = \mathbf{Sp} \\ [n] &\longmapsto \mathcal{N}_*(\mathrm{Iso}(\mathcal{C}^{[n]})) \end{aligned}$$

and for $f : [n] \to [m]$, it induces a map $\vec{f} : \vec{n} \to \vec{m}$, which gives a map $f_* : \mathcal{C}^{[m]} \to \mathcal{C}^{[n]}$ and thus a map

$$\mathcal{N}^R f_* : \mathcal{N}_*(\mathrm{Iso}(\mathcal{C}^{[m]})) \to \mathcal{N}_*(\mathrm{Iso}(\mathcal{C}^{[n]})).$$

By convention, $\mathcal{N}^R(\mathcal{C})_* : \Delta^{op} \to \mathbf{Sp}$ looks like

$$\mathcal{N}^R(\mathcal{C})_* = \left\{ \mathcal{N}^R(\mathcal{C})_0 \leftleftarrows \mathcal{N}^R(\mathcal{C})_1 \mathrel{\substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow}} \mathcal{N}^R(\mathcal{C})_2 \cdots \right\}.$$

Thus we have the Rezk nerve functor

$$\begin{aligned} \mathcal{N}^R : \mathbf{Cat} &\longrightarrow \mathbf{sSp} \\ \mathcal{C} &\longmapsto \mathcal{N}^R(\mathcal{C}) \end{aligned}$$

The components look as follows.

- $\mathcal{N}^R(\mathcal{C})_0 = \mathcal{N}(\mathrm{Iso}(\mathcal{C}))$ contains only the information of isomorphisms in $\mathcal{C}$.
- $\mathcal{N}^R(\mathcal{C})_1 = \mathcal{N}(\mathrm{Iso}(\mathcal{C}^{[1]}))$, where $\mathrm{Iso}(\mathcal{C}^{[1]}) = \mathrm{Mor}(\mathcal{C})$,
  - objects are morphisms in $\mathcal{C}$ and,
  - morphisms are pairs of isomorphisms $(\varphi, \phi) : f \to g$ such that

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\varphi"', "\cong"] & y \arrow[d, "\phi", "\cong"'] \\ z \arrow[r, "g"] & w \end{tikzcd}$$

    commutes

- In general, $\mathcal{N}^R(\mathcal{C})_{n,m}$ is the set of diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "\cong"] & \bullet \arrow[r] \arrow[d, "\cong"] & \bullet \arrow[r, dashed, no head] \arrow[d, "\cong"] & \bullet \arrow[r] \arrow[d, "\cong"] & \bullet \arrow[d, "\cong"] \\
\bullet \arrow[r] \arrow[d, dashed, no head] & \bullet \arrow[r] \arrow[d, dashed, no head] & \bullet \arrow[r, dashed, no head] \arrow[d, dashed, no head] & \bullet \arrow[r] \arrow[d, dashed, no head] & \bullet \arrow[d, dashed, no head] \\
\bullet \arrow[r] \arrow[d, "\cong"] & \bullet \arrow[r] \arrow[d, "\cong"] & \bullet \arrow[r, dashed, no head] \arrow[d, "\cong"] & \bullet \arrow[r] \arrow[d, "\cong"] & \bullet \arrow[d, "\cong"] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r, dashed, no head] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

  where horizontal ones are n-composable morphisms and vertical ones are m-composable isomorphisms.

Comparison map. The adjunction

$$i : \mathbf{Gpd} \rightleftarrows \mathbf{Cat} : \mathrm{Iso}$$

yields the unit of the adjunction $\eta : \mathrm{id}_{\mathbf{Gpd}} \to \mathrm{Iso}\, i$, which extends to a natual morphism

$$\eta : \mathcal{N} \longrightarrow \mathcal{N}^R$$

of functors $\mathbf{Gpd} \to \mathbf{sSp}$. Explicitly, for any gropuoid $\mathcal{G}$, there is a natural map

$$\eta_{\mathcal{G}} : \mathcal{N}(\mathcal{G})_* \to \mathcal{N}^R(\mathcal{G})_*$$

of simplicial spaces, where $\mathcal{N}(\mathcal{G})_*$ is regarded as a constant (in horizontal direction) simplicial space.

$$\begin{tikzcd}
\mathbf{Gpd} \arrow[rr, "\mathcal{N}^R"] \arrow[dr, "\mathcal{N}"'] & {} & \mathbf{sSp} \\
& \mathbf{sSet} \arrow[u, phantom, "\Uparrow\eta"] \arrow[ur, hook] &
\end{tikzcd}$$

Definition 3.14. Unless named otherwise, by a weak equivalence of simplicial spaces we mean an degreewise weak equivalences of simplicial sets, i.e. $f : X_* \to Y_*$ is a weak equivalence if and only if $f_n : X_n \to Y_n$ is a weak equivalence in $\mathbf{Sp} = \mathbf{sSet}$.

Remark. This class of weak equivalences can be extended to different model structures, namely, the Reedy model structure and BK model structure of $\mathbf{sSp}$.

Proposition 3.15 ([R1]). *The map $\eta_{\mathcal{Y}} : \mathcal{N}(\mathcal{Y}) \to \mathcal{N}^R(\mathcal{Y})$ is a weak equivalence in* $\mathbf{sSp}$ *for any groupoid $\mathcal{Y}$.*

Proof. If $\mathcal{Y}$ is a groupoid, then so is all $\mathcal{Y}^{[n]}$ for all $n \geq 0$, and hence $\mathrm{Iso}(\mathcal{Y}^{[n]}) = \mathcal{Y}^{[n]}$. On the other hand, $[n] \to [0]$ induces a functor

$$\mathcal{Y} = \mathcal{Y}^{[0]} \xrightarrow{\sim} \mathcal{Y}^{[n]}$$

which is an equivalence of categories.

This map is fully faithful and is essentially surjective since $\mathcal{Y}$ is a groupoid,

$$\mathcal{Y} = \mathcal{Y}^{[0]} \simeq \mathcal{Y}^{[n]} = \mathrm{Iso}(\mathcal{Y}^{[n]}),$$

thus $\eta_{\mathcal{Y}} : \mathcal{N}(\mathcal{Y}) \to \mathcal{N}^R(\mathcal{Y})$ is a weak equivalence. □

Example 3.16. Ig $\mathcal{G}$ is a connected groupoid (i.e. there is only one iso-class of objects) then the choice of such any object in $\mathcal{G}$ gives an equivalence of categories,

$$\mathrm{Aut}_{\mathcal{G}}(x) \xrightarrow{\sim} \mathcal{G}$$

thus

$$\mathcal{N}^R(\mathcal{G}) \simeq \mathcal{N}\mathcal{G} \simeq \mathcal{N}\,\mathrm{Aut}_{\mathcal{G}}(x).$$

If $\mathcal{G}$ is not connected, we have

$$\mathcal{N}^R(\mathcal{G}) \simeq \mathcal{N}\mathcal{G} \simeq \coprod_{\langle x\rangle \in \mathrm{Isoclass}(\mathcal{G})} \mathrm{Aut}_{\mathcal{G}}(x).$$

Now let $\mathcal{C}$ be a category (not a groupoid),

- $\mathcal{N}^R(\mathcal{C})_0 = \mathcal{N}(\mathrm{Iso}(\mathcal{C}))$
- $\mathcal{N}^R(\mathcal{C})_1 = \coprod_{\langle x\rangle,\langle y\rangle \in \mathrm{Isoclass})\mathcal{C})} \coprod_{\langle \varphi : x \to y\rangle} \mathcal{N}\,\mathrm{Aut}_{\mathcal{C}^{[n]}}(\langle \varphi \rangle)$

Example 3.17. 1. Let $\mathcal{C} = \overrightarrow{1} = \{0 \xrightarrow{\varphi} 1\}$. Then

   - $\mathrm{Aut}_{\mathcal{C}}(0) = \mathrm{Aut}_{\mathcal{C}}(1) = \mathrm{Aut}_{\mathcal{C}}(\varphi) = e$ the trivial group.
   - $\mathcal{N}^R\mathcal{C}_0 = \mathcal{N}e \coprod \mathcal{N}e$.
   - $\mathcal{N}^R\mathcal{C}_1 = \mathcal{N}e \coprod \mathcal{N}e \coprod \mathcal{N}e$, and $|\mathcal{N}^R\mathcal{C}_1| = 3$ points.

2. Let $\mathcal{C} = \overrightarrow{0}$. We see that $\mathcal{N}^R(\overrightarrow{0})_0 \not\simeq \mathcal{N}^R(\overrightarrow{1})_0$, so $\mathcal{N}^R(\overrightarrow{0}) \not\simeq \mathcal{N}^R(\overrightarrow{1})$.

3. Let $\mathcal{C} = \overleftrightarrow{1}$, then $\mathcal{N}^R(\overleftrightarrow{1})_0 \simeq \mathcal{N}^R(\overrightarrow{1})_0$, but $\mathcal{N}^R(\overleftrightarrow{1})_1 \simeq \mathcal{N}e \coprod \mathcal{N}e$, thus $|\mathcal{N}^R(\overleftrightarrow{1})_1| \simeq 2\mathbf{pt} \not\simeq |\mathcal{N}^R(\overrightarrow{1})_1|$.

Theorem 3.18 ([R1]). *1. The functor $\mathcal{N}^R : \mathbf{Cat} \to \mathbf{sSp}$ is fully faithful.*

2. *$F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if $\mathcal{N}^R(F) : \mathcal{N}^R\mathcal{C} \to \mathcal{N}^R\mathcal{D}$ is a weak equivalences of simplicial spaces.*

Proof. For part (2), notice that $\mathrm{Iso}(\mathcal{C}^{[n]})$ are groupoids for $n \geq 0$. Hence $\mathcal{N}^R(F)_n : \mathcal{N}^R\mathcal{C}_n \to \mathcal{N}^R\mathcal{D}_n$ is a weak equivalence of simplicial sets if and only if $\mathrm{Iso}(F^{[n]}) : \mathrm{Iso}(\mathcal{C}^{[n]}) \to \mathrm{Iso}(\mathcal{D}^{[n]})$ is an equivalence of categories.

Of course, if $F^{[n]}$ is an equivalence of categories, then $\mathrm{Iso}(F^{[n]})$ is an equivalence of categories. But check

Lemma 3.19. *$F^{[n]}$ is an equivalence of categories if and only if F is an equivalence of categories.*

thus $\mathcal{N}^R(F) : \mathcal{N}^R\mathcal{C} \to \mathcal{N}^R\mathcal{D}$ is a weak equivalences of simplicial spaces.
(To be continued later.) □

3.3 Reedy Model Structure on Simplicial Spaces

Goal. We want a model structure on **sSp** such that $\mathcal{N}^R(\mathcal{C})$ are (special) fibrant objects. [9]

Remark. Properties of Rezk nerve:

1. For any $\mathcal{C}$, $\mathcal{N}^R(\mathcal{C})$ is determined by $\mathcal{N}^R(\mathcal{C})_0$ and $\mathcal{N}^R(\mathcal{C})_1$. Indeed, we have

$$\mathcal{N}^R(\mathcal{C})_n \cong \underbrace{\mathcal{N}^R(\mathcal{C})_1 \times_{\mathcal{N}^R(\mathcal{C})_0} \cdots \times_{\mathcal{N}^R(\mathcal{C})_0} \mathcal{N}^R(\mathcal{C})_1}_{n}.$$

   We will relax the isomorphisms to be weak equivalences, which gives us the definition of Segal spaces.

2. The subspace of $\mathcal{N}^R(\mathcal{C})_1$ arising from isomorphisms in $\mathcal{C}$ is equivalent to $\mathcal{N}^R(\mathcal{C})_0$. More precisely,

$$i : \mathrm{Iso}(\mathcal{C}) \hookrightarrow \mathcal{C}$$

   gives us

$$\begin{tikzcd} \mathcal{N}^R(\mathrm{Iso}(\mathcal{C}))_1 \arrow[r, "i"] & \mathcal{N}^R(\mathcal{C})_1 \\ \mathcal{N}^R(\mathrm{Iso}(\mathcal{C}))_0 \arrow[u, "s_0"', "\simeq"] \arrow[r, equal] & \mathcal{N}^R(\mathcal{C})_0 \arrow[u, "s_0"] \end{tikzcd}$$

By mimicking the above properties, we give the definition of complete Segal spaces as below.

Definition 3.20. Let X_* be a simplicial space,

1. if the Segal maps $X_n \simeq X_1 \times_{X_0} \cdots \times_{X_0} X_1$ are a weak equiavlence, i.e. there is a composition operation well defined up to coherent homotopy, then X_* is called a **Segal space**.

[9] Prof. Yuri Berest quoted "The route is curved, but our goals are trivial". Here is the same situation.

2. if furthermore, the sub-simplicial object $\mathrm{Core}(X_*)$ on the invertible morphisms in each degree is homotopy constant: it has all face and degeneracy maps being homotopy equivalences (this says that if a morphism is an equivalence under the explicit composition operation then it is already a morphism in X_0), then X_* is a **complete Segal space**.

Remark. One cannot just put a model structure on the category of models (e.g. the category of all quasi-categories) because this is not closed under limits or/and colimits. Instead, we put a model structure on a larger category so that the fibrant objects are precisely our models. For instance, we have the Joyal model structure on **sSet** where quasi-categories are fibrant objects.

For Complete Segal spaces (and other models), we need to work with $\mathbf{sSp} = \mathbf{sSet}_\Delta$ and modify the (canonical) Reedy model structure by localizing (Bousfield localization) twice to get our desired model structures.

Remark. This "double" Bousfield localization is a common phenomena.

Example 3.21 (Derived stacks)**.** Let k be a commutative ring. Let

$$\mathbf{dAff}_k = (\mathbf{sComm}_k)^{op}$$

be the category of derived affine schemes over k. The category

$$\mathrm{sPr}(\mathbf{dAff}_k) = \mathrm{Fun}(\mathbf{sComm}_k, \mathbf{sSet})$$

has a canonical model structure, and by localizing once we get

$$\mathrm{sPr}(\mathbf{dAff}_k^{\checkmark})$$

prestack model structure, where fibrant objects are derived prestacks. And after the second localization (stackification) we get

$$\mathrm{sPr}(\mathbf{dAff}_k^{\sim})$$

stack model structure where fibrant objects are derived stacks.

3.3.1 Promoting Model Structures

There is a very useful procedure to lift or promote model structure from one category to another.

Theorem 3.22 (Kan, [H2])**.** [10] *Let* $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ *be a pair of adjoint functors. Let* $\mathcal{C}$ *be a cofibrantly generated model category with generating cofibrations* I *and generating acyclic cofibrations* J*. Define weak equivalences, cofibrations and fibrations in* $\mathcal{D}$ *as follows*

[10] see old notes.

- $f \in \mathsf{WE}(\mathcal{D}) \Leftrightarrow G(f) \in \mathsf{WE}(\mathcal{C})$.
- $f \in \mathtt{Fib}(\mathcal{D}) \Leftrightarrow G(f) \in \mathtt{Fib}(\mathcal{C})$.
- $\mathtt{Cof}(\mathcal{D}) \coloneqq \mathsf{LLP}(\mathsf{WE}(\mathcal{D}) \cap \mathsf{WE}(\mathcal{D}))$.

Assume

1. G *preserves sequential colimits, i.e. given*

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n \longrightarrow \cdots$$

   the natural map

$$\mathrm{colim}_n(GX_n) \xrightarrow{\sim} G(\mathrm{colim}_n X_n)$$

   is an isomorphism in $\mathcal{C}$.

2. $\mathtt{Cof}(\mathcal{D}) \cap \mathsf{LLP}(\mathtt{Fib}(\mathcal{D})) \subseteq \mathsf{WE}(\mathcal{D})$

Then $(\mathsf{WE}(\mathcal{D}), \mathtt{Fib}(\mathcal{D}), \mathtt{Cof}(\mathcal{D}))$ *makes* $\mathcal{D}$ *a cofibrantly generated model category with generating cofibrations* $F(I)$ *and generating acyclic cofibrations* $F(J)$.

Remark. In practice, (1) is easy to check and for (2) we need the following.

Lemma 3.23. *Suppose in* $\mathcal{D}$ *the following holds,*

1. *there is a functorial fibrant replacement* $R : \mathcal{D} \to \mathcal{D}$

$$(X \to *) \longrightarrow (X \to RX \twoheadrightarrow *)$$

   In particular, $R = \mathrm{id}$ *if* $\mathcal{D}$ *is fibrant, i.e. every morphism* $X \to *$ *is a fibration.*

2. *for any object* X *in* $\mathcal{D}$*, there is a canonical path object*

$$\begin{tikzcd} X \arrow[rr, "\Delta"] \arrow[dr, "\sim"'] & & X \times X \\ & P(X) \arrow[ur, two heads] & \end{tikzcd}$$

Then $\mathcal{D}$ *satisfies property (2).*

Example 3.24. Let k be a commutative ring.

$$\mathrm{Sym} : \mathbf{sMod}_k \rightleftarrows \mathbf{sComm}_k : U$$

gives the model structure on $\mathbf{sComm}_k$ and this becomes a Quillen adjunction.

Theorem 3.25 (BK, standard projective model structure on diagrams)**.** *Let $\mathcal{C}$ be a small category, and $\mathcal{M}$ a cofibrantly generated model category with generating cofibrations I and generating acyclic cofibrations J. Let $\mathcal{M}^{\mathcal{C}} = \mathrm{Fun}(\mathcal{C}, \mathcal{M})$ be the category of all $\mathcal{C}$-diagrams in $\mathcal{M}$. Then $\mathcal{M}^{\mathcal{C}}$ has a cofibrantly generated model structure with*

- $\mathcal{WE}(\mathcal{M}^{\mathcal{C}}) :=$ *objectwise weak equivalences of diagrams, i.e.*

$$f : X \to Y \in \mathcal{WE}(\mathcal{M}^{\mathcal{C}}) \Leftrightarrow f(c) : X(c) \to Y(c) \in \mathcal{WE}(\mathcal{M}), \forall c \in \mathrm{Ob}(\mathcal{C}).$$

- $\mathtt{Fib}(\mathcal{M}^{\mathcal{C}}) :=$ *objectwise fibrations of diagrams.*

- $\mathtt{Cof}(\mathcal{M}^{cC}) := \mathrm{LLP}(\mathcal{WE}(\mathcal{M}^{\mathcal{C}}) \cap \mathtt{Fib}(\mathcal{M}^{\mathcal{C}}))$.

[H2], *11.6.1.* Let $\mathcal{C}^{\delta}$ be the discrete category underlying $\mathcal{C}$,

- objects are the same as objects in $\mathcal{C}$, and
- morphisms are only identity morphisms.

then

$$\mathcal{M}^{\mathcal{C}^{\delta}} = \mathrm{Fun}(\mathcal{C}^{\delta}, \mathcal{M}) = \prod_{c \in \mathrm{Ob}(\mathcal{C})} \mathcal{M}$$

has the obvious model structure with $\mathcal{WE}, \mathtt{Fib}, \mathtt{Cof}$ defined objectwise. Then the inclusion $\mathcal{C}^{\delta} \hookrightarrow \mathcal{C}$ induces a functor U

$$\mathbb{F} : \mathcal{M}^{\mathcal{C}^{\delta}} \rightleftarrows \mathcal{M}^{\mathcal{C}} : U$$

with left adjoint $\mathbb{F} : \mathcal{M}^{\mathcal{C}^{\delta}} \to \mathcal{M}^{\mathcal{C}}$ defined by for $X = \{X(c)\}_{c \in \mathrm{Ob}(\mathcal{C})}$,

$$\mathbb{F}(X) := \coprod_{c \in \mathrm{Ob}(\mathcal{C})} X(c) \bigotimes h^{c}$$

where $h^{c} = \mathrm{Hom}_{\mathcal{C}}(c.-)$. Note $\mathcal{M}$ is tensored over **Set**. More explicitly,

$$\mathbb{F}(X) = \coprod_{c \in \mathrm{Ob}(\mathcal{C})} \coprod_{f \in \mathrm{Hom}_{\mathcal{C}}(c,d)} X(c).$$

Check this adjunction satisfies the lifting theorem. □

Example 3.26. Let $\mathcal{M} = \mathbf{sSet}$, $\mathcal{C} = \Delta^{op}$, $\mathcal{M}^{\mathcal{C}} = \mathbf{sSp}$ has a model structure where weak equivalences are objectwise weak equivalences.

Reedy Model Structures Reedy Model Structure (RMS) [[H2], Chapter 15] is a natural model structure on $\mathcal{M}^{\mathcal{C}}$ ($\mathcal{M}$ a (simplicial) model category) defined only for special $\mathcal{C}$ - called Reedy categories - with the same weak equivalences as in BK but more cofibrations (and hence less fibrations than in BK).

The advantage of this model structure is, if $\mathcal{M}$ is a simplicial model category (e.g. $\mathcal{M} = \mathbf{s}\mathcal{N}$) there is an internal realization functor

$$|-|_{\mathcal{M}} : \mathbf{s}\mathcal{M} \to \mathcal{M}$$

which is left Quillen with respect to Reedy model structure. For instance, $\mathcal{M} = \mathbf{Top}$ recovers our classical geometric realization functor.

Definition 3.27. A small category $\mathcal{C}$ is **Reedy** if it contains two wide subcategories $\overrightarrow{\mathcal{C}}$ (called direct) and $\overleftarrow{\mathcal{C}}$ (called inverse) such that

1. $\mathcal{C} = \overrightarrow{\mathcal{C}} \bowtie \overleftarrow{\mathcal{C}}$ is a crossed category, i.e. any morphism f in $\mathcal{C}$ has a unique decomposition
$$f = \overrightarrow{f} \circ \overleftarrow{f},$$
where $\overrightarrow{f} \in \mathrm{Mor}(\overleftarrow{\mathcal{C}})$ and $\overleftarrow{f} \in \mathrm{Mor}(\overleftarrow{\mathcal{C}})$.

2. there is a degree function
$$\deg : \mathrm{Ob}(\mathcal{C}) \to \mathbb{Z}_{\geq 0}$$
such that
   - every nonidentity morphism in $\overrightarrow{\mathcal{C}}$ raises degree, i.e.
   $$\deg(f(c)) \geq \deg(c), \forall c \in \mathrm{Ob}(\mathcal{C}), f \in \mathrm{Mor}(\overrightarrow{\mathcal{C}}),$$
   - every nonidentity morphism in $\overleftarrow{\mathcal{C}}$ lowers degree, i.e.
   $$\deg(f(c)) \leq \deg(c), \forall c \in \mathrm{Ob}(\mathcal{C}), f \in \mathrm{Mor}(\overleftarrow{\mathcal{C}}).$$

Remark. Notice

1. If $\mathcal{C}$ is Reedy, so is $\mathcal{C}^{op}$ (swap the two subcategories).

2. If $\mathcal{C}, \mathcal{D}$ are Reedy, so is $\mathcal{C} \times \mathcal{D}$.

Example 3.28. 1. Δ is Reedy, with $\deg[n] = n$. $\overrightarrow{\Delta} = \Delta^+$ all injective maps, and $\overleftarrow{\Delta} = \Delta^-$ all surjective maps.

2. Δ^{op} is Reedy.

3. Let X be a simplicial set, $\Delta X = \Delta \downarrow X$ the category of simplices is a Reedy category, with $\deg(x) = n$ if $x \in X_n$. Then
$$\overrightarrow{\Delta X} = \Delta^+ X, \quad \overleftarrow{\Delta X} = \Delta^- X.$$

3.3.2 Skeletons and Coskeletons

Recall (cf. odd notes) if $\mathcal{M}$ is a category with all (small) limits and colimits, then for any $f : I \to J \in \mathrm{Mor}(\mathbf{Cat})$, we have

$$\begin{tikzcd} \mathcal{M}^J \arrow[d, "f^*"] \\ \mathcal{M}^I \arrow[u, bend left=60, "f_!"] \arrow[u, bend right=60, "f_*"'] \end{tikzcd}$$

where

$$f_! : \mathcal{M}^I \to \mathcal{M}^J$$

is given by left Kan extension [11]

$$\begin{tikzcd} I \arrow[r, "X"] \arrow[d, "f"'] & \mathcal{M} \\ J \arrow[ur, "L_f(X) =: f_!(X)"'] \end{tikzcd}$$

which can be computed by

$$f_!X_(j) \cong \mathrm{colim}(f \downarrow j \xrightarrow{u} I \xrightarrow{X} \mathcal{M}).$$

And dually,

$$f_*X_(j) \cong \lim(j \downarrow f \xrightarrow{u} I \xrightarrow{X} \mathcal{M}).$$

Example 3.29. Let $J = \Delta$ and $I = \Delta_{\leq n}$ the full subcategory of Δ with objects $\{[0], \cdots, [n]\}$. There is a natural inclusion functor

$$i_n : \Delta^{op}_{\leq n} \hookrightarrow \Delta^{op}$$

For any $\mathcal{M}$, $\mathbf{s}\mathcal{M} = \mathrm{Fun}(\Delta^{op}, \mathcal{M})$ and $\mathbf{s}_{\leq n}\mathcal{M} = \mathrm{Fun}(\Delta^{op}_{\leq n}, \mathcal{M})$ the n-**truncated simplicial objects** in $\mathcal{M}$, we have the following diagram of adjoints.

$$\begin{tikzcd} \mathbf{s}\mathcal{M} \arrow[d, "(i_n)^*"] \\ \mathbf{s}_{\leq n}\mathcal{M} \arrow[u, bend left=60, "(i_n)_!"] \arrow[u, bend right=60, "(i_n)_*"'] \end{tikzcd}$$

For any simplicial set X, we define

[11] This is in fact pointwise left Kan extension.

- $\mathrm{sk}_n(X) \coloneqq (i_n)_! i_n^* X$, and
- $\mathrm{cosk}_n(X) \coloneqq (i_n)_* i_n^* X$.

Observe that adjunction morphisms

$$\mathrm{sk}_n(X) \to X, \quad X \to \mathrm{cosk}_n(X)$$

give

1. the **skeleton filtration** $\mathrm{sk}_0 X \to \mathrm{sk}_0 X \to \mathrm{sk}_2 X \to \cdots \to X$ of X,
2. the **coskeleton tower** $X \to \cdots \to \mathrm{cosk}_2 X \to \mathrm{cosk}_1 X \to \mathrm{cosk}_0 X$ of X,

From now on we will mainly focus on the skeleton filtration, the dual statements hold for coskeleton towers.

By the colimit formula of left Kan extension,

$$\mathrm{sk}_n(X)_m \cong \operatorname*{colim}_{\left(\substack{\varphi:[m]\to[k], \\ k\leq n}\right)\in\Delta} \varphi^*(X_k) \operatorname*{colim}_{\left(\substack{\varphi:[m]\to[k], \\ k\leq n}\right)\in\Delta^{-mn,,,,,,,,nnnnnnnnnnnnnnnnnnnnnnn}} \varphi^*(X_k)$$

since any $\varphi : [m] \to [k]$ can be factored uniquely as $[m] \twoheadrightarrow [k'] \hookrightarrow [k]$.

Notice if $m \leq n$, $\mathrm{sk}_n X_m = X_m$. In general this formula says $\mathrm{sk}_n X_m$ is generated by images of degeneracy in X of dimension m, and

$$\bigcup_{n\geq 0} \mathrm{sk}_n(X) \cong X$$

i.e. the skeleton filtration is exhausted.

More generally, we define for $f : X_* \to Y_* \in \mathrm{Mor}(\mathbf{s}\mathcal{M})$, the n**-th relative skeleton** of f is defined to be the pushout in $\mathbf{s}\mathcal{M}$

$$\begin{tikzcd}
\mathrm{sk}_n X \arrow[r] \arrow[d, "\mathrm{sk}_n f"'] & X \arrow[d] \arrow[dr, "f"] & \\
\mathrm{sk}_n(Y) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \mathrm{sk}_n^X(Y) \arrow[r] & Y
\end{tikzcd}$$

3.3.3 Latching and Matching Objects

Given $X \in \mathrm{Ob}(\mathbf{s}\mathcal{M})$, we define the n**-th latching object** in $\mathcal{M}$

$$L_n(X) \coloneqq \mathrm{sk}_{n-1} X_n \in \mathrm{Ob}(\mathcal{M})$$

and the n**-th matching object** in $\mathcal{M}$

$$M_n(X) \coloneqq \mathrm{cosk}_{n-1} X_n \in \mathrm{Ob}(\mathcal{M}).$$

These come with natural maps

$$s : L_n(X) \to X, d : X \to M_n(X)$$

and in a similar manner we can define the n**-th relative latching** and **matching objects** for any map $f : X \to Y$ in $\mathbf{s}\mathcal{M}$.

Theorem 3.30 (Reedy)**.** *Let $\mathcal{C} = \Delta^{\mathrm{op}}$ and $\mathcal{M}$ a model category. There is a Reedy model structure on $\mathcal{M}^{\mathcal{C}}$ with*

1. $\mathrm{WE}(\mathbf{s}\mathcal{M}) =$ *objectwise weak equivalences.*

2. $\mathrm{Cof}(\mathbf{s}\mathcal{M}) =$ *are morphisms $f : X_* \to Y_*$ such that for any $n \geq 0$, the natural maps $L_n^X(Y) \hookrightarrow Y$ is a cofibration in $\mathcal{M}$.*

$$\begin{tikzcd} L_n(X) \arrow[r] \arrow[d, "L_n f"'] & X \arrow[d] \arrow[dr, "f"] & \\ L_n(Y) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & L_n^X(Y) \arrow[r, hook] & Y \end{tikzcd}$$

3. $\mathrm{Fib}(\mathbf{s}\mathcal{M}) =$ *morphisms $f : X_* \to Y_*$ such that for any $n \geq 0$, the natural maps $X_n \twoheadrightarrow M_n^X(Y)$ are fibrations in $\mathcal{M}$.*

Remark.
1. This theorem extends to all Reedy categories.

2. Every cofibrant object $X \in \mathrm{Ob}(\mathcal{M})$ gives a cofibrant object in $\mathbf{s}\mathcal{M}$ with respect to the Reedy model structure, but this is not true for fibrant objects.

3. Unlike Bousfield-Kan model structure, $\mathbf{s}\mathcal{M}$ is not a simplicial model category (natural simplicial structure on $\mathbf{s}\mathcal{M}$ does not agree with Reedy model structure). For instance, given a cofibration $i : K \hookrightarrow L$ in **sSet** and a cofibration $j : X \hookrightarrow Y$ in $\mathbf{s}\mathcal{M}$,

$$i \boxtimes j : (K \boxtimes Y) \coprod_{K \boxtimes X} L \boxtimes X \hookrightarrow L \boxtimes Y$$

$i \boxtimes j$ is a cofibration, but when i is a acyclic cofibration, $i \boxtimes j$ is not necessarily an acyclic cofibration.

Recall the notion of Quillen functors.

Lemma 3.31 (see old notes)**.** *Given a pair of adjoint functors between model categories*

$$F : \mathcal{M} \rightleftarrows \mathcal{N} : G$$

the following are equivalent:

1. G *preserves fibrations and acyclic fibrations,*

2. F *preserves cofibrations and acyclic cofibrations,*

3. G *preserves fibrations and* F *preserves cofibrations.*

These are called **Quillen pairs***.*

Assume that $\mathcal{M}$ is a simplicial model category, then there is a natural realization functor

$$|-|_{\mathcal{M}} : \mathbf{s}\mathcal{M} \longrightarrow \mathcal{M}$$

defined as follows. We will use

$$(-) \boxtimes_{\mathcal{M}} (-) : \mathbf{sSet} \times \mathcal{M} \longrightarrow \mathcal{M}$$

for the internal tensor product on $\mathcal{M}$.

Definition 3.32. For any $X \in \mathrm{Ob}(\mathbf{s}\mathcal{M})$, we define

$$|X| \coloneqq \mathrm{coeq}\left[\coprod_{\varphi:[m]\to[n]\in\Delta} \Delta[n] \boxtimes X_m \overset{\varphi^*}{\underset{\varphi_*}{\rightrightarrows}} \coprod_{[n]\in\Delta} \Delta[n] \boxtimes X_n \right] \in \mathcal{M}$$

Example 3.33. When $\mathcal{M} = \mathbf{Top}$ the category of compact generated weakly Hausdorff topological spaces,

$$|-|_{\mathbf{Top}} : \mathbf{sTop} \longrightarrow \mathbf{Top}$$

is the classical geometric realization.

Notice $|-|_{\mathcal{M}}$ is left adjoint to the exponential object functor

$$(-)^{\Delta} : \mathcal{M} \to \mathbf{s}\mathcal{M}$$

Theorem 3.34 (Reedy)**.** *If $\mathcal{M}$ is a simplicial model category,a nd $\mathbf{s}\mathcal{M}$ is equipped with Reedy model structure, then*

$$|-|_{\mathcal{M}} : \mathbf{s}\mathcal{M} \rightleftarrows \mathcal{M} : (-)^{\Delta}$$

is a Quillen pair. In particular,

Example 3.35. $\mathcal{M} = \mathbf{sSet}$ and $\mathbf{s}\mathcal{M} = \mathbf{sSp}$ with Reedy model structure.

4 Derived Functors

Given a model category $\mathcal{M}$ there is an associated homotopy category

$$\mathrm{Ho}(\mathcal{M}) := \mathcal{M}[\mathcal{W}^{-1}], \mathcal{W} = \mathcal{W}E(\mathcal{M}).$$

When $\mathcal{M}$ is a model category, the homotopy category $\mathrm{Ho}(\mathcal{M})$ has good properties.

Our goal is to construct localization theory [12] for $\mathrm{Ho}(\mathcal{M})$ in terms of $\mathcal{M}$.

4.1 Yoga of derived functors

[DHKS]

We work with homotopical categories $(\mathcal{M}, \mathcal{W})$ where $\mathcal{W} \subset \mathcal{M}$ is the full subcategory of weak equivalences satisfying

1. $\mathcal{W}$ contains all isomorphisms.
2. $\mathcal{W}$ satisfies 2-of-6 property.
3. $\mathcal{W}$ is saturated, i.e. $f \in \mathcal{W}$ if and only if $\delta f \in \mathrm{Iso}(\mathrm{Ho}(\mathcal{M}))$, where

$$\delta : \mathcal{M} \longrightarrow \mathrm{Ho}(\mathcal{M}) = \mathcal{M}\,[\mathcal{W}]\,.$$

Example 4.1. $(\mathcal{M}, \mathcal{W})$ is homotopical when $\mathcal{M}$ is a model category with $\mathcal{W}$ the class of weak equivalences.

Remark. For homotopical category $\mathcal{M}$, $\mathrm{Ho}(\mathcal{M})$ can be viewed as a minimal homotopical category with $\mathcal{W}_{\mathrm{Ho}(\mathcal{M})} = \mathrm{Iso}(\mathrm{Ho}(\mathcal{M}))$.

Definition 4.2. Let $\mathcal{M}, \mathcal{N}$ be homotopical categories, $F : \mathcal{M} \to \mathcal{N}$ is homotopical if $F(\mathcal{W}_{\mathcal{M}}) \subseteq \mathcal{W}_{\mathcal{N}}$.

Lemma 4.3. *For a homotopical functor* $F : \mathcal{M} \to \mathcal{N}$, *there is a unique functor* $\overline{F} : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$ *such that*

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"] & \mathcal{N} \arrow[r, "\delta"] & \mathrm{Ho}(\mathcal{N}) \\ \mathrm{Ho}(\mathcal{M}) \arrow[urr, "\mathbb{L}F := R_{\gamma}(\delta F)"'] & & \end{tikzcd}$$

commutes.

Proof. Apply UMP of localization. ☐

[12] There are left and right localization theory because there are left and right Quillen functors.

Definition 4.4. The **total left derived functor** $\mathbb{L}F$ of F is defined as the right Kan extension

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{N} \arrow[r, "\delta"] & \mathrm{Ho}(\mathcal{N}) \\
\mathrm{Ho}(\mathcal{M}) \arrow[ur, Rightarrow, "\varepsilon"] \arrow[urr, "\mathbb{L}F := R_\gamma(\delta F)"'] & &
\end{tikzcd}$$

This comes with a natural transformation $\varepsilon : \mathbb{L}F \circ \gamma \Rightarrow \delta F$, which is universal in the sense that for any (G, ε'), there is a unqiue factorization

$$\begin{tikzcd}
G \circ \gamma \arrow[rr, Rightarrow, "\varepsilon'"] \arrow[dr, Rightarrow, "\exists!", "\mathbb{L}F"'] & & \delta F \\
& \mathbb{L}F \circ \gamma \arrow[ur, Rightarrow, "\varepsilon"'] &
\end{tikzcd}$$

{def:ldf}

Definition 4.5 ([DHKS]). A **left derived functor** of $F : \mathcal{M} \to \mathcal{N}$ is a functor $\mathbf{L}F : \mathcal{M} \to \mathrm{Ho}(\mathcal{N})$ together with a comparision map $\varepsilon : \mathbf{L}F \Rightarrow \delta F$ satisfying

1. $\mathbf{L}F$ is homotopical.
2. $(\mathbf{L}F, \varepsilon)$ is universal (terminal) among all homotopical functors $G : \mathcal{M} \to \mathrm{Ho}(\mathcal{N})$ with $\varepsilon' : G \Rightarrow \delta F$, i.e.

$$\begin{tikzcd}
G \arrow[rr, Rightarrow, "\varepsilon'"] \arrow[dr, Rightarrow, "\exists!", "\eta"'] & & \delta F \\
& \mathbf{L}F \arrow[ur, Rightarrow, "\varepsilon"'] &
\end{tikzcd}$$

Definition 4.6 ([S3]). A **pointwise left derived functor** LF of $F : \mathcal{M} \to \mathcal{N}$ is a functor $LF : \mathcal{M} \to \mathcal{N}$ with $\varepsilon : LF \to F$ such that $(\mathbf{L}F = \delta LF, \delta\varepsilon : \delta \mathbf{L}F \to \delta F)$ is a left derived functor as in the definition 4.5.

Remark. 1. If LF exists, it is unique up to homotopy.

2. Even if $\mathbf{L}F$ exists, LF may not exist. [13]

Later we will define realizable left derived functors which refines LF.

4.2 Derived Functors as Deformations

Definition 4.7. A **left deformation** of a homotopical category $\mathcal{M}$ is an endofunctor $Q : \mathcal{M} \to \mathcal{M}$ with natural weak equivalence $q : Q \Rightarrow \mathrm{id}_{\mathcal{M}}$.

[13]For instance, when a model category does not have functorial cofibrant replacement.

Remark. Q is always a homotopical functor. For any $f : X \to Y \in \mathrm{Mor}(\mathcal{M})$, the following commutative diagram

$$\begin{tikzcd} QX \arrow[r, "Qf"] \arrow[d, "q_X"'] & QY \arrow[d, "q_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

shows that $f \in \mathcal{W}$ if and only if $Qf \in \mathcal{W}$.

Lemma 4.8. *Let $\mathcal{M}_Q$ be the full subcategory of $\mathcal{M}$ such that $\mathcal{M}_Q \supseteq \mathrm{Im}(Q)$ contains essential image of $\mathcal{Q}$ for a left deformation (Q, q) of $\mathcal{M}$. Then $i : \mathcal{M}_Q \to \mathcal{M}$ induces an equivalence of categories*

$$\mathrm{Ho}(\mathcal{M}_Q) \simeq \mathrm{Ho}(\mathcal{M}).$$

We call any $\mathcal{M}_Q$ with these properties a left deformation category of $\mathcal{M}$ associated to (Q, q).

Proof. $\mathcal{M}_Q$ is homotopical with $\mathcal{W}_{\mathcal{M}_Q} = \mathcal{W}_{\mathcal{M}} \cap \mathcal{M}_Q$, and $\mathrm{Ho}(\mathcal{M}_Q) = \mathcal{M}_Q[\mathcal{W}_Q^{-1}]$, then i is homotopical, which induces a functor

$$\bar{i} : \mathrm{Ho}(\mathcal{M}_Q) \longrightarrow \mathrm{Ho}(\mathcal{M})$$

with inverse induced by $Q : \mathcal{M} \to \mathrm{Im}(Q) \subseteq \mathcal{M}_Q$. □

Definition 4.9. Let $F : \mathcal{M} \to \mathcal{N}$ be any functor. A left deformation (Q, q) of $\mathcal{M}$ is called a **left F-deformation** if there is a left deformation retract $\mathcal{M}_Q$ such that $F|_{\mathcal{M}_Q}$ is homotopical.

Exercise 4.10. Show that a left deformation (Q, q) is an F-deformation if and only if {ex:F-def}

1. $FQ : \mathcal{M} \to \mathcal{N}$ is a homotopical functor, and
2. $FqQ : FQ^2 \xrightarrow{\sim} FQ$ is a natural weak equivalence, i.e. for any $X \in \mathrm{Ob}(\mathcal{M})$,

   $$F(q_{QX}) : F(QQX) \xrightarrow{\sim} FQX$$

   is a weak equivalence in $\mathcal{N}$.

4.2.1 Main Exmaples

Cofibrant Replacement Let $\mathcal{M}$ be a model category, $\mathcal{W} = WE(\mathcal{M}) \subset \mathcal{M}$. Assume $\mathcal{M}$ has functorial factorization

$$\varnothing \hookrightarrow QX \overset{q_X}{\twoheadrightarrow} X$$

then $Q : \mathcal{M} \to \mathcal{M}$ the cofibrant replacement functor together with $q : Q \xrightarrow{\sim} \mathrm{id}$ is a left deformation, and $\mathcal{M}_Q$ is the category of all cofibrant objects in $\mathcal{M}$.

Classical homological algebra Let A be an associative unital ring. Let $\mathcal{M} = \mathbf{Ch}^+(A)$ be the category of nonnegatively graded chain complexes of left A-modules. $\mathcal{M}$ is a model category with

- $\mathcal{WE}(\mathcal{M})$ = quasi-isomorphisms.
- $\mathsf{Fib}(\mathcal{M})$ = levelwise epimorphisms $f_n : M_n \twoheadrightarrow N_n$ for $n > 0$.
- $\mathsf{Cof}(\mathcal{M})$ = levelwise injective morphisms with projective cokernels.

$\mathcal{M}_Q$ consists of chain complexes with projective components.

Classical homotopy theory Let $\mathcal{M} = \mathbf{sGr}$ the category of simplicial groups.

Definition 4.11. A simplicial group Γ_* is **semifree** if the restriction

$$\begin{tikzcd} \Gamma_*|_{(\Delta^+)^{op}} : (\Delta^+)^{op} \arrow[rr] \arrow[dr] & & \mathbf{Gr} \\ & \mathbf{Set} \arrow[ur, "\mathbb{F}\langle - \rangle"'] & \end{tikzcd}$$

factors through **Set**. Explicitly, Γ_* is semifree if and only if there exists a sequence of subsets $B_n \subset \Gamma_n$ such that

1. $\Gamma_n = \mathbb{F}\langle B_n \rangle, n \geq 0$, and
2. $B = \cup_{n \geq 0} B_n$ is closed under degeneracy maps, i.e. $s_j^{\Gamma}(B_{n-1}) \subseteq B_n, n \geq 1, 0 \leq j \leq n$.

Theorem 4.12 (Kan)**.** *A* **cofibrant simplicial group** *is a retract of a semifree simplicial group.*

Kan Loop Group Construction is a functor

$$\mathbb{G} : \mathbf{sSet}_0 \to \mathbf{sGr}$$
$$X_* \longmapsto \mathbb{G}(X)_*$$

where

$$\mathbb{G}(X)_n = \mathbb{F}\langle X_{n+1} \rangle / (s_0 x = 1, \forall x \in X_n).$$

with degeneracy maps

$$s_j^{\mathbb{G}(X)} : \mathbb{G}(X)_n \to \mathbb{G}(X)_{n+1}$$

induced by $s_j^X : X_{n+1} \to X_{n+2}$ and face maps

$$d_i(x) = \begin{cases} d_0^X(x) d_1^X(x)^{-1}, & i = 0, \\ \text{induced by } d_1^X : X_{n+1} \to X_n, & i > 0 \end{cases}$$

Remark. Note we have

$$\begin{tikzcd} B_n \coloneqq X_{n+1} \backslash s_0(X_n) \arrow[r, hook] \arrow[dr] & X_{n+1} \arrow[r, hook] & (\mathbb{G}X)_n \\ & \mathbb{F}(B_n) \arrow[ur, "\cong"'] & \end{tikzcd}$$

which shows that $\mathbb{G}(X)$ is semifree with basis $B = \bigcup_{n\geq 0} B_n$.

Theorem 4.13. *The Kan loop group functor has a right adjoint*

$$\mathbb{G} : \mathbf{sSet}_0 \rightleftarrows \mathbf{sGr} : \overline{W}$$

where $\overline{W}$ is simplicial classifying space functor,

$$(\overline{W}\Gamma)_n = \Gamma_{n-1} \times \cdots \times \Gamma_0$$

with corresponding face and degeneracy maps. In particular, when we restrict it to **Gr**, *this is usual classifying space functor $\overline{W}|_{\mathbf{Gr}} = B_*$. This is a Quillen equivalence and we have*

$$\mathbb{G} : \mathrm{Ho}(\mathbf{sSet}_0) \simeq \mathrm{Ho}(\mathbf{sGr}) : \overline{W}$$

where both functors are homotopical.

Combining this with the geometric realization Quillen adjunction

$$|-| : \mathbf{sSet}_0 \rightleftarrows \mathbf{Top}_{0,*} : ES$$

we see that simplicial groups models homotopy types of pointed connected topological spaces.

In this case, $Q = \mathbb{G}\overline{W}$ and the counit $q : Q \Rightarrow \mathrm{id}$ is a left deformation of simplicial groups.

4.3 Properties of Derived Functors

Proposition 4.14. *If F is left deformable with respect to (Q, q), then F has a left derived functor $\mathbf{L}F : \mathcal{M} \to \mathrm{Ho}(\mathcal{N})$ given by*

$$\mathbf{L}F = \delta FQ : \mathcal{M} \xrightarrow{Q} \mathcal{M} \xrightarrow{F} \mathcal{N} \xrightarrow{\delta} \mathrm{Ho}(\mathcal{N})$$

with comparison map

$$\varepsilon = \delta Fq : \mathbf{L}F \Longrightarrow \delta F.$$

Proof. Recall the definition of left derived functors, we need to show two things.

1. It is easy to see that $\mathbf{L}F = \delta FQ$ is homotopical.

$$\begin{tikzcd} \mathbf{L}F = \delta FQ : \mathcal{M} \arrow[r, "Q"] \arrow[d, "Q"] & \mathcal{M} \arrow[r, "F"] & \mathcal{N} \arrow[r, "\delta"] & \mathrm{Ho}(\mathcal{N}) \\ \mathcal{M}_Q \arrow[ur, hook] \arrow[urr, "F|_{\mathcal{M}_Q}"'] & & & \end{tikzcd}$$

   Thus $\mathbf{L}F = \delta \circ F|_{\mathcal{M}_Q} \circ Q$ is homotopical as a composition of homotopical functors.

2. For the universal property, let $(G, \tilde{\varepsilon} : G \Rightarrow \delta F)$ be another pair. Observe that if G is homotopical and $q : Q \Rightarrow \mathrm{id}$ is a natural weak equivalence, then $Gq : GQ \Rightarrow G$ is an isomorphism of functors. Indeed,

$$Gq = \{GQX \xrightarrow{\cong} G(X), X \in \mathrm{Ob}(\mathcal{M})\}$$

   since weak equivalences in $\mathrm{Ho}(\mathcal{N})$ are isomorphisms. Thus it has an inverse

$$(Gq)^{-1} : G \xRightarrow{\cong} GQ$$

   then the composite

$$\tilde{\varepsilon} : G \xRightarrow{(Gq)^{-1}} GQ \xRightarrow{\tilde{\varepsilon}Q} \delta FQ = \mathbf{L}F \xRightarrow{\varepsilon} \delta F$$

   is equal to $\tilde{\varepsilon}$ since the diagram

$$\begin{tikzcd} G \arrow[d, Rightarrow, "\tilde{\varepsilon}"'] & GQ \arrow[l, Rightarrow, "Gq"', "\cong"] \arrow[d, Rightarrow, "\tilde{\varepsilon}Q"] \\ \delta F & \delta FQ = \mathbf{L}F \arrow[l, Rightarrow, "\varepsilon"] \end{tikzcd}$$

   commutes. Hence $\eta = (\tilde{\varepsilon}Q) \circ (Gq)^{-1}$ satisfies

$$\begin{tikzcd} G \arrow[rr, Rightarrow, "\tilde{\varepsilon}"] \arrow[dr, Rightarrow, "\eta"'] & & \delta F \\ & \mathbf{L}F \arrow[ur, Rightarrow, "\varepsilon"'] & \end{tikzcd}$$

   It remains to show the uniqueness of the factorization. Take another factorization (instead of η):

$$\begin{tikzcd} G \arrow[rr, Rightarrow, "\tilde{\varepsilon}"] \arrow[dr, Rightarrow, "\xi"'] & & \delta F \\ & \delta FQ = LF \arrow[ur, Rightarrow, "\varepsilon"'] & \end{tikzcd}$$

Let's precompose with Q

$$\begin{tikzcd} GQ \arrow[rr, Rightarrow, "\tilde{\varepsilon}Q"] \arrow[dr, Rightarrow, "\xi Q"'] & & \delta FQ \\ & \delta FQ^2 = (LF)Q \arrow[ur, Rightarrow, "\cong", "\varepsilon Q"'] & \end{tikzcd}$$

where

$$\begin{tikzcd} \xi Q : \mathcal{M} \arrow[r, "Q"] & \mathcal{M} \arrow[r, bend left, "G"] \arrow[r, bend right, "\delta F"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & \mathrm{Ho}(\mathcal{N}) \end{tikzcd}$$

Notice that εQ is an isomorphism of functors. Indeed, by exercise 4.10, $FqQ : FQ^2 \Rightarrow FQ$ is a natural weak equivalences, so

$$\varepsilon Q = \delta FqQ : \delta FQ^2 \Longrightarrow \delta FQ$$

is a natural isomorphism.

This means that ξQ is uniquely determined by $\tilde{\varepsilon}Q$, i.e. $\xi Q = (\varepsilon Q)^{-1} \circ \tilde{\varepsilon}Q$. But ξ is a natural transformation implies we have the following commutative diagram

$$\begin{tikzcd} GQ \arrow[r, Rightarrow, "\xi Q"] \arrow[d, Rightarrow, "Gq"', "\cong"] & \delta FQ^2 \arrow[d, Rightarrow, "\varepsilon Q", "\cong"'] \\ G \arrow[r, Rightarrow, "\xi"] & \delta FQ \end{tikzcd}$$

so $\xi = (\varepsilon Q) \circ (\xi Q) \circ (Gq)^{-1}$ is uniquely determined by ξQ thus by $\tilde{\varepsilon}Q$.

□

Remark. This argument shows that $LF = FQ : \mathcal{M} \to \mathcal{N}$ together with $Fq : LF \Rightarrow F$ is a pointwise derived functor.

Example 4.15. Let $\mathcal{M}$ be a model category, $\mathcal{W} = WE(\mathcal{M}) \subset \mathcal{M}$. Assume $\mathcal{M}$ has functorial factorization

$$\varnothing \hookrightarrow QX \xrightarrow{q_X} X$$

then the cofibrant replacement functor $Q : \mathcal{M} \to \mathcal{M}$ together with $q : Q \xrightarrow{\sim} \mathrm{id}$ is a left deformation, and $\mathcal{M}_Q$ is the category of all cofibrant objects in $\mathcal{M}$. It follows that any left Quillen functor $F : \mathcal{M} \to \mathcal{N}$ is left deformable with respect to (Q, q) and hence $\mathbf{L}F : \mathcal{M} \to \mathrm{Ho}(\mathcal{N})$ exists.

More generally, any functor $F : \mathcal{M} \to \mathcal{N}$ that maps acyclic cofibrations between cofibrant objects to weak equivalences is left deformable with respect to (Q, q).

$$F(\mathrm{Cof}(\mathcal{M}_Q) \cap WE(\mathcal{M}_Q)) \subseteq WE(\mathcal{N}). \tag{4.1}$$ {{eq:def-cond}}

Proof. Recall a left Quillen functor maps cofibrations to cofibrations and acyclic cofibrations to acyclic cofibrations, so it satisfies condition 4.1. We need to show the following lemma. ☐

Lemma 4.16 (Brown's lemma)**.** *If $F : \mathcal{M} \to \mathcal{N}$ is a functor between model categories satisfies condition 4.1, i.e. it maps acyclic cofibrations between cofibrant objects to weak equivalences, then $F|_{\mathcal{M}_Q}$ is homotopical.*

Proof. Take $A, B \in \mathrm{Ob}(\mathcal{M}_Q)$ and $f : A \xrightarrow{\sim} B$ a weak equivalence, we need to show that $F(f) \in \mathcal{WE}(\mathcal{N})$. Consider a factorization of $f \coprod \mathrm{id}_B = pq : A \coprod B \hookrightarrow C \overset{\sim}{\twoheadrightarrow} B$ and

$$\begin{tikzcd} A \arrow[d, hook, "i_A"] & & \\ A \coprod B \arrow[r, hook, "q"'] & C \arrow[r, two heads, "p", "\sim"'] & B \\ B \arrow[u, hook, "i_B"'] & & \end{tikzcd}$$

Observe that

- $qi_A : A \hookrightarrow A \coprod B \hookrightarrow C \in \mathrm{Cof}(\mathcal{M})$, and
- $qi_B : B \hookrightarrow A \coprod B \hookrightarrow C \in \mathrm{Cof}(\mathcal{M})$,

and

- $p : C \to B \in \mathcal{WE}(\mathcal{M})$, and
- $pqi_A = f : A \xrightarrow{\sim} B \in \mathcal{WE}(\mathcal{M})$, and
- $pqi_B = \mathrm{id}_B \in \mathcal{WE}(\mathcal{M})$,

so by 2-of-3 property,

$$qi_B, qi_A \in \mathrm{Cof}(\mathcal{M}) \cap \mathcal{WE}(\mathcal{M}).$$

Apply F to these maps, we see

$$F(qi_A), F(qi_B), F(pqi_B) = F(\mathrm{id}_B \in \mathcal{WE}(\mathcal{N}).$$

Again by 2-of-3 property, $F(p) \in \mathcal{WE}(\mathcal{N})$, so

$$F(f) = F(qi_A) \circ F(p) \in \mathcal{WE}(\mathcal{N}).$$

☐

Example 4.17 (Quillen homology)**.** Let $\mathcal{M}$ be a model category. Let $\mathcal{M}_{ab}$ be the category of abelian group objects, i.e. $A \in \mathrm{Ob}(\mathcal{M}_{ab})$ if and only if $\mathrm{Hom}_{\mathcal{M}}(B, A)$ is an abelian group for any $B \in \mathrm{Ob}(\mathcal{M})$, i.e. $h_A : \mathcal{M} \to$ **Set** factors through **Ab**. We have $i : \mathcal{M}_{ab} \to \mathcal{M}$ the forgetful functor (which is faithful but not full).

Example 4.18. For $\mathcal{M} =$ **sGr** or **sSet**, $\mathcal{M}_{ab} =$ **sAb**.

In many cases, $i : \mathcal{M}_{ab} \to \mathcal{M}$ has a left adjoint

$$(-)_{ab} : \mathcal{M} \rightleftarrows \mathcal{M}_{ab} : i. \tag{4.2}$$ {{eq:ab}}

the abelianization functor.

$\mathcal{M}_{ab}$ can be given a model structure so that 4.2 is a Quillen adjunction and we have the derived functor

$$\mathbf{L}(-)_{ab} : \mathcal{M} \longrightarrow \mathrm{Ho}(\mathcal{M}_{ab})$$

which is the Quillen homology.

We can also define Quillen cohomology as follows.

Assume $\mathcal{M}_{ab}$ has stable model structure so that $\varnothing = *$ and there is a (invertible) suspension functor

$$\Sigma : \mathrm{Ho}(\mathcal{M}_{ab}) \longrightarrow \mathrm{Ho}(\mathcal{M}_{ab})$$

Then we can define the Quillen cohomology of $\mathcal{M}$ with coefficients in $A \in \mathrm{Ho}(\mathcal{M}_{ab})$ as

$$H^{\mathcal{M}}_n(X, A) \coloneqq \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M}_{ab})}(\mathbb{L}(X)_{ab}, \Sigma^{-n}A).$$

Example 4.19. Let $\mathcal{M} =$ **sSet** and $\mathcal{M}_{ab} =$ **sAb**. We have

$$(-)_{ab} = \mathbb{Z}[-] : \mathbf{sSet} \rightleftarrows \mathbf{sAb} : i.$$

Note $\mathbb{L}(-)_{ab} \cong (-)_{ab} \cong \mathbb{Z}[-]$, so we have

$$\mathbb{L}_*(X)_{ab} = \pi_*\mathbb{Z}[X] \eqqcolon H_*(X, \mathbb{Z})$$

which is usual simplicial homology. In particular when X is the simplicial complex associated to a topological space, this is the usual singular homology of spaces.

Example 4.20. Let $\mathcal{M} =$ **sGr** and $\mathcal{M}_{ab} =$ **sAb**. We have

$$(-)_{ab} : \mathbf{sGr} \rightleftarrows \mathbf{sAb} : i.$$

where $(\gamma)_{ab} = \Gamma/[\Gamma, \Gamma]$. Then

$$\mathbb{L}(\Gamma)_{ab} \cong Q\Gamma/[Q\Gamma, Q\Gamma]$$

where $Q = \mathbb{G}\overline{W}$ is the semifree deformation. Then

$$\pi_* \mathbb{L}(\Gamma)_{ab} = \pi_*(Q\Gamma)_{ab} \cong H_{*+1}(B\Gamma, \mathbb{Z}) = H_{*+1}(\Gamma, \mathbb{Z})$$

is the group homology.

Example 4.21. Assume k is a field of characteristic 0. Let $\mathcal{M} = \mathbf{DGCA}_k$ be the category of commutative DG algebras over k. Let $A \in \mathcal{M}_{ab}$, then $\mathrm{Hom}_{\mathcal{M}}(B, A) \in \mathbf{Ab}$ implies $A = \{0\}$, so there is no nontrivial abelian group objects in this category. We will fix this by considering the follwoing category.

Fix $A \in \mathbf{DGCA}_k$ and consider

$$\mathcal{M} \downarrow A = \mathbf{DGCA}_k / A$$

Then [14]

$$(\mathcal{M} \downarrow A)_{ab} \cong \mathbf{DGMod}(A)$$

and we have the following adjunction

$$\Omega^1 : (-/A) : \mathbf{DGCA}_k / A \rightleftarrows \mathbf{DGMod}(A) : A \ltimes (-)$$

where

$$A \ltimes M = A \oplus M$$

is the semidirect product (square-zero extension), with

$$(a, m)(b, n) = (ab, an + mb).$$

The left adjoint is the relative Kähler differential

$$\Omega^1(B/A) \coloneqq A \otimes_B \Omega^1(B)$$

and its left derived functor is

$$\mathbf{L}\Omega^1(-/A) : \mathbf{DGCA}_k / A \longrightarrow \mathcal{D}(\mathbf{DGMod}(A))$$

defined by $\mathbf{L}\Omega^1(B/A) = A \otimes_{QB} \Omega^1(QB)$.

Take $f = \mathrm{id}_A : A \to A$,

$$\mathbb{L}_{k\backslash A} \coloneqq \mathbf{L}\Omega^1(\mathrm{id}_A)$$

is the cotangent complex of A. And its associated homology is André-Quillen homology

$$H_*^{AQ}(A) = H_*(\mathbb{L}_{k\backslash A})$$

and André-Quillen homology of A with coefficients in M

$$H_*^{AQ}(A, M) = H_*(\mathbb{L}_{k\backslash A} \bigotimes_A M).$$

[14] When A is an associative algebra, we need to replace modules by bimodules, then this recovers Hochschild homology.

Example 4.22 (Lie-Hodge Homology)**.** Assume k is a field with $\mathrm{char}(k) = 0$, Let $\mathbf{DGLA}_k$ [15] be the category of DG Lie algebras, where objects are $\mathfrak{a} = \oplus_{i \in \mathbb{Z}} \mathfrak{a}_i \in \mathbf{DGLA}_k$ equipped with a DG Lie bracket

$$[-,-] : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$$

satisfying commutativity

$$[x, y] = (-1)^{|x||y|}[y, x]$$

and Jacobi identity

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]].$$

This category is equipped with a projective model structure, with

- WE = quasi-isomorphisms, and
- Fib = degreewise surjective maps.

In this case, the abelianization $(\mathbf{DGLA}_k)_{ab} \cong \mathbf{Ch}_k$ and we have the adjunction

$$(-)_{ab} : \mathbf{DGLA}_k \rightleftarrows \mathbf{Ch}_k : i$$

where the left adjoint is given by

$$\mathfrak{a}_{ab} = \mathfrak{a}/[\mathfrak{a}, \mathfrak{a}].$$

Then the left derived functor is given by

$$\mathbf{L}(\mathfrak{a})_{ab} = Q\mathfrak{a}/[Q\mathfrak{a}, Q\mathfrak{a}]$$

and Quillen showed

$$H_*(\mathbf{L}\mathfrak{a}_{ab}) \cong H^{CE}_{*+1}(\mathfrak{a}, k).$$

where H^{CE}_* is the Chevalley-Eilenberg homology of DG Lie algebras,

Recall if $\mathfrak{a}$ is a Lie algebra, an adjoint-invariant symmetric bilinear form on $\mathfrak{a}$ is

$$\begin{tikzcd} \langle -,- \rangle : \mathfrak{a} \times \mathfrak{a} \arrow[rr] \arrow[dr] & & V \\ & \mathrm{Sym}^2(\mathfrak{a}) \arrow[ur] & \end{tikzcd}$$

which factors through $\mathrm{Sym}^2(\mathfrak{a})$, i.e $\langle x, y \rangle = \langle y, x \rangle$, and satisfies

$$\langle [x, y], z \rangle = \langle x, [y, z] \rangle.$$

[15]We can replace $\mathbf{DGLA}_k$ by $\mathbf{sLA}$ when $\mathrm{char}(k) \neq 0$, but there are some interesting properties that requires this assumption.

Example 4.23. Let $\mathfrak{a}$ be a semisimple finite dimensional complex Lie algebra, the killing form on $\mathfrak{a}$ is an adjoint-invariant symmetric bilinear form.

Among all such forms, there is a universal one

$$\begin{tikzcd} \mathfrak{a}\times\mathfrak{a} \arrow[rr] \arrow[dr] & & V \\ & \lambda^{(2)}(\mathfrak{a}) \arrow[ur] & \end{tikzcd}$$

where $\lambda^{(2)} : \mathrm{LieAlg}_k \to \mathbf{Vect}_k$ is given by

$$\lambda^2(\mathfrak{a}) = \frac{\mathrm{Sym}^2(\mathfrak{a})}{\langle [x,y]z - x[y,z]\rangle}$$

which is introduced by Drinfeld (1991), and is used by Ginzburg and Gan (2003).

A natural question that we can ask is

Question 4.24. *What's cyclic homology of Lie algebras?*

The generalization we will discuss goes back to Getzler and Kapranov (following Kontsevich, 1998), which works for algebras over any cyclic operad.

Theorem 4.25 (Feigin-Tsygan)**.** *The functor*

$$(-)_\# : \mathbf{DGA} \longrightarrow \mathbf{Ch}_k$$

$$A \longmapsto A/(k\cdot 1_A + [A,A])$$

is left deformable (not a Quillen functor), which implies it has a left derived functor and

$$H_*(\mathbf{L}(A)_\# \cong \overline{HC}_*(A).$$

This is one application of cyclic homology by viewing it as homology of derived functor of abelianization. If we look at the expression, this is the derived functor of universal trace (vanishes on commutators).

In the case of $\mathbf{DGA}_k$, we have (by accident) this coincidence that (reduced) cyclic homology agrees with homology of derived functor of abelianization, but for DG Lie algebras, the cyclic homology is defined as the homology of derived functor of $\lambda^{(2)}$,

$$HC^{\mathrm{Lie}}(\mathfrak{a}) = H_*(\mathbb{L}\lambda^{(2)}(\mathfrak{a})).$$

Exercise 4.26. Show that the derived functor exists.

Now instead of considering only 2-forms, let's consider the universal ad-invariant symmetric p-multilinear form (coinvariants of the adjoint representation of $\mathfrak{a}$ in $\mathrm{Sym}^p(\mathfrak{a})$)

$$\mathfrak{a}\times\cdots\times\mathfrak{a} \longrightarrow \lambda^{(p)}(\mathfrak{a}) = \frac{\mathrm{Sym}^p(\mathfrak{a})}{[\mathfrak{a},\mathrm{Sym}^p(\mathfrak{a})]}$$

For every $p \geq 1$, we have

$$\lambda^{(p)} : \mathbf{DGLA}_k \longrightarrow \mathbf{Ch}_k$$
$$\mathfrak{a} \longmapsto \lambda^{(p)}(\mathfrak{a})$$

When $p = 1$, $\lambda^{(1)}(\mathfrak{a}) = \mathfrak{a}_{ab}$.

Remark. When $\dim_k(\mathfrak{a}) < \infty$, the natural pairing

$$\begin{tikzcd} \mathrm{Sym}^p(\mathfrak{a}) \times \mathrm{Sym}^p(\mathfrak{a}^*) \arrow[rr] \arrow[dr] & & k \\ & \lambda^{(p)}(\mathfrak{a}) \times \mathrm{Sym}^p(\mathfrak{a}^*)^{ad} \arrow[ur] & \end{tikzcd}$$

is nondegenerate (here $\mathrm{Sym}^p(\mathfrak{a}^*)^{ad} = k[\mathfrak{a}]^{ad}$), so we have

$$\lambda^{(p)}(\mathfrak{a}) = \mathrm{Hom}_k(\mathrm{Sym}^p(\mathfrak{a}^*)^{ad}, k).$$

Theorem 4.27 ([BFP+]). *λ^p is left deformable, thus $\mathbf{L}\lambda^{(p)}$ exists and has nice properties, and we call*

$$HC^{(p)}(\mathfrak{a}) = H_*(\mathbf{L}\lambda^{(p)}(\mathfrak{a}))$$

Lie-Hodge homology of $\mathfrak{a}$. In particular,

$$\begin{aligned} HC^{(1)}(\mathfrak{a}) &= H^{CE}_{*+1}(\mathfrak{a}, k) \\ HC^{(2)}(\mathfrak{a}) &= HC_*(\mathfrak{a}). \end{aligned}$$

Proposition 4.28. *For any DG Lie algebra $\mathfrak{a}$, there is a natural direct sum decomposition*

$$\overline{HC}_*(U\mathfrak{a}) \cong \bigoplus_{p \geq 1} \overline{HC}^{(p)}(\mathfrak{a})$$

called Lie-Hodge decomposition.

Remark (Loday,1988). For a commutative DG algebra A, there is a similar decomposition

$$HC_*(A) = \bigoplus_{p \geq 0} HC^{(p)}(A).$$

But these two decompositions are very different.

Sketch of proof. Recall for any $\mathfrak{a} \in \mathbf{DGLA}_k$, by the PBW theorem

$$\bigoplus_{p \geq 0} \mathrm{Sym}^p(\mathfrak{a}) \cong \mathcal{U}\mathfrak{a}$$

as $\mathfrak{a}$-modules. Thus we have

$$\mathrm{Sym}^p(\mathfrak{a}) \hookrightarrow \mathcal{U}\mathfrak{a}$$
$$x_1 \cdots x_p \mapsto \frac{1}{p!} \sum_{\sigma \in \Sigma_p} \pm x_{\sigma(1)} \cdots x_{\sigma(p)}$$

which induces

$$\lambda^{(p)}(\mathfrak{a}) \hookrightarrow (\mathcal{U}\mathfrak{a})_{\#}$$

and a natural transformation of functors

$$\mathbf{L}\lambda^{(p)}(\mathfrak{a}) \hookrightarrow \mathbf{L}(\mathcal{U}\mathfrak{a})_{\#}$$

which induces the decomposition

$$\bigoplus_{p\geq 0} HC^{(p)}(\mathfrak{a}) \to \overline{HC}(\mathcal{U}\mathfrak{a})$$

□

Topological interpretation (rational homotopy theory [Q1]) Let X be a simply-connected space, X is called **rational** if $\pi_n(X,x)$ is a $\mathbb{Q}$-vector space for $\forall n \geq 2$. Then

$$f : X \xrightarrow{\sim} X_{\mathbb{Q}}$$

is called the **rationalization** (BK localization at 0), and it induces a map

$$\begin{tikzcd} f_* : \pi_*(X) \arrow[rr] \arrow[dr] & & \pi_*(X_{\mathbb{Q}}) \\ & \pi_*(X) \otimes \mathbb{Q} \arrow[ur, "\cong"'] & \end{tikzcd}$$

Theorem 4.29 (Quillen)**.** *There is an equivalence of homotopy categories*

$$\mathrm{Ho}(\mathbf{Top}^{\mathbb{Q}}_{0,*}) \overset{\cong}{\to} \mathrm{Ho}(\mathbf{DGLA}^{0}_{K})$$

$$X \longmapsto \mathfrak{a}_X$$

where the left hand side corresponds to the homotopy types of rational simply connected spaces, and the right hand side corresponds to the homotopy types of rational non-negatively (homological) graded DL Lie algebras.

Theorem 4.30 (Goodwille-Jones, ...)**.** *Let X be a simply connected space of finite $\mathbb{Q}$-type ($\dim_{\mathbb{Q}} H_*(X,\mathbb{Q}) < \infty$), let $\mathcal{L}X = \mathrm{Map}(S^1, X)$ be the free loop space of X. Notice that there is a natural S^1-action on $\mathcal{L}X$, then for $\overline{H}^{S^1}_*(\mathcal{L}X,\mathbb{Q})$ the reduced S^1-equivariant homology with coefficients in $\mathbb{Q}$, there is a natural isomorphism*

$$\alpha : \overline{H}^{S^1}_*(\mathcal{L}X,\mathbb{Q}) \cong \overline{HC}_*(\mathcal{U}\mathfrak{a}_X).$$

Consider finite coverings

$$\begin{aligned} \varphi_n : S^1 &\to S^1, \qquad n \geq 0 \\ e^{i\varphi} &\longmapsto e^{in\varphi} \end{aligned}$$

which induces Frobenius operations

$$\Phi_n : \overline{H}_*^{S^1}(\mathcal{L}X, \mathbb{Q}) \longrightarrow \overline{H}_*^{S^1}(\mathcal{L}X, \mathbb{Q}).$$

Theorem 4.31 (Bugheler, Gajda, Frobenius)**.** *For each* $p \geq 0$*, define*

$$\overline{H}_*^{S^1,(p)}(\mathcal{L}X, \mathbb{Q}) = \bigcap_{n \geq 0} \ker(\Phi_n - n^p \mathrm{id})$$

then we have another decomposition

$$\overline{H}_*^{S^1}(\mathcal{L}X, \mathbb{Q}) \cong \bigoplus_{p \geq 0} \overline{H}_*^{S^1,(p)}(\mathcal{L}X, \mathbb{Q})$$

and the Goodwille isomorphisms restricts to isomorphisms

$$\overline{H}_*^{S^1,(p)}(\mathcal{L}X, \mathbb{Q}) \cong HC_*^{(p+1)}(\mathfrak{a}_X)$$

String Topology [CS] Let X be a close smooth manifold. There is a (geometrically defined) Lie bracket

$$\overline{H}_*^{S^1}(\mathcal{L}X, \mathbb{Q}) \times \overline{H}_*^{S^1}(\mathcal{L}X, \mathbb{Q}) \longrightarrow \overline{H}_*^{S^1}(\mathcal{L}X, \mathbb{Q}).$$

Question 4.32. *Does the Chas-Sullivan bracket preserves the Hodge decomposition?*

Theorem 4.33. *Suppose* X *is rationally diptic space (i.e.* $\dim_{\mathbb{Q}}(\bigoplus_{n \geq 2} \pi_n(X) \otimes \mathbb{Q}) < \infty$*), then CS bracket is compatible with the Hodge decomposition.*

4.3.1 Derived Adjunctions

Perhaps the most important and useful application of model categories are

Question 4.34. *Given a functor* $F : \mathcal{M} \to \mathcal{N}$*, when does its derived functor* $\mathbb{L}F$ *or* $\mathbb{R}F$ *induces*

$$\mathrm{Ho}(\mathcal{M}) \simeq \mathrm{Ho}(\mathcal{N})?$$

Quillen divided this into two questions.

Question 4.35. *Given a pair of adjoint functors*

$$F : \mathcal{M} \rightleftarrows \mathcal{N} : G$$

when does (F, G) *induce an adjunction*

$$\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{N}) : \mathbb{R}G \tag{4.3}$$ {{eq:Quillen-adj}}

The answer to this question is Quillen's adjunction theorem.

Question 4.36. *If 4.3 holds, when does* $(\mathbb{L}F, \mathbb{R}G)$ *gives equivalences?*

The answer to this question is Quillen's equivalence theorem.

Recently, Maltsimistis [M1] [16] realized that the universal properties of left/right Kan extensions are not enough, and we need stronger notions, namely absolute derived functors.

More on Kan Extensions Let's consider the universal property of Kan extensions. Given two functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{D} \\ \mathcal{E} \arrow[ur, dashed] & \end{tikzcd}$$

The existence of left/right Kan extensions amounts to the existence of the following left/right adjunctions

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{E}, \mathcal{D}) \arrow[d, "G^*"] \\ \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \arrow[u, bend left=50, "L_G(F)"] \arrow[u, bend right=50, "R_G(F)"'] \end{tikzcd}$$

There are natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(H \circ G, F) &\cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{E},\mathcal{D})}(H, R_G F), \\ \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, H \circ G) &\cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{E},\mathcal{D})}(L_G F, H) \end{aligned}$$

which describes exactly the universal properties of left/right Kan extensions.

Definition 4.37. A right Kan extension $R_G(F)$ is **pointwise** if for any $e \in \mathrm{Ob}(\mathcal{E})$,

$$R_G(F)(e) \cong \lim(e \downarrow G \xrightarrow{u} \mathcal{C} \xrightarrow{F} \mathcal{D}).$$

[16] Deligne [D1] noticed this in the context of triangulated categories, namely derived categories of abelian categories.

Remark. When $\mathcal{D}$ is complete, every right Kan extension is pointwise.

In homotopical algebra,

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{N} \arrow[r, "\delta"] & \mathrm{Ho}(\mathcal{N}) \\ \mathrm{Ho}(\mathcal{M}) \arrow[urr, "\mathbb{L}F = R_\gamma(\delta F)"'] & & \end{tikzcd}$$

$\mathrm{Ho}(\mathcal{N})$ is neither complete nor cocomplete.

Proposition 4.38 (MacLane)**.** *A right Kan extension is pointwise if and only if it is preserved by all representatble functors*

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, "G"] & \mathcal{D} \arrow[r, "h^d"] & \mathbf{Set} \\ \mathcal{E} \arrow[ur, "R_G(F)"'] \arrow[urr, bend right, "R_G(h^d \circ F)"'] & & \end{tikzcd}$$

where $h^d = \mathrm{Hom}_{\mathcal{D}}(d, -) : \mathcal{D} \to \mathbf{Set}$ *for any* $d \in \mathcal{D}$*. In other words,*

$$R_G(h^d \circ F) \cong h^d \circ R_G(F).$$

Proof. It is clear that when the right Kan extension is pointwise, it commutes with h^d (since h^d commutes with all limits). On the other direction, we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}}(d, R_G(F)(e)) &= h^d(R_G(F)(e)) \\ &= h^d \circ R_G(F)(e) \\ &\cong R_G(h^d \circ F)(e) \\ &\cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{E},\mathbf{Set})}(h^e, R_G(h^d \circ F)) \\ &\cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathbf{Set})}(h^e \circ G, h^d \circ F) \\ &\cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathbf{Set})}(\mathrm{Hom}(e, G(-)), \mathrm{Hom}(d, F(-)) \\ &\cong \{\text{cones under } d \text{ of the functor } FU : e \downarrow G \xrightarrow{FU} \mathbf{Set}\} \end{aligned}$$

Consider the diagram of $e \downarrow G$, it is clear that $R_G(F)(e)$ represents $\varprojlim_{e\downarrow G}(FU)$. $\square$

Remark. If $\mathbf{L}F$ is not pointwise, then it is usually illy behaved.

However, for us pointwise Kan extension is not enough, and we will introduce the notion of absolute Kan extensions.

Definition 4.39. Let $F : \mathcal{M} \to \mathcal{N}$ be a functor between homotopical categories. The (total) left derived functor $\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$ is called absolute if the corresponding right Kan extension is absolute in the sense that it is preserved by any functors

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{N} \arrow[r, "\delta"] & \mathrm{Ho}(\mathcal{N}) \arrow[r] & \mathcal{E} \\
\mathrm{Ho}(\mathcal{M}) \arrow[urr, "\mathbb{L}F = R_\gamma(\delta F)"'] \arrow[urrr, bend right, "R_\gamma(H\delta F)"'] & & &
\end{tikzcd}$$

for any $H : \mathrm{Ho}(\mathcal{N}) \to \mathcal{E}$, $R_G(H\delta F) \cong H\delta R_G(F)$.

Proposition 4.40 ([M1])**.** *The left derived functor $\mathbb{L}F$ of any left deformable functor is absolute.*

Corollary 4.41. *The left derived functor $\mathbb{L}F$ of any left Quillen functor $F : \mathcal{M} \to \mathcal{N}$ between model categories is absolute.*

Quillen Adjunction

Theorem 4.42 ([M1], [D1])**.** [17] *Given an adjoint pair*

$$F : \mathcal{M} \rightleftarrows \mathcal{N} : G$$

such that $\mathbb{L}F$ and $\mathbb{R}G$ exists, then there is a unique adjunction

$$\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{M}) : \mathbb{R}G$$

compatible with localization in the sense

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{N}}(FX, Y) \arrow[r, "\cong"] \arrow[d, "\delta"] & \mathrm{Hom}_{\mathcal{M}}(X, GY) \arrow[d, "\gamma"] \\
\mathrm{Hom}_{\mathrm{Ho}(\mathcal{N})}(\delta F(X), \delta Y) \arrow[d, "\varepsilon_X^*"] & \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(\gamma X, \gamma GY) \arrow[d, "\eta_Y^*"] \\
\mathrm{Hom}_{\mathrm{Ho}(\mathcal{N})}(\mathbb{L}F(\gamma X), \delta Y) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(\gamma X, \mathbb{R}G(\delta Y))
\end{tikzcd}$$

commutes, where $\varepsilon : \mathbb{L}F\gamma \Rightarrow \delta F$ and $\eta : \gamma G \Rightarrow \mathbb{R}G\delta$ are the corresponding natural transformations equipped in the Kan extension constructions.

Proof. See notes. □

Corollary 4.43 (Quillen Adjunction Theorem)**.** *If*

$$F : \mathcal{M} \rightleftarrows \mathcal{M} : G$$

is a Quillen pair, then $\mathbb{L}F$ and $\mathbb{R}G$ exist and we have an adjunction

$$\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{M}) : \mathbb{R}G.$$

[17] See Keller's survey on derived categoreis.

DHKS Adjunction Theorem

Definition 4.44. An adjunction

$$\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{M}) : \mathbb{R}G$$

is called **deformable** if F is left deformable and G is right deformable.

{thm:DHKS-adjunction}

Corollary 4.45 ([DHKS] Adjunction Theorem, Section 42)**.** *For any deformable adjunction*

$$\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{M}) : \mathbb{R}G$$

$\mathbb{L}F$ *and* $\mathbb{R}G$ *exist and we have an adjunction*

$$\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{M}) : \mathbb{R}G.$$

Quillen Equivalence

Definition 4.46. Let

$$\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{M}) : \mathbb{R}G$$

be a Quillen pair between model catgories. We say (F, G) is a **Quillen equivalence** if for any $A \in \mathrm{Ob}(\mathcal{M}^{cof})$ and $X \in \mathrm{Ob}(\mathcal{N}^{fib})$, $f : A \xrightarrow{\sim} GX$ is weak equivalence in $\mathcal{M}$ if and only if $f^\# : FA \to X$ is a weak equivalence in $\mathcal{N}$.

Theorem 4.47 (Quillen Equivalence Theorem)**.** *If* (F, G) *is a Quillen equivalence, then*

$$\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{M}) : \mathbb{R}G$$

is an equivalence of categories.

Proof. It suffices to show the unit and counit are natural isomorphisms.

1. Consider the unit $\eta : \mathrm{id}_{\mathrm{Ho}(\mathcal{M})} \Rightarrow \mathbb{R}G \circ \mathbb{L}F$.

   Take $A \in \mathrm{Ob}(\mathcal{M}^{cof})$ and apply F to fibrant replacement $r_{F(A)} : F(A) \xrightarrow{\sim} \mathcal{L}F(A)$, which is a weak equivalence in $\mathcal{N}$ if and only if $f^\# : A \xrightarrow{\sim} G(\mathcal{L}F(A))$ is a weak equivalence in $\mathcal{M}$, if and only if $\gamma r^\#_{F(A)} : rA \xrightarrow{\cong} \gamma G(\mathcal{L}F(A)) = \mathbb{R}G(\mathbb{L}F(\gamma A))$ is an isomorphisms in $\mathrm{Ho}(\mathcal{M})$. For any $\overline{B} \in \mathrm{Ob}(\mathrm{Ho}(\mathcal{M}))$, there is an isomorphism $\overline{B} \xrightarrow{\cong} \gamma A$ where $A \in \mathrm{Ob}(\mathcal{M}^{cof})$, so we have

$$\begin{tikzcd} \overline{B} \arrow[r, "\eta_B"] \arrow[d, "\cong"'] & \mathbb{R}G\mathbb{L}F(\overline{B}) \arrow[d, "\cong"] \\ \gamma A \arrow[r, "\eta_{\gamma A}", "\cong"'] & \mathbb{R}G\mathbb{L}F(\gamma A) \end{tikzcd}$$

   thus η_B is also an isomorphism.

2. The proof for the counit $\varepsilon : \mathbb{L}F \circ \mathbb{R}G \Rightarrow \mathrm{id}_{\mathrm{Ho}(\mathcal{N})}$ is dual.

□

Nonexample In practice, many interesting adjunctions are not deformable, so Theorem 4.45 does not apply.

Let G be an affine algebraic group scheme over a field k. Take (for instance) $G = GL_n, n \geq 1$, the functor

$$\begin{aligned} GL_n : \mathbf{Comm}_k &\longrightarrow \mathbf{Gr} \\ A &\longmapsto GL_n(A) \end{aligned}$$

has a left adjoint

$$(-)_G : \mathbf{Gr} \rightleftarrows \mathbf{Comm}_k : G(-)$$

$$\Gamma \longmapsto (\Gamma)_G = \mathcal{O}(\mathrm{Rep}_G(\Gamma))$$

$$G(A) \longleftarrow\!\shortmid A$$

which extends levelwise to a simplicial adjunction

$$(-)_G : \mathbf{sGr} \rightleftarrows \mathbf{sComm}_k : G(-)$$

Notice there are natural projective model structures on **sGr** and $\mathbf{sComm}_k$ but the adjunction is not a Quillen pair.

For instance, when $n = 1$, $G = GL_1 = \mathbb{G}_m$, we have

$$(-)_{\mathbb{G}_m} : \mathbf{Gr} \rightleftarrows \mathbf{Comm}_k : \mathbb{G}_m(-)$$

$$A^\times \longleftarrow\!\shortmid A$$

$$\mathbb{F}_1 \longmapsto k[\mathbb{F}_1] = k\left[x, x^{-1}\right]$$

where $\mathbb{F}_1$ is cofibrant in **sGr** but $k[x, x^{-1}]$ is not cofibrant in $\mathbf{sComm}_k$.

Claim 4.48 ([BFP$^+$]). *This pair is not Quillen, nor deformable as adjunction between homotopical categories. Nevertheless, there exists derived adjunction*

$$\mathbb{L}(-)_G : \mathrm{Ho}(\mathbf{sGr}) \rightleftarrows \mathrm{Ho}(\mathbf{sComm}_k) : \mathbb{R}G$$

and are absolute.

Model Approximations [CCS]

Definition 4.49. A **left model approximation** of a homotopical (or more generally, a relative) category $\mathcal{C}$ is a model category $\mathcal{M}$ given together with a pair of adjoint functor

$$l : \mathcal{M} \rightleftarrows \mathcal{C} : r$$

satisfies

1. r is homotopical, i.e. $r(\mathcal{W}) \subseteq WE(\mathcal{M})$.
2. l is homotopical on cofibrant objects in $\mathcal{M}$.
3. (l, r) is almost Quillen equivalence: for any $A \in Ob(\mathcal{C})$ and $X \in Ob(\mathcal{M}^{cof})$,

$$(X \xrightarrow{f} r(A) \in WE(\mathcal{M}) \iff (l(X) \xrightarrow{f^\#} A) \in WE(\mathcal{C}) \tag{4.4}$$ {{eq:model-approximation}}

Example 4.50 (Prototypical)**.** If $l : \mathcal{M} \rightleftarrows \mathcal{N} : r$ is a Quillen equivalence with r being homotopical, then $\mathcal{M}$ is a model approximation of $\mathcal{N}$.

Remark. By definition, Quillen equivalence $l : \mathcal{M} \rightleftarrows \mathcal{N} : r$ is characterized by the following fact: for any fibrant object A in $\mathcal{N}$ and cofibrant object X in $\mathcal{M}$,

$$(X \xrightarrow{f} r(A) \in WE(\mathcal{M}) \quad \iff \quad (l(X) \xrightarrow{f^\#} A) \in WE(\mathcal{N})$$

and we have derived adjunction

$$\mathbb{L}l : Ho(\mathcal{M}) \rightleftarrows Ho(\mathcal{N}) : \mathbb{R}r$$

and 4.4 implies both unit and counit

$$\eta : id \xrightarrow{\sim} \mathbb{R}r\mathbb{L}l \tag{4.5}$$ {{eq:cond1}}

$$\varepsilon : \mathbb{L}l\mathbb{R}r \xrightarrow{\sim} id \tag{4.6}$$ {{eq:cond2}}

are isomorphisms.

When r is homotopical, $\mathbb{R}r = \bar{r}$. In this care, condition 4.6 follows from condition 4.4 without assuming A being fibrant.

Moral From homotopical point of view, being a model category or having a model approximation should not have much difference.

{lem:model-approx-adjunctio

Proposition 4.51. *Let* $l : \mathcal{M} \rightleftarrows \mathcal{C} : r$ *be a left model approximation of a homotopical category, then* (l, r) *induces derived adjunction*

$$\mathbb{L}l : Ho(\mathcal{M}) \rightleftarrows Ho(\mathcal{C}) : \bar{r}$$

with counit $\varepsilon : \mathbb{L}l \circ \bar{r} \xrightarrow{\sim} id_{Ho(\mathcal{C})}$ *being a natural isomorphism so that* $\bar{r} : Ho(\mathcal{C}) \hookrightarrow Ho(\mathcal{M})$ *is fully faithful.*

Proof. By definition of left model approximation, (l, r) is deformable, so $(\mathbb{L}l, \bar{r})$ exists and are adjoint by DHKS theorem. By formal properties, we have

Lemma 4.52. *The counit* $\varepsilon : FG \to id$ *of an adjunction*

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

is an isomorphism if and only if G *is fully faithful.*

Proof. Observe for any $d, d' \in \mathrm{Ob}(\mathcal{D})$ and any $f : d \to d'$ in $\mathcal{D}$, commutativity of

$$\begin{tikzcd}
FG(d) \arrow[r, "\varepsilon_d"] \arrow[d, "FG(f)"] & d \arrow[d, "f"] \\
FG(d') \arrow[r, "\varepsilon_{d'}"] & d'
\end{tikzcd}$$

is equivalent to the commutativity of the following diagram

$$\begin{tikzcd}
\mathrm{Hom}(d, d') \arrow[r, "G"] \arrow[d, "h_{d'}(\varepsilon_d)"'] & \mathrm{Hom}_{\mathcal{C}}(G(d), G(d')) \arrow[d, "F"] \arrow[dl, "\Psi_{Gd,Gd'}"', "\cong"] \\
\mathrm{Hom}(FG(d), d') & \mathrm{Hom}_{\mathcal{D}}(FG(d), FG(d')) \arrow[l, "h^{FG(d)}(\varepsilon_{d'})"]
\end{tikzcd}$$

Notice $h^{FG(d)}(\varepsilon_{d'}) \circ F$ is the adjunction isomorphism associated to ε. Thus G is a bijection if and only if

$$h_{d'}(\varepsilon_d) : \mathrm{Hom}(d, d') \longrightarrow \mathrm{Hom}(FG(d), d')$$

is an isomorphism, if and only if ε_d is an isomorphism. $\square$

To prove proposition, it suffices to show that $\overline{r} : \mathrm{Ho}(\mathcal{C}) \hookrightarrow \mathrm{Ho}(\mathcal{M})$ is fully faithful, i.e for any X, Y in $\mathcal{C}$, write $\overline{X} = \gamma(X)$ and $\overline{Y} = \gamma Y$ in $\mathrm{Ho}(\mathcal{C})$,

$$\overline{r}_{\overline{X},\overline{Y}} : \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(\overline{X}, \overline{Y}) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(\overline{r(X)}, \overline{r(Y)})$$

We construct explicitly the inverse map.

Let's fix $Q, R : \mathcal{M} \to \mathcal{M}$ be the cofibrant and fibrant replacement functor together with $q : Q \Rightarrow \mathrm{id}_{\mathcal{M}}$ amd $s : \mathrm{id}_{\mathcal{M}} \Rightarrow R$ the corresponding natural weak equivalences. Given any $\overline{f} : \overline{r(X)} \to \overline{r(Y)}$, we can represent it by $f : Qr(X) \to RQr(Y)$. Then we can form zigzags of maps

$$X \xleftarrow[\simeq]{q^{\#}_{r(X)}} lQr(X) \xrightarrow{l(f)} lRQr(Y) \xleftarrow[\simeq]{ls_{r(Y)}} lQr(Y) \xrightarrow[\simeq]{q_{r(Y)}} Y$$

where $q^{\#}_{r(X)} : lQr(X) \to X$ is a weak equivalence in $\mathcal{C}$ since $q_{r(X)} : Qr(X) \to r(X)$ is a weak equivalence in $\mathcal{M}$, and similarly every arrow except for $l(f)$ are weak equivalences. Apply $\mathcal{C} \to \mathrm{Ho}(\mathcal{C})$ and we get a map $\overline{\psi f} : \overline{X} \to \overline{Y}$ by inverting all isomorphisms, and $\overline{f} \mapsto \overline{\psi f}$ is the inverse of $\overline{r}$.

Check this. $\square$

Definition 4.53. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between homotopical categories. A left deformation $l : \mathcal{M} \rightleftarrows \mathcal{C} : r$ is called **good** for F is

$$Fl : \mathcal{M} \longrightarrow \mathcal{C} \longrightarrow \mathcal{D}$$

is homotopical on cofibrant objects.

Lemma 4.54. *If (l,r) is a good for* F*, then* F *is left deformable, hence* $\mathbb{L}F$ *exists and is absolute.*

Example 4.55 ([CCS], Bousfield-Kan model approximation)**.** Given a model category $\mathcal{M}$ and any small category $\mathcal{I}$, the functor category $\mathcal{M}^{\mathcal{I}} = \mathrm{Fun}(\mathcal{I}, \mathcal{M})$ does not have, in general a model structure. But it has a natural class of weak equivalences, namely the class of objectwise weak equivalences, and we can treat it as a homotopical category. There is a "universal" model approximation of $\mathcal{M}^{\mathcal{I}}$ described as follows.

Take $\Delta\mathcal{I}$ the simplex category of $\mathcal{I}$,

$$\Delta\mathcal{I} = \Delta \downarrow \mathcal{N}_*(\mathcal{I})$$

with

- objects $\sigma \in \bigcup_{n\geq 0} \mathrm{Hom}(\Delta[n]_*, \mathcal{N}_*(\mathcal{I})) = \bigcup_{n\geq 0} \mathcal{N}_n(\mathcal{I})$, and
- morphisms are

  $$\mathrm{Hom}_{\Delta\mathcal{I}}(\sigma', \sigma) = \{f : \Delta[n]_* \to \Delta[m]_* | \sigma' = \sigma f : \Delta[n]_* \xrightarrow{f} \Delta[m]_* \xrightarrow{\sigma} \mathcal{N}_*(\mathcal{I})\}.$$

  We denote such maps by $\tilde{f} : \sigma f \to \sigma$.

Consider the full subcategory $\mathcal{M}_b^{\Delta\mathcal{I}} \subseteq \mathcal{M}^{\Delta\mathcal{I}}$ of bounded $\Delta\mathcal{I}$-diagrams, i.e. $X : \Delta\mathcal{I} \to \mathcal{M}$ with the property $X(\tilde{s}^j)$ are isomorphisms in $\mathcal{M}$ for all $\tilde{s}^j : \sigma s^j \to \sigma \in \Delta\mathcal{I}$.

Consider

$$\begin{aligned} \tau : \Delta\mathcal{I} &\longrightarrow \mathcal{I} \\ (i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n) &\mapsto i_0 \end{aligned}$$

a map of small categories which yields

$$\begin{tikzcd} \mathcal{M}^{\mathcal{I}} \arrow[rr] \arrow[dr, "\tau^*"'] & & \mathcal{M}^{\Delta\mathcal{I}} \\ & \mathcal{M}_b^{\Delta\mathcal{I}} \arrow[ur, hook] & \end{tikzcd}$$

which factors through $\mathcal{M}_b^{\Delta\mathcal{I}}$, and we have an adjunction

$$\tau_! : \mathcal{M}_b^{\Delta\mathcal{I}} \rightleftarrows \mathcal{M}^{\mathcal{I}} : \tau^*$$

where the left adjoint is given by left Kan extension along τ.

Theorem 4.56. *For any model category* $\mathcal{M}$ *and small category* $\mathcal{I}$*,*

1. $\mathcal{M}_b^{\Delta\mathcal{I}}$ *has a natural BK model structure.*

2. $(\tau_!, \tau^*)$ *is a left model approximation.*

More generally, any left model approximation $l : \mathcal{M} \rightleftarrows \mathcal{C} : r$ gives rise to

$$\mathcal{M}_b^{\Delta \mathcal{I}} \underset{\tau^*}{\overset{\tau_!}{\rightleftarrows}} \mathcal{M}^{\mathcal{I}} \underset{r^{\mathcal{I}}}{\overset{l^{\mathcal{I}}}{\rightleftarrows}} \mathcal{C}^{\mathcal{I}}$$

which is a left model approximation of $\mathcal{C}^{\mathcal{I}}$.

Example 4.57. Consider **sGr** the category of simplicial groups viewed as a homotopical category (forgetting the model structure) and **sMon** the category of simplicial monoids viewed as a model category, then the adjunction

$$l : \mathbf{sMon} \rightleftarrows \mathbf{sGr} : r$$

is a left model approximation.

Proof. Think of **sMon** as a simplicial category with one object and l is DK localization. $\square$

Theorem 4.58 ([BRY], Appendix)**.** *Let* $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ *be adjoint pair between homotopical categories. Assume that*

$$\begin{tikzcd}
\mathcal{M} \arrow[d, shift left, "l"] \arrow[dr, shift left, "\widehat{F}"] & \\
\mathcal{C} \arrow[u, shift left, "r"] \arrow[r, shift left, "F"] & \mathcal{D} \arrow[l, shift left, "G"] \arrow[ul, shift left, "\widehat{G}"]
\end{tikzcd}$$

where (l, r) *is a left model approximation given with adjunction* $(\widehat{F}, \widehat{G})$ *such that*

1. $(\widehat{F}, \widehat{G})$ *is deformable (e.g. Quillen pair),*

2. *there is a natural transformation* $\eta : \widehat{F} \Rightarrow Fl$ *which is a natural weak equivalence on cofibrant objects in* $\mathcal{M}$*,*

3. *essential images* $Im(\mathbb{R}\widehat{G}) \subseteq Im(\overline{r})$ *in* $\mathrm{Ho}(\mathcal{M})$.

then $\mathbb{L}F$ *and* $\mathbb{R}G$ *exist and are given by*

$$\mathbb{L}F = \mathbb{L}\widehat{F} \circ \overline{r}, \quad \mathbb{R}G = \mathbb{L}L \circ \mathbb{R}\widehat{G}.$$

Moreover, these are absolute derived functors and we have derived adjunction

$$\mathbb{L}F : \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D}) : \mathbb{R}G.$$

Remark. If $\mathcal{D}$ is a model category and $(\widehat{F}, \widehat{G})$ is a Quillen pair, we call $(\widehat{F}, \widehat{G})$ a left model approximation of (F, G).

Sketch of proof. By [DHKS], condition (1) implies that $\mathbb{L}\widehat{F}, \mathbb{R}\widehat{G}$ exists and are absolute, and are adjoint. Condition (2) implies that

$$\mathbb{L}\widehat{F} \cong \mathbb{L}(F \circ l) \cong \mathbb{L}F \circ \mathbb{L}l$$

Since $\widehat{F}$ is left deformable, (l, r) is good for F, so by proposition 4.51 $\mathbb{L}l \circ \bar{r} \cong \mathrm{id}$ thus $\mathbb{L}F \cong \mathbb{L}\widehat{F} \circ \bar{r}$. By condition (3), $\mathbb{R}\widehat{G}$ can be factored as

$$\mathbb{R}\widehat{G} : \mathrm{Ho}(\mathcal{D}) \xrightarrow{\overline{G}_0} \overline{\mathcal{C}} \xrightarrow{\bar{\imath}} \mathrm{Ho}(\mathcal{M})$$

where $\overline{\mathcal{C}} = \mathrm{Im}(\bar{r})$. By proposition,

$$\bar{r} : \mathrm{Ho}(\mathcal{C}) \xrightarrow{\bar{r}_0} \overline{\mathcal{C}} \xrightarrow{\bar{\imath}} \mathrm{Ho}(\mathcal{M})$$

where $\bar{r}_0$ is an equivalence with inverse $\bar{l}_0 := \mathbb{L}l \circ \bar{\imath} : \overline{\mathcal{C}} \xrightarrow{\sim} \mathrm{Ho}(\mathcal{C})$.

Consider $\mathbb{L}l \circ \mathbb{R}\widehat{G} \cong \mathbb{L}l \circ \bar{\imath} \circ \overline{G}_0 \cong \bar{l}_0 \circ \overline{G}_0$, take any $X \in \mathrm{Ho}(\mathcal{C})$ and $A \in \mathrm{Ho}(\mathcal{D})$,

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, \mathbb{L}l \circ \mathbb{R}\widehat{G}(A)) &\cong \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, \bar{l}_0 \circ \overline{G}_0(A)) \\
&\cong \mathrm{Hom}_{\overline{\mathcal{C}}}(\bar{r}_0(X), \overline{G}_0(A)) \\
&\cong \mathrm{Hom}_{\overline{\mathcal{C}}}(\bar{\imath}\bar{r}_0(X), \bar{\imath}\overline{G}_0(A)) \\
&\cong \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(\bar{r}(X), \mathbb{R}\widehat{G}(A)) \\
&\cong \mathrm{Hom}_{\mathrm{Ho}(\mathcal{D})}(\mathbb{L}\widehat{F}\bar{r}(X), A) \\
&\cong \mathrm{Hom}_{\mathrm{Ho}(\mathcal{D})}(\mathbb{L}F(X), A)
\end{aligned}$$

Hence $\mathbb{L}l \circ \mathbb{R}\widehat{G}$ is right adjoint to $\mathbb{L}F$. Theorem follows from the following lemma. ☐

Lemma 4.59. *Let* $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ *be an adjunction between homotopical categories. Assume*

1. F *has an absolute left derived functor* $\mathbb{L}F$ *and*
2. $\mathbb{L}F$ *has a right adjoint* $\tilde{G}$.

Then $\tilde{G}$ *is the absolute right derived functor of* G, *i.e.* $\mathbb{R}G$ *exists and* $\mathbb{R}G \cong \tilde{G}$.

Proof. We check that $\tilde{G}$ satisfies the UMP of absolute right derived functor $\mathbb{R}G$.

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "G"] \arrow[d, "\gamma_{\mathcal{D}}"'] & \mathcal{C} \arrow[r, "\gamma_{\mathcal{C}}"] & \mathrm{Ho}(\mathcal{C}) \arrow[r, "E"] \arrow[dll, dashed, bend left=5, "\mathbb{L}F"'] & \mathcal{E} \\
\mathrm{Ho}(\mathcal{D}) \arrow[urr, dashed, bend right=5, "\mathbb{R}G"'] \arrow[urrr, bend right, "H"'] & & &
\end{tikzcd}$$

For any E, H we want to show

$$\mathrm{Hom}(E \circ \mathbb{R}G), H) \cong \mathrm{Hom}(E\gamma_{\mathcal{C}}G, H\gamma_{\mathcal{D}}) \tag{4.7}$$ {{eq:nat-iso}}

where Hom is the set of natural transformations. The following observations implies the functorial isomorphism 4.7.

1. By UMP of $\mathbb{L}F$ being an absolute left derived functor
$$\mathrm{Hom}(E, H \circ \mathbb{L}F) \cong \mathrm{Hom}(E\gamma_{\mathcal{C}}, H\gamma_{\mathcal{D}}F)$$
2. $\tilde{G}$ is right adjoint to $\mathbb{L}F$ implies $\tilde{G}^* = (-) \circ \tilde{G}$ is left adjoint to $\mathbb{L}F^* = (-) \circ \mathbb{L}F$.
$$\mathrm{Hom}(E \circ \tilde{G}, H) \cong \mathrm{Hom}(E, H \circ \mathbb{L}F)$$
3. G is right adjoint to F so $G^* = (-) \circ G$ is left adjoint to $F^* = (-) \circ F$
$$\mathrm{Hom}(E\gamma_{\mathcal{C}}, H\gamma_{\mathcal{D}}F) \cong \mathrm{Hom}(E\gamma_{\mathcal{C}}G, H\gamma_{\mathcal{D}})$$

□

Example 4.60. [BKR] Let **Ring** be the category of associative unital rings. Consider

$$\mathrm{GL}_n : \mathbf{Ring} \longrightarrow \mathbf{Gr}$$

$$A \longmapsto \mathrm{GL}_n(A)$$

where $\mathrm{GL}_n(A)$ is the group of invertible $n \times n$ matrices with entrics in A. This functoe has a left adjoint which can be described as follows

$$\mathbf{Gr} \underset{(-)^{\times}}{\overset{\mathbb{Z}[-]}{\rightleftarrows}} \mathbf{Ring} \underset{u}{\overset{\mathbb{M}_n(\mathbb{Z})*(-)}{\rightleftarrows}} \mathbb{M}_n(\mathbb{Z}) \downarrow \mathbf{Ring} \underset{\mathbb{M}_n(-)}{\overset{(-)^{\mathbb{M}_n}}{\rightleftarrows}} \mathbf{Ring}$$

where

$$(f : A \to B)^{A} := \{b \in B | [f(a), b] = 0, \forall a \in A\}.$$

We have $(\Gamma)_n = (\mathbb{Z}[\Gamma] * \mathbb{M}_n(\mathbb{Z}))^{\mathbb{M}_n(\mathbb{Z})}$. Thus we have adjunction

$$(-)_n : \mathbf{Gr} \rightleftarrows \mathbf{Ring} : \mathrm{GL}_n(-)$$

which extends levelwise to an adjunction

$$(-)_n : \mathbf{sGr} \rightleftarrows \mathbf{sRing} : \mathrm{GL}_n$$

This is not a Quillen pair. For instance, for $n = 1$, take $\Gamma = \mathbb{F}_1$,

$$(\mathbb{F}_1)_1 \cong k[x, x^{-1}]$$

which is not cofibrant.

It is not hard to see that $(-)_n$ is left deformable (take $Q = \mathbb{G}\overline{W} : \mathbf{sGr} \to \mathbf{sGr}$ so $\mathbb{L}(-)_n$ exists and is absolute., but GL_n is not right deformable. By a theorem of Shulman (2011), if G is both left and right deformable, $\mathbb{L}G \cong \mathbb{R}G$. And by a theorem of Swan, GL_n is left deformable, but $\mathbb{R}G \not\cong \mathbb{L}G$. So this is a case where the theorem applies.

Consider the adjunction

$$\begin{tikzcd}
\mathbf{sMon} \arrow[d, shift left, "l"] \arrow[drr, shift left, "\widehat{(-)}_n"] & & \\
\mathbf{sGr} \arrow[u, shift left, "r"] \arrow[rr, shift left, "(-)_n"] & & \mathbf{sRing} \arrow[ll, shift left, "\mathrm{GL}_n"] \arrow[ull, shift left, "\widehat{\mathrm{GL}}_n"]
\end{tikzcd}$$

where l is the Dywer-Kan localization, and $\widehat{\mathrm{GL}}_n$ is Waldhausen's construction of matrics invertible up to homotopy and $\mathbb{R}\,\mathrm{GL}_n = \mathbb{L}l \circ \widehat{\mathrm{GL}}_n$.

Remark. In 1985- 1986, Waldhausen defined algebraic K-theory of spaces

$$A : \mathbf{Top}_* \longrightarrow \mathbf{Top}_*$$

which has many properties as Quillen's algebraic K-theory

$$K_Q : \mathbf{Ring} \longrightarrow \mathbf{Top}_*$$

Take $X \in \mathbf{sSet}_0$ and $\mathbb{Z}[\mathbb{G}(X)] \in \mathbf{sRing}$, $A(X) \not\simeq K_Q(\mathbb{Z}[\mathbb{G}(X)])$.

Note $K_Q(R_*) \coloneqq \mathbb{Z} \times |B\,\mathrm{GL}_\infty(R_*)|^+$ is not homotopy invariant since GL_∞ is not homotopy invariant.

Waldhusen generalized GL_n as follows. Note we have

$$\pi_* \mathbb{M}_n(R_*) \cong \mathbb{M}_n(\pi_i(R_*)), \forall i \geq 0$$

so the functor $\mathbb{M}_n : \mathbf{sRing} \to \mathbf{sRing}$ is homotopic, and we can define $\widehat{\mathrm{GL}}_n(R)$ as the pullback

$$\begin{tikzcd}
\widehat{\mathrm{GL}}_n(R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{M}_n(R) \arrow[d, "\pi_0"] \\
\mathrm{GL}_n(\pi_0 R) \arrow[r, hook] & \mathbb{M}_n(\pi_0 R)
\end{tikzcd}$$

then we see

$$\pi_i \widehat{\mathrm{GL}}_n(R) = \begin{cases} \mathrm{GL}_n(\pi_0 R), & i > 0 \\ \mathbb{M}_n(\pi_i R), & i = 0 \end{cases}$$

and we can define $\widehat{\mathrm{GL}}_\infty(R) = \lim \widehat{\mathrm{GL}}_n(R)$ and

$$K_W(R_*) \coloneqq \mathbb{Z} \times |B(\widehat{\mathrm{GL}}_\infty(R_*)|^+.$$

Theorem 4.61. *For any* $X \in \mathbf{sSet}_0$, $A(X) \to K_W(\mathbb{Z}[\mathbb{G}X])$ *is a rational homotopy equivalence.*

For arbitrary affine algebraic group scheme G viewed from the functor point of view.

The question is how to make G or BG homotopy invariant?

Galatius and Venkatesh [GV] define

$$\Delta \to \mathbf{Comm}_k \xrightarrow{Q} \mathbf{sComm}$$

$$[n] \mapsto \mathcal{O}(\mathcal{N}G) \mapsto Q\mathcal{O}(\mathcal{N}G) \cong \mathcal{O}(G)^{\otimes n}$$

And

$$\widehat{G} : \mathbf{sComm}_k \longrightarrow \mathbf{sSp}_0$$

$$A \longmapsto \mathrm{Map}^{\Delta}(Q\mathcal{O}(\mathcal{N}G), A)$$

In this case, we have left model approximation

$$\begin{tikzcd}
\mathcal{L}\mathbf{sSp}_0 \arrow[d, shift left] \arrow[dr, shift left] & \\
\mathbf{sGr} \arrow[u, shift left] \arrow[r, shift left] & \mathbf{sComm}_k \arrow[l, shift left] \arrow[ul, shift left]
\end{tikzcd}$$

5 Localizations

Let $\mathcal{M}$ be a model category and $\mathcal{S} \subset \mathrm{Mor}(\mathcal{M})$ be a class of maps.

Definition 5.1. A **left Bousfield localization** of $\mathcal{M}$ with respect to $\mathcal{S}$ is a model category $L_{\mathcal{S}}\mathcal{M}$ given together with left Quillen functor $j : \mathcal{M} \to L_{\mathcal{S}}\mathcal{M}$ satisfying

1. $\mathbb{L}j : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(L_{\mathcal{S}}\mathcal{M})$ maps every map in $\overline{\mathcal{S}} = \{\gamma(f) \in \mathrm{Ho}(\mathcal{M}), f \in \mathcal{S}\}$ to isomorphisms in $\mathrm{Ho}(L_{\mathcal{S}})\mathcal{M})$.
2. j is initial among all left Quillen functors $F : \mathcal{M} \to \mathcal{N}$ such that $\mathbb{L}F(\overline{\mathcal{S}}) \subset \mathrm{Iso}(\mathrm{Ho}(\mathcal{N}))$, i.e. there is a unique $\overline{F} : L_{\mathcal{S}}\mathcal{M} \to \mathcal{N}$ so that we have the following factorization.

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "F"] \arrow[dr, "j"'] & & \mathcal{N} \\ & L_{\mathcal{S}}\mathcal{M} \arrow[ur, "\overline{F}"'] & \end{tikzcd}$$

Remark. If it exists, j is unique up to unique isomorphism.

Dually we cam define

Definition 5.2. A **right Bousfield localization** of $\mathcal{M}$ with respect to $\mathcal{S}$ is a model category $R_{\mathcal{S}}\mathcal{M}$ given together with right Quillen functor $j : \mathcal{M} \to R_{\mathcal{S}}\mathcal{M}$ satisfying

1. $\mathbb{R}j : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(R_{\mathcal{S}}\mathcal{M})$ maps every map in $\overline{\mathcal{S}} = \{\gamma(f) \in \mathrm{Ho}(\mathcal{M}), f \in \mathcal{S}\}$ to isomorphisms in $\mathrm{Ho}(R_{\mathcal{S}})\mathcal{M})$.
2. j is initial among all right Quillen functors $F : \mathcal{M} \to \mathcal{N}$ such that $\mathbb{L}F(\overline{\mathcal{S}}) \subset \mathrm{Iso}(\mathrm{Ho}(\mathcal{N}))$, i.e. there is a unique $\overline{F} : R_{\mathcal{S}}\mathcal{M} \to \mathcal{N}$ $\begin{tikzcd} \mathcal{M} \arrow[rr, "F"] \arrow[dr, "j"'] & & \mathcal{N} \\ & R_{\mathcal{S}}\mathcal{M} \arrow[ur, "\overline{F}"'] & \end{tikzcd}$

Question 5.3. *1. Given a Quillen pair* $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ *and a class of morphisms* $\mathcal{S} \subset \mathrm{Mor}(\mathcal{M})$ *we want to specify conditions (in terms of* F*) where* $\mathbb{L}F$ *maps* $\overline{\mathcal{S}}$ *to isomorphisms in* $\mathrm{Ho}(\mathcal{N})$.

2. How to describe weak equivalences in $L_{\mathcal{S}}\mathcal{M}$ *(*$\mathcal{S}$*-local weak equivalences).*

We need a notion of localization of categories.

5.1 Simplicial Localization

Recall the notion of a simplicial category

Definition 5.4. A simplicial category is a category enriched in **sSet**, i.e.

1. For any $X, Y \in \mathrm{Ob}(\mathcal{C})$, there is a mapping space $\mathrm{Map}_{\mathcal{C}}(X, Y) \in \mathrm{oB}(\mathbf{sSet})$.
2. For any $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, there is a composition law
$$\mathrm{Map}_{\mathcal{C}}(X, Y) \times \mathrm{Map}_{\mathcal{C}}(Y, Z) \xrightarrow{\circ} \mathrm{Map}(X, Z)$$
which is a morphism of simplicial sets.
3. For any $X \in \mathrm{Ob}(\mathcal{C})$, there is a unit map
$$i_X : \Delta[0]_* \longrightarrow \mathrm{Map}(X, X)$$
which is a morphism of simplicial sets.
4. For any $X, Y \in \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}}(X, Y) \cong \mathrm{Map}_{\mathcal{C}}(X, Y)_0$.

These satisfy axioms of associativity and left/right indentities.

Remark. A simplicial set is not the same thing as a simplicial object in **Cat**. Any simplicial category $\mathcal{C}$ gives rise to an simplicial object $\mathcal{C}_* : \Delta^{op} \to \mathbf{Cat}$
$$\mathcal{C}_0 \; \substack{\longleftarrow\\ \Longrightarrow} \; \mathcal{C}_1 \; \substack{\longleftarrow\\ \Longrightarrow \\ \longleftarrow \\ \longrightarrow} \; \mathcal{C}_2 \cdots$$
such that

- objects are discrete $\mathrm{Ob}(\mathcal{C}_n) = \mathrm{Ob}(\mathcal{C})$, for any $n \geq 0$, and d_i and s_j act as identities on objects,
- morphisms are $\mathrm{Hom}_{\mathcal{C}_n}(X, Y) \coloneqq \mathrm{Map}_{\mathcal{C}}(X, Y)_n$.

Definition 5.5 (Quillen)**.** A simplicial model category $\mathcal{M}$ is a model category and a simplicial category with two structures satisfying the two compatibility axioms:

MC6 $\mathcal{M}$ is tensored and cotensored over **sSet**, i.e.
$$\boxtimes : \mathbf{sSet} \times \mathcal{M} \longrightarrow \mathcal{M} \qquad \text{tensor or copower}$$
$$(-)^{-} : \mathcal{M} \times \mathbf{sSet}^{op} \to \mathcal{M} \qquad \text{cotensor or power}$$
together with natural isomorphisms in **sSet**
$$\mathrm{Map}_{\mathcal{M}}(K \boxtimes X, Y) \cong \mathrm{Map}_{\mathbf{sSet}}(K, \mathrm{Map}_{\mathcal{C}}(X, Y)) \cong \mathrm{Map}_{\mathcal{M}}(X, Y^K).$$

MC7 For any $j : A \hookrightarrow A$ a cofibration in $\mathcal{M}$, and any $p : X \twoheadrightarrow Y$ a fibration in $\mathcal{M}$,
$$\mathrm{Map}(j, p) : \mathrm{Map}_{\mathcal{M}}(B, X) \longrightarrow \mathrm{Map}_{\mathcal{M}}(A, X) \times_{\mathrm{Map}_{\mathcal{M}}(A,Y)} \mathrm{Map}_{\mathcal{M}}(B, Y)$$
(called pullback corner map) is a Kan fibration in **sSet**. Moreover, when j, p are weak equivalences, so is $\mathrm{Map}(j, p)$.

Lemma 5.6 ([H2], 9.3.7)**.** *MC7 is equivalent to the following. For any* $i : A \hookrightarrow B \in \mathtt{Cof}(\mathcal{M})$ *and any* $j : K \hookrightarrow L \in \mathtt{Cof}(\mathbf{sSet})$ *(i.e. inclusion), the following pushout corner map*

$$i \boxtimes j : A \boxtimes L \coprod_{A \boxtimes K} B \boxtimes K \longrightarrow B \boxtimes L$$

is a cofibration in $\mathcal{M}$*. Moreover, if* i *and* j *are weak equivalences, so is* $i \boxtimes j$*.*

Example 5.7. Let $\mathcal{M} = \mathbf{sSet}$ then the mapping space is defined by

$$\mathrm{Map}_{\mathbf{sSet}}(K, L)_n = \mathrm{Hom}_{\mathbf{sSet}}(K \times \Delta[n]_*, L)$$

for any $n \geq 0$.

Example 5.8. Let $\mathcal{M} = \mathbf{s}\mathcal{C}$ where $\mathcal{C}$ is an algebraic category, (thus cocomplete and complete),

$$\mathrm{Map}_{\mathbf{s}\mathcal{C}}(X, Y)_n = \mathrm{Hom}_{\mathbf{s}\mathcal{C}}(X \boxtimes \Delta[n]_*, Y).$$

For instance, take $\mathcal{M} = \mathbf{sGr}$, the tensor product is given by $G \boxtimes K = \{\coprod_{K_n} G_n\}_{n \geq 0}$.

Example 5.9 (Counterexample)**.** The reedy model structure on $\mathbf{s}\mathcal{C}$, where $\mathcal{C}$ is cofibrantly generated model category, is not a simplicial model category, note $i \boxtimes j \in W\!E(\mathbf{s}\mathcal{C})$ if $i \in W\!E(\mathbf{s}\mathcal{C})$.

5.2 Monad and Comonad

In order to introduce the standard simplicial resolution, let's start with a brief review of (co)monad.

5.2.1 (Co)monad

Definition 5.10. A **monad** (**triple**) on a category $\mathcal{C}$ is given by an endofunctor $T : \mathcal{C} \to \mathcal{C}$ with two morphisms $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow T$ and $\mu : T \circ T \Rightarrow T$ satisfying

1. associativity, i.e. the following diagram commutes

$$\begin{tikzcd} (T \circ T) \circ T = T \circ (T \circ T) \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"] & T \circ T \arrow[d, Rightarrow, "\mu"] \\ T \circ T \arrow[r, "\mu"] & T \end{tikzcd}$$

2. unitality, i.e. the following diagram commutes

$$\begin{tikzcd} T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, equal] & T \circ T \arrow[d, Rightarrow, "\mu"] & T \arrow[l, Rightarrow, "T\eta"'] \arrow[dl, equal] \\ & T & \end{tikzcd}$$

Remark. This can be regarded as a "generalized" associative, unital algebras.

Definition 5.11. A **comonad** (**cotriple**) on a category $\mathcal{C}$ is given by an endofunctor $\perp : \mathcal{C} \Rightarrow \mathcal{C}$ with two morphisms $\varepsilon : \perp \Rightarrow \mathrm{id}_{\mathcal{C}}$ and $\delta : \perp \Rightarrow \perp \circ \perp$ satisfying coassociative diagram

$$\begin{tikzcd} \perp \arrow[rr, Rightarrow, "\delta"] \arrow[d, Rightarrow, "\delta"'] & & \perp \circ \perp \arrow[d, Rightarrow, "\perp\delta"] \\ \perp \circ \perp \arrow[r, "\delta\perp"] & (\perp \circ \perp) \circ \perp \arrow[r, phantom, "="] & \perp \circ (\perp \circ \perp) \end{tikzcd}$$

and counital diagram

$$\begin{tikzcd} \perp & \perp \circ \perp \arrow[l, Rightarrow, "\varepsilon\perp"'] \arrow[r, Rightarrow, "\perp\varepsilon"] & \perp \\ & \perp \arrow[ul, equal] \arrow[u, Rightarrow, "\delta"] \arrow[ur, equal] & \end{tikzcd}$$

Primary Examples: (Topological) Operads The main reference is [M2].

Example 5.12 (Operads in Spaces)**.** An **operad** $\mathcal{O}$ in $\mathbf{Top}_*$ is given by a collection of spaces $\{\mathcal{O}(j)\}_{j\geq 0}$ such that

1. $\mathcal{O}(0) = \{*\}$,
2. there is a unit element $1 \in \mathcal{O}(1)$,
3. for any $j \geq 0$, $\mathcal{O}(j)$ is a right Σ_j-space: $cO(j) \times \Sigma_j \to \mathcal{O}(j)$.

satisfying composition laws: for any $k \geq 0$, and any $j_1, \cdots, j_k \geq 0$,

$$\gamma : \mathcal{O}(k) \times \mathcal{O}(j_1) \times \cdots \times \mathcal{O}(j_k) \longrightarrow \mathcal{O}(j_1 + \cdots + j_k)$$

and Σ-equivalence and associativity and unitality.

Remark. $\mathcal{O}(j)$ is called the space of j-nary operation.

Definition 5.13. We say that a space X carries an action of $\mathcal{O}$ (X is a $\mathcal{O}$-space) if we are given

$$\Theta_j : \mathcal{O}(j) \times X \longrightarrow X, \qquad j \geq 0$$

such that

1. $\Theta_0 : * \to X$ is the basepoint of X,
2. $\Theta_1 : \mathcal{O}(1) \times X \to X$ satisfies $\Theta(1)(1, x) = x$.

Equivalently, for any $X \in \mathbf{Top}_*$, there is the **endomorphism operad** $\mathcal{E}_X$ where

$$\mathcal{E}_X(j) = \mathrm{Hom}_{\mathbf{Top}_*}(X^j, X)$$

Then an $\mathcal{O}$-space is a morphism of operad $\mathcal{O} \to \mathcal{E}_X$.

Example 5.14. 1. Let $\mathcal{N}$ be a discrete operad with $\mathcal{N}(j) = *, \forall j \geq 0$. An $\mathcal{N}$-space is a commutative monoid in $\mathbf{Top}_*$

2. Let $\mathcal{M}$ be a discrete operad with $\mathcal{M}(j) = \Sigma_j \forall j \geq 0$, with right Σ-action given by multiplication

$$\mathcal{M}(j) \times \Sigma_j \xrightarrow{m} \Sigma_j$$

and composition laws are (Σ-equivariant thus) determined by the values at e

$$\gamma : \Sigma_k \times \Sigma_{j_1} \times \cdots \times \Sigma_{j_k} \to \Sigma_j$$

$$(e_k, e_{j_1}, \cdots, e_{j_k}) \longmapsto e_j$$

where $j = j_1 + \cdots + j_k$. An $\mathcal{M}$-space is an associative monoid (i.e. associative H-space).

Example 5.15 ($\mathcal{M}$-space)**.** The (reduced) **James monoid** is given by

$$\mathcal{J}(X) = (\coprod_{k \geq 0} X^k)/\sim$$

where $(x_1, \cdots, x_i, *, x_{i+1}, \cdots, x_k) \sim (x_1, \cdots, x_i, x_{i+1}, \cdots, x_k)$. This is the free topological monoid generated by X.

Example 5.16 ($\mathcal{N}$-space)**.** The **Dold-Thom space** (or infinite symmetric product space) $SP^\infty X$ of X is defined as follows. Let $SP^N(X) = X^N/\Sigma_N$, there is natural inclusions

$$SP^1(X) \longrightarrow SP^2(X) \longrightarrow \cdots$$

and we define $SP^\infty X = \varinjlim_N SP^N X$.

Theorem 5.17 (Dold-Thom)**.** $\pi_* SP^\infty(X) \cong \tilde{H}_*(X; \mathbb{Z})$.

Theorem 5.18 ([M2])**.** $SP^\infty X$ *has homotopy type of an abelian simplicial group*

$$|A| \simeq \prod_{n \geq 0} K(n, \tilde{H}_n(X)).$$

Example 5.19 (little cube operad)**.** Fix $n \geq 1$, $\mathcal{C}_n(j)$ = the space of j-tuples of "little n-cubes", i.e. linear embedding with parallel axes and disjoint interior.

Note $\mathcal{C}_n(j) \simeq \mathrm{Conf}_j(\mathbb{R}^n)$ whic is Σ_j-homotopy equivalent.

Key fact: For any $X \in \mathbf{Top}_*$, the n-th loop space

$$Y \simeq \Omega^n(X) = \mathrm{Map}_*(S^n, X)$$

carries a natural action of $\mathcal{C}_n$

$$\Theta_{n,j} : \mathcal{C}_n(j) \times \Omega^n(X) \longrightarrow \Omega^n(X)$$

Exercise 5.20. Write explicit formulas.

Definition 5.21. An A_∞**-operad** is a Σ-free operad $\mathcal{C}$ (i.e. $\forall j \geq 0$, $C(j)$ is a free Σ_j-space) in $\mathbf{Top}_*$ given with augmentation (a map of operad) $\varepsilon : \mathcal{C} \to \mathcal{M}$ such that

$$\pi_0(\varepsilon) : \pi_0 \mathcal{C} \simeq \mathcal{M}$$

is a local Σ-equivalence, i.e. for any $j \geq 0$, the components of $C(j) \cong \Sigma_j$ and each component is contractible.

Example 5.22. $\mathcal{C}_1$ is an A_∞-operad.

Proposition 5.23 ([M2])**.** *Every operad $\mathcal{O}$ determines a monad $T_\mathcal{O}$ in $\mathbf{Top}_*$ so that an $\mathcal{O}$-space is equivalent to an algebra over $T_\mathcal{O}$.*

Proof. There is functor

$$\mathrm{Operad}(\mathbf{Top}_*) \longrightarrow \mathrm{Monad}(\mathbf{Top}_*)$$

$$\mathcal{O} \longmapsto (T_\mathcal{O} : \mathbf{Top}_* \to \mathbf{Top}_{ast})$$

where

$$T_\mathcal{O}(X) = \coprod_{j \geq 0} (\mathcal{O}(j) \times_{\Sigma_j} X^j) / \sim$$

and the equivalence relation $\sim$ is given by

$$(e, x_1, \cdots, x_{i-1}, *, x_{i+1}, \cdots, x_j) \sim (\delta(c), x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_j)$$

When $\mathcal{O} = \mathcal{N}$, $T_\mathcal{N}(X) = SP^\infty X$ and when $\mathcal{O} = \mathcal{M}$, $T_\mathcal{M}(X) = \mathcal{J}(X)$. $\square$

(Co)monad arising from adjunctions. Given a pair of adjoint functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ with unit $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow UF$ and counit $\varepsilon : FU \Rightarrow \mathrm{id}_{\mathcal{D}}$, we can define

$$\begin{aligned} T &= UF : \mathcal{C} \to \mathcal{C}, & \mu &= U\varepsilon F : T \circ T = UFUF \Rightarrow T = UF, \\ \perp &= FU : \mathcal{D} \to \mathcal{D}, & \delta &= F\eta U : \perp = FU \to \perp \circ \perp = FUFU. \end{aligned}$$

Claim 5.24. $(T = UF, \eta, \mu)$ *is a monad on* $\mathcal{C}$ *and* $(\perp = FU, \varepsilon, \delta)$ *is a comonad in* $\mathcal{D}$.

Proof. Use identities for adjunction morphisms, we have

$$\left(FU \overset{F\eta U}{\Rightarrow} FUFU \overset{\varepsilon FU}{\Rightarrow} FU\right) = \mathrm{id}_{FU}$$
$$\left(UF \overset{\eta UF}{\Rightarrow} UFUF \overset{U\varepsilon F}{\Rightarrow} UF\right) = \mathrm{id}_{UF}$$

which gives the unitality diagrams. The associativity diagram follows from naturality of the unit and counit functors, i.e. for any $x \in \mathrm{Ob}(\mathcal{C})$ we have

$$\begin{tikzcd} UFUFUFx \arrow[r, "UFU\varepsilon_{Fx}"] \arrow[d, "U\varepsilon_{FUFx}"] & UFUFx \arrow[d, "U\varepsilon_{Fx}"] \\ UFUFx \arrow[r, "U\varepsilon_{Fx}"] & UFx \end{tikzcd}$$

and for any $y \in \mathrm{Ob}(\mathcal{D})$

$$\begin{tikzcd} FUy \arrow[r, "F\eta_{Uy}"] \arrow[d, "F\eta_{Uy}"] & FUFUy \arrow[d, "FUF\eta_{Uy}"] \\ FUFUy \arrow[r, "F\eta_{UFUy}"] & FUFUFUy. \end{tikzcd}$$

□

5.2.2 (Co)bar Construction

Claim 5.25. *Every monad in* $\mathcal{C}$ *gives a functor* $\mathcal{C} \to \mathbf{c}\mathcal{C}$ *and every comonad gives a functor* $\mathcal{D} \to \mathbf{s}\mathcal{D}$.

Proof. Given $(\perp, \varepsilon, \delta)$ on $\mathcal{D}$, and $A \in \mathrm{Ob}(\mathcal{D})$, we define

$$\begin{aligned} \perp_* : \mathcal{D} &\longrightarrow \mathbf{s}\mathcal{D} \\ A &\longmapsto \perp_* A = \{\perp_n A\}_{n \geq 0} \end{aligned}$$

where $\perp_n A = \perp^{n+1} A$ and

$$d_i = \perp^i \varepsilon \perp^{n-i} : \perp^{n+1} A \longrightarrow \perp^n A,$$

$$s_j = \perp^j \delta \perp^{n-j} : \perp^{n+1} A \to \perp^{n+2} A.$$

Explicitly, $\perp_* A$ can be expressed as follows

$$\perp A \leftleftarrows \perp^2 A \leftleftarrows \perp^3 A \cdots$$

The simplicial identities are satisfied because of the functoriality of units and counits and the identity for adjunction morphisms. In particular, $d_i s_j = \mathrm{id}$ for $i = j, j+1$ follows from

$$\left(FU \overset{F\eta U}{\Rightarrow} FUFU \overset{\varepsilon FU}{\Rightarrow} FU\right) = \mathrm{id}_{FU},$$
$$\left(FU \overset{F\eta U}{\Rightarrow} FUFU \overset{FU\varepsilon}{\Rightarrow} FU\right) = \mathrm{id}_{FU}.$$

□

We apply this to the following adjunction

$$F : \mathrm{Graph} \rightleftarrows \mathbf{Cat} : U$$

where Graph is the category of reflexive discrete graphs, i.e. every vector is equipped with a (identity) loop. The above construction gives us a functor

$$(FU)_* : \mathbf{Cat} \longrightarrow \mathbf{sCat}$$
$$\mathcal{C} \longmapsto (FU)_*(\mathcal{C})$$

which is the simplicial resolution of $\mathcal{C}$ such that the image of every category $\mathcal{C}$ is a simplicial category.

Example 5.26 (Homotopy Coherent Nerve)**.** Recall the adjunction

$$c : \mathbf{sSet} \rightleftarrows \mathbf{Cat} : \mathcal{N}$$

gives a simplicial resolution functor

$$Q : \mathbf{Cat} \longrightarrow \mathbf{sCat}$$

and applying this to Δ viewed as a category, we get a cosimplicial simplicial category

$$\Delta \to \mathbf{Cat} \overset{Q}{\to} \mathbf{sCat}$$
$$[n] \mapsto \overrightarrow{n} \mapsto Q\overrightarrow{n}$$

which by 2.1 gives the adjunction

$$\mathcal{C} : \mathbf{sSet} \rightleftarrows \mathbf{sCat} : \mathcal{N}$$

where $\mathcal{C}$ is the rigidification functor and $\mathcal{N}$ is homotopy coherent nerve functor.

Example 5.27. The adjunction

$$\mathrm{Sym} : \mathbf{Mod}_k \rightleftarrows \mathbf{Comm}_k : U$$

induces the simplicial resolution functor

$$Q : \mathbf{Comm}_k \longrightarrow \mathbf{sComm}_k.$$

Then consider the group scheme functor

$$\begin{aligned} G : \mathbf{Comm}_k &\to \mathbf{Gr} \xrightarrow{\mathcal{N}} \mathbf{sSet} \\ A &\longmapsto G(A) \mapsto \mathcal{N}_*(A) \end{aligned}$$

and thus we have

$$\begin{aligned} \Delta &\xrightarrow{\mathcal{O}(\mathcal{N}_*(G))} \mathbf{Comm}_k \xrightarrow{Q} \mathbf{sComm}_k \\ [n] &\longmapsto \mathcal{O}(\mathcal{N}_n G) \mapsto Q\mathcal{O}(\mathcal{N}_n G) \end{aligned}$$

which gives us an adjunction

$$\mathbf{sSet} \rightleftarrows \mathbf{sComm}_k : \mathbb{G}$$

where $\mathbb{G}$ is called homotopy coherent group [18] (it is not a group).

Question 5.28. *When does this construction give a resolution?*

Extra Degeneracy Let Δ_+ be the category defined by adding to Δ an initial object $[-1]$ so that $\Delta \subset \Delta_+$ is a full subcategory such that

- $\mathrm{Ob}(\Delta_+) = \{[n]\}_{n \geq -1}$,
- $\mathrm{Hom}_{\Delta_+}([n],[m]) = \mathrm{Hom}_{\Delta}([n],[m])$ for any $n, m \geq 0$, and
- $\mathrm{Hom}_{\Delta_+}([n],[-1]) = \emptyset$ for any $n \geq 0$, and
- $\mathrm{Hom}_{\Delta_+}([-1],[n]) = \{*\}$ for any $n \geq -1$.

Definition 5.29. An **augmented simplicial object** in $\mathcal{C}$ is a functor $X : \Delta_+ \to \mathcal{C}$ which is usually presented as follows.

$$X_{-1} \xleftarrow{\varepsilon} X_0 \underset{d_1}{\overset{d_0}{\leftleftarrows}} X_1 \; \substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow} \; \cdots$$

[18]One can ask what is the relation between $\mathbb{G}$ and $\mathbb{R}\widehat{G}$, and the answer is yes, but with some modifications.

Note given a simplicial object $X_* \in \mathrm{Ob}(\mathbf{s}\mathcal{C})$, to give an augmentation is equivalent to either give a map $\varepsilon : X_0 \to X_{-1}$ in $\mathcal{C}$ such that $\varepsilon d_0 = \varepsilon d_1$ in $\mathrm{Hom}_{\mathcal{C}}(X_1, X_{-1})$, or as a map of simplicial objects

$$\varepsilon_* : X_* \longrightarrow X_{-1} \tag{5.1}$$ {{eq:augmentation}}

where X_{-1} is viewed as a discrete simplicial set.

Example 5.30. If $\mathcal{C} = \mathbf{Set}$, there are two canonical augmentation for a simplicial set $X_* \in \mathrm{Ob}(\mathbf{sSet})$. Consider the inclusion $i : \Delta \hookrightarrow \Delta_+$, which induces a functor $i^* : \mathbf{sSet}_+ \to \mathbf{sSet}$ that has both left and right adjoint

$$\begin{tikzcd} \mathbf{sSet}_+ \arrow[d, "i^*"] \\ \mathbf{sSet} \arrow[u, bend left=60, "\pi_0"] \arrow[u, bend right=60, "triv"'] \end{tikzcd}$$

where

$$\pi_0 X = \mathrm{coeq}\ \{X_0 \underset{d_1}{\overset{d_0}{\leftleftarrows}} X_1 \Lleftarrow \cdots\}.$$

Question 5.31. *When is 5.1 a weak equivalence in* $\mathbf{s}\mathcal{C}$*?*

Definition 5.32. An **extra degeneracy** for an augmentation $\varepsilon_* : X_* \longrightarrow X_{-1}$ is given by a sequence of maps $s_{-1} : X_n \to X_{n+1}$ such that

1. $\varepsilon s_{-1} = \mathrm{id}_{X_{-1}}$,
2. $d_0 s_{-1} = \mathrm{id}_{X_0}$ for all $n \geq 0$,
3. $d_{i+1} s_{-1} = s_{-1} d_i$ for all $i \geq 0$,
4. $s_{j+1} s_{-1} = s_j s_{-1}$ for all $i \geq 0$.

Lemma 5.33. *If X is a simplicial object in $\mathcal{C}$ where $\mathcal{C}$ is cocomplete, giving an augmentation $\varepsilon_* : X_* \to X_{-1}$ is equivalent to giving a retract diagram*

$$X_{-1} \xrightarrow{s_{-1}} X_* \xrightarrow{\varepsilon_*} X_{-1}$$

so that these maps give a simplicial homotopy equivalence $X_* \simeq X_{-1}$.

Corollary 5.34. *If a simplicial object X in a simplicial model category admits an augmentation $\varepsilon_* : X_* \to X_{-1}$ with extra degeneracy, then*

$$|\varepsilon_*| : |X_*| \xrightarrow{\sim} X_{-1}$$

is a weak equivalence.

Example 5.35 (Bar construction)**.** $X = B(*, I, F)$.

For the functor $(FU)_* : \mathcal{D} \to \mathbf{s}\mathcal{D}$ associated to an adjunction (F, U), the natural augmentation $(FU)_*(A) \to A$ does not have an extra degeneracy in general.

If we apply to $(FU)_*(A)$ the functor U, then we can see the natural augmentation

$$\begin{tikzcd} UA \arrow[r, bend left] & UFU(A) \arrow[l] \arrow[r, bend left] & UFUFUA \arrow[l, shift left] \arrow[l, shift right] \arrow[r, bend left] & \cdots \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] \end{tikzcd}$$

always has an extra degeneracy given by

$$s_{-1} = \eta U(FU)^{n+1} : U(FU)^{n+1}A \longrightarrow U(FU)^{n+2}A.$$

Moral. When $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ is a forgetful adjunction such that

$$(f_* : X_* \to Y_*) \in WE(\mathbf{s}\mathcal{D}) \iff (Uf_* : UX_* \to UY_*) \in WE(\mathbf{s}\mathcal{C})$$

In this case, we have that 5.1 is a weak equivalence for any object $A \in \mathbf{s}\mathcal{D}$.

Example 5.36. Let G be a group. Consider the adjunction

$$G \times - : \mathbf{Set} \rightleftarrows \mathbf{Set}^G : U$$

Apply the above construction to the trivial G set $\{*\}$ gives us the classical bar construction of a group.

5.3 Simplicial Localizations

There are really two constructions [19]

- simplicial localization,
- hammock localization.

5.3.1 Simplicial Localization

Recall the adjunction

$$F : \mathrm{Graph} \rightleftarrows \mathbf{Cat} : U$$

where the category $F(A)$ associated to a graph A is free in the sense that every morphism in $F(A)$ can be written uniquely as a composition of atomic (i.e. it admits no nontrivial factorization, $f = f_1 \circ f_2$ implies either f_1 or f_2 is id) morphisms in A.

[19] See Dywer-Kan's series of papers.

The above construction gives us a functor

$$\begin{aligned}(\mathsf{FU})_* : \mathbf{Cat} &\longrightarrow \mathbf{sCat}\\ \mathcal{C} &\longmapsto (\mathsf{FU})_*(\mathcal{C})\end{aligned}$$

which is the simplicial resolution of $\mathcal{C}$ such that the image of every category $\mathcal{C}$ is a simplicial category $\mathbb{F}(\mathcal{C})$.

Definition 5.37 ([DK2])**.** Given a relative category $(\mathcal{C}, \mathcal{W})$, define its simplicial localization to be

$$\mathsf{L}(\mathcal{M}, \mathcal{W}) = \mathbb{F}_*(\mathcal{M})[\mathbb{F}_*(\mathcal{W})^{-1}].$$

which is achieved by the usual localization applied degreewise.

The universal property of simplicial localization is that, for any functor $\mathsf{T} : \mathcal{M} \to \mathcal{D}$ (where $\mathcal{D}$ is a simplicial model category) such that $\mathsf{T}(\mathcal{W}) \subseteq \mathcal{W}\mathsf{E}(\mathcal{D})$, there exists a unique simplicial functor $\overline{\mathsf{T}} : \mathsf{L}(\mathcal{M}, \mathcal{W}) \to \mathcal{D}$ such that we have the unique factorization

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "\mathsf{T}"] \arrow[dr, "can"'] & & \mathcal{D}. \\ & \mathsf{L}(\mathcal{W}, \mathcal{W}) \arrow[ur, "\overline{\mathsf{T}}"'] & \end{tikzcd}$$

In general, $\mathsf{L}(\mathcal{M}, \mathcal{W})$ need not to be locally small.

Definition 5.38. Let $\mathcal{S}$ be a simplicial category, we define $\pi_0(\mathcal{S})$ the **category of components** of $\mathcal{S}$ by

- $\mathsf{Ob}(\pi_0\mathcal{S}) = \mathsf{Ob}(\mathcal{S})$ and,
- $\mathrm{Hom}_{\pi_0\mathcal{S}}(x, y) = \pi_0[\mathrm{Map}_{\mathcal{S}}(x, y)]$.

Theorem 5.39 (DK)**.** $\pi_0(\mathsf{L}(\mathcal{M}, \mathcal{W})) \cong \mathcal{M}[\mathcal{W}^{-1}]$.

Consider the category $\mathcal{SC}$ of (small) simplicial categories (categories enriched in **sSet**) and simplicial functors $\mathsf{F} : \mathcal{C} \to \mathcal{D}$ such that $\mathsf{F}_{\mathsf{X},\mathsf{Y}} : \mathrm{Map}_{\mathcal{C}}(\mathsf{X}, \mathsf{Y}) \to \mathrm{Map}_{\mathcal{D}}(\mathsf{FX}, \mathsf{FY})$ is a morphisms of simplicial sets. Then we have a functor

$$\pi_0 : \mathcal{SC} \longrightarrow \mathbf{Cat}$$

There is natural functor

$$\mathrm{Hom}_{\Pi_0(\mathcal{C})}(\mathsf{X}, \mathsf{Y}) = \pi_0 \,\mathrm{Map}_{\mathcal{C}}(\mathsf{X}, \mathsf{Y}).$$

Simplicial localization solves the following universal problem.

Given a relative category $(\mathcal{M}, \mathcal{W})$, there is a (unique) simplicial category $L\mathcal{M} = L(\mathcal{M}, \mathcal{W})$ with a functor

$$\mathrm{sloc} : \mathcal{M} \longrightarrow L\mathcal{M}$$

satisfying

1. sloc induces an equivalence of categories

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\mathrm{sloc}"] \arrow[dr, "\mathrm{loc}"'] & L\mathcal{M} \arrow[r, "\pi_0"] & \pi_0 L\mathcal{M} \\
& \mathcal{M}\left[\mathcal{W}^{-1}\right] \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

2. sloc is initial among all functors $T : \mathcal{M} \to \mathcal{D}$ where $\mathcal{D}$ is a simplicial category and $\pi_0 T(f)$ is an isomorphisms in $\pi_0(\mathcal{D})$ for all $f \in \mathcal{W}$.

$$\begin{tikzcd}
\mathcal{M} \arrow[rr, "T"] \arrow[dr, "\mathrm{sloc}"'] & & \mathcal{D} \\
& L(\mathcal{W}, \mathcal{W}) \arrow[ur, "\overline{T}"'] &
\end{tikzcd}$$

5.3.2 Hammock Localization

Definition 5.40 (DK2). Given a relative category $(\mathcal{M}, \mathcal{W})$, a **hammock localization** $L^H(\mathcal{M}, \mathcal{W})$ is the simplicial category with

- $\mathrm{Ob} L^H(\mathcal{M}, \mathcal{W}) = \mathrm{Ob}(\mathcal{M}) = \mathrm{Ob}(\mathcal{W})$, and
- mapping spaces $\mathrm{Map}_{L^H(\mathcal{M},\mathcal{W})}(x, y)$ having reduced hammocks of width k and (any) length $n \geq 0$ in simplicial degree k

$$\begin{tikzcd}
& c_{0,1} \arrow[d, "\wr"] & c_{0,2} \arrow[l, "\sim"'] \arrow[r] & \cdots \arrow[r] & c_{0,n-1} & \\
& c_{1,1} \arrow[d, "\wr"] & c_{1,2} \arrow[l, "\sim"'] \arrow[r] & \cdots \arrow[r] & c_{1,n-1} & \\
x \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] & c_{k-1,1} \arrow[d, "\wr"] & c_{k-1,2} \arrow[l, "\sim"'] \arrow[r] & \cdots \arrow[r] & c_{k-1,n-1} & y \arrow[uul, "\sim"'] \arrow[ul, "\sim"'] \arrow[l, "\sim"] \arrow[dl, "\sim"] \\
& c_{k,1} & c_{k,2} \arrow[l, "\sim"'] \arrow[r] & \cdots \arrow[r] & c_{k,n-1} &
\end{tikzcd}$$

such that

1. all vertical arrows look downwards and are in $\mathcal{W}$,

2. in each column, all horizontal arrows starting at that column look in the same direction and the arrows looking to the left are weak equivalences.
3. in any adjacent columns has arrows look in opposite directions.
4. in each column, the horizontal arrows cannot all be the identity maps.

Theorem 5.41 ([DK1]). *If $(\mathcal{M}, \mathcal{W})$ is a model category, then the mapping space in $L^H(\mathcal{M}, \mathcal{W})$ can be defined only using hammocks of length 2*

$$x \longrightarrow c_{0,1} \xleftarrow{\sim} c_{0,2} \longrightarrow y$$

or

$$x \xleftarrow{\sim} c_{0,1} \longrightarrow c_{0,2} \xleftarrow{\sim} y.$$

Question 5.42. *What is the relation between $L(\mathcal{M}, \mathcal{W})$ and $L^H(\mathcal{M}, \mathcal{W})$?*

Answer. They are DK equivalent.

Definition 5.43. A simplicial functor $F : \mathcal{C} \to \mathcal{D}$ is called **Dwyer-Kan equivalence** if

DK1 F is homotopically fully faithful, i.e.

$$F_{X,Y} : \mathrm{Map}_{\mathcal{C}}(X, Y) \to \mathrm{Map}_{\mathcal{D}}(FX, FY)$$

is a weak equivalences in the Kan model structure of **sSet**.

DK2 $\pi_0 F : \pi_0 \mathcal{C} \to \pi_0 \mathcal{D}$ is an equivalence of categories.

Remark. In presence of DK1, condition (DK2) is equivalent to the following.

DK2' F is homotopically essentially surjective, i.e. $\pi_0 F : \pi_0 \mathcal{C} \to \pi_0 \mathcal{D}$ is essentially surjective.

Theorem 5.44 (DK).

1. *If $\mathcal{M}$ is a model category, with $\mathcal{W} = WE(\mathcal{M})$ being weak equivalences in $\mathcal{M}$, then $L\mathcal{M}$ and $L^H\mathcal{M}$ are DK equivalent.*
2. *If $\mathcal{M}$ is a simplicial model category, the simplicial functor $\mathcal{M} \to L^H\mathcal{M}$ is a DK equivalence.*
3. *Up to DK equivalences, every simplicial category is of the form $L(\mathcal{M}, \mathcal{W})$ for some relative category $(\mathcal{M}, \mathcal{W})$.*

Motto If we think of relative categories as abstract homotopy theories, then simplicial categories can be regarded as models for homotopy theories. Then the category $\mathcal{SC}$ of all simplicial categories with DK-equivalences as weak equivalences is a homotopy theory of homotopy theories (in place of the non-existence of model category of model categories).

Goal Put a model structure on $\mathcal{SC}$ with DK-equivalences being precisely weak equivalences.

5.3.3 Example and Application of Simplicial Localization

Example 5.45 ([DK1]). Consider the diagram of discrete group homomorphisms

$$H \xleftarrow{\alpha} G \xrightarrow{\beta} K$$

Define $\mathcal{M}$ to be the category with 3 objects X, Y, Z and

$$\begin{tikzcd} Y \arrow[loop above, "1"] & X \arrow[l, "H"'] \arrow[r, "K"] \arrow[loop above, "G"] & Z \arrow[loop above, "1"] \end{tikzcd}$$

1. $\mathrm{Hom}_{\mathcal{M}}(X, X) = G$ as a monoid,
2. $\mathrm{Hom}_{\mathcal{M}}(X, Y) = H$ as a set,
3. $\mathrm{Hom}_{\mathcal{M}}(X, Z) = K$ as a set,
4. $\mathrm{Hom}_{\mathcal{M}}(Y, Y) = \mathrm{Hom}_{\mathcal{M}}(Z, Z) = \{\mathrm{id}\}$.

with composition

$$\begin{aligned} \mathrm{Hom}_{\mathcal{M}}(X, Y) \times \mathrm{Hom}_{\mathcal{M}}(X, X) &\overset{\circ}{\to} \mathrm{Hom}_{\mathcal{M}}(X, Y) \\ (h, g) &\longmapsto h\alpha(g) \\ \mathrm{Hom}_{\mathcal{M}}(X, Z) \times \mathrm{Hom}_{\mathcal{M}}(X, X) &\overset{\circ}{\to} \mathrm{Hom}_{\mathcal{M}}(X, Z) \\ (k, g) &\longmapsto k\beta(g) \end{aligned}$$

Let $\mathcal{W}$ be the subcategory of $\mathcal{M}$ with

1. $\mathrm{Hom}_{\mathcal{M}}(X, X) = G$ as a monoid,
2. $\mathrm{Hom}_{\mathcal{M}}(X, Y) = e_H$ as a set,
3. $\mathrm{Hom}_{\mathcal{M}}(X, Z) = e_K$ as a set,
4. $\mathrm{Hom}_{\mathcal{M}}(Y, Y) = \mathrm{Hom}_{\mathcal{M}}(Z, Z) = \{\mathrm{id}\}$.

Note after taking nerve functor we get

$$\mathcal{N}_* H \xleftarrow{\mathcal{N}_* \alpha} \mathcal{N}_* G \xrightarrow{\mathcal{N}_* \beta} \mathcal{N}_* K$$

in **sSet**.

Lemma 5.46. $\mathcal{N}_*\mathcal{M} \cong \operatorname{hocolim}(\ \mathcal{N}_*H \xleftarrow{\mathcal{N}_*\alpha} \mathcal{N}_*G \xrightarrow{\mathcal{N}_*\beta} \mathcal{N}_*K \ ).$

Corollary 5.47. $\pi_1(X) \cong H *_G K = \operatorname{hocolim}(\ H \xleftarrow{\alpha} G \xrightarrow{\beta} K \ ).$

Theorem 5.48. *1.* $\mathcal{M}[\mathcal{W}^{-1}] \simeq$ *connected discrete groupoid with objects* X, Y, Z *where*

$$\mathrm{Aut}_{\mathcal{M}[\mathcal{W}^{-1}]}(X) \cong \mathrm{Aut}_{\mathcal{M}[\mathcal{W}^{-1}]}(Y) \cong \mathrm{Aut}_{\mathcal{M}[\mathcal{W}^{-1}]}(Z) \cong \pi_1(X).$$

2. $L(\mathcal{M}, \mathcal{W}) \cong$ *simplicial connected groupoid on objects* X, Y, Z *with simplicial mapping spaces*

$$\mathrm{Map}_{L(\mathcal{M})}(X,X) \simeq \mathrm{Map}_{L(\mathcal{M})}(Y,Y) \simeq \mathrm{Map}_{L(\mathcal{M})}(Z,Z) \cong \mathbb{G}X.$$

Abstract Morita Theory Given a functor $f : \mathcal{C} \to \mathcal{D}$ a morphism in **Cat**, there is an adjoint pair

$$f_! : \mathbf{Set}^{\mathcal{C}} \rightleftarrows \mathbf{Set}^{\mathcal{D}} : f^*$$

Theorem 5.49 (Borceax)**.** *The adjunction is an equivalence if and only if* f *is fully faithful and essentially surjective (up to retracts), i.e. there exists* $\mathcal{D}' \subset \mathcal{D}$ *such that* f *is essentially surjective onto* $\mathcal{D}'$*.*

Simplicial Morita Thoery Given a simplicial functor $F : \mathcal{C} \to \mathcal{D}$ which is a morphism in $\mathcal{SC}$, there is an adjoint pair

$$F_! : \mathbf{Set}^{\mathcal{C}} \rightleftarrows \mathbf{Set}^{\mathcal{D}} : F^*$$

Theorem 5.50. *The induced Quillen pair* $(F_!, F^*)$ *for the projective model structure on diagram categories is a Quillen pair if and only of*

1. F *satisfies (DK1),*
2. *F satisfies (DK2') homotopically up to retracts, i.e.* $\pi_0(F)$ *is essentially surjective up to retracts.*

Definition 5.51. If $\mathcal{C}$ is a simplicial category, $f : X \to Y$ is called a **homotopy equivalence** in $\mathcal{C}$ if $\pi_0 f$ is an isomorphism in $\pi_0\mathcal{C}$.

Theorem 5.52 ([B3])**.** *The category* $\mathcal{SC}$ *of small simplicial categories has a cofibrantly generated model category structure with* $F : \mathcal{C} \to \mathcal{D}$ *being a*

- *weak equivalences if and only if* F *is a DK equivalence,*
- *fibration if for any homotopy equivalence in* $\mathcal{D}$ *of the form*

$$f : FX_1 \xrightarrow{\sim} Y$$

there is a homotopy equivalence $g : X_1 \to X_2$ *in* $\mathcal{C}$ *such that* $F(g) = f$*.*

6 Simplicial Categories

There are Quillen equivalences between different models of ∞ -categories.

$$\begin{tikzcd} \mathcal{SC} \arrow[r, shift left] \arrow[d, shift right, "\mathfrak{C}"'] & \mathbf{SeCat} \arrow[l, shift left] \arrow[r, shift left] & \mathbf{CSS} \arrow[l, shift left] \arrow[d, shift right] \\ \mathbf{QCat} \arrow[u, shift right, "\mathfrak{N}"'] & & \mathbf{RCat} \arrow[u, shift right] \end{tikzcd}$$

QCat the category of quasi-categories.
$\mathcal{SC}$ the category of simplicial categories.
SeCat the category of Segal categories.
CSS the category of complete Segal spaces.
RCat the category of relative categories.
In this section we will focus on simplicial categories.

6.1 Simplicial Categories and Simplicial Functors

Definition 6.1. A **simplicial functor** $F : \mathcal{C} \to \mathcal{D}$ between simplicial categories is an assignment such that

- for every object X in $\mathcal{C}$, there is an assigned object FX in $\mathcal{D}$, and
- for every pair of object X, Y in $\mathcal{C}$, a map between simplicial sets $F_{X,Y}$: $\mathrm{Map}_{\mathcal{C}}(X, Y) \to \mathrm{Map}_{\mathcal{C}}(FX, FY)$ such that the following associativity diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(Y,Z) \times \mathrm{Map}_{\mathcal{C}}(X,Y) \arrow[r, "\circ_{X,Y,Z}"] \arrow[d, "F_{Y,Z}\times F_{X,Y}"'] & \mathrm{Map}_{\mathcal{C}}(X,Z) \arrow[d, "F_{X,Z}"] \\ \mathrm{Map}_{\mathcal{D}}(FY,FZ) \times \mathrm{Map}_{\mathcal{D}}(FX,FY) \arrow[r, "\circ_{FX,FY,FZ}"] & \mathrm{Map}_{\mathcal{D}}(FX,FZ) \end{tikzcd}$$

and the identity diagram

$$\begin{tikzcd} \Delta[0]_* \arrow[rr, "i_X"] \arrow[dr, "i_{FX}"'] & & \mathrm{Map}_{\mathcal{C}}(X,X) \arrow[dl, "F_{X,X}"] \\ & \mathrm{Map}_{\mathcal{D}}(FX,FX) & \end{tikzcd}$$

commutes.

Question 6.2. *Given a functor $F : \mathcal{C} \to \mathcal{D}$ on the underlying categories of simplicial categories, when does F extend to a simplicial functor $\tilde{F}$ so that $(\tilde{F}_{X,Y})_0 = F_{X,Y}$.*

$$(F_{X,Y})_0 : \mathrm{Map}_{\mathcal{C}}(X,Y)_0 \cong \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Map}_{\mathcal{D}}(FX,FY)_0 \cong \mathrm{Hom}_{\mathcal{D}}(FX,FY)$$

Assume that $\mathcal{C}, \mathcal{D}$ are tensored over **sSet**, so that

$$\mathrm{Map}_{\mathcal{C}}(K \bigotimes X, Y) \cong \mathrm{Map}_{\mathbf{sSet}}(K, \mathrm{Map}_{\mathcal{C}}(X, Y)). \tag{6.1}$$ {{eq:tensor-hom}}

Take $K = \Delta[n]_*$ and apply $(-)_0$ we see

$$\mathrm{Hom}_{\mathcal{C}}(\Delta[n]_* \bigotimes X, Y) \cong \mathrm{Hom}(\Delta[n]_*, \mathrm{Map}_{\mathcal{C}}(X, Y)) \cong \mathrm{Map}_{\mathcal{C}}(X, Y)_n$$

Hence for a functor $F : \mathcal{C} \to \mathcal{D}$ to be simplicial, we need to an n-simplex in $\mathrm{Map}_{\mathcal{C}}(X, Y)_n$, i.e. $\alpha : \Delta[n]_* \otimes X \to Y$, assign an n-simplex in $\mathrm{Map}_{\mathcal{D}}(FX, FY)$, i.e. $\Delta[n]_* \otimes FX \to FY$. In order to get such a map, we may consider

$$\Delta[n]_* \otimes F(X) \xrightarrow{\sigma_{n,X}} F(\Delta[n]_* \otimes X) \xrightarrow{F\alpha} F(Y)$$

and more generally, we need maps $\sigma_{K,X} : K \otimes F(X) \to F(K \otimes X)$ for all X in $\mathcal{C}$ and all finite simplicial sets (i.e. simplicial sets with finitely many nondegenerate simplices) K, which are natural in K and X, satisfying

$$K \otimes F(X) \xrightarrow{\sigma_{K,X}} F(K \otimes X) \longrightarrow F(Y)$$

1. for $K = \Delta[0]_*$, $\sigma_{0,X}$ is an isomorphism such that

$$\begin{tikzcd}
\Delta[0]_* \otimes FX \arrow[rr, "\cong"] \arrow[dr, "\cong"'] & & F(\Delta[0]_* \otimes X) \arrow[dl, "\cong"] \\
& F(X) &
\end{tikzcd}$$

   commutes.

2. (Associativity condition) this is needed for extension of F to preserve composition. For any K, L finite simplicial set and X in $\mathcal{C}$,

$$\begin{tikzcd}
(K \times L) \otimes FX \arrow[r, "\cong"] \arrow[d, "\sigma_{K\times L,X}"'] & K \otimes (L \otimes FX) \arrow[d, "\sigma_{K,L\otimes X}"] \\
F((K \times L) \otimes X) \arrow[r, "\cong"] & F(K \otimes (L \otimes X))
\end{tikzcd}$$

Theorem 6.3 ([H2]). *A functor $F : \mathcal{C} \to \mathcal{D}$ extends to a simplicial functor if and only if there exists $\sigma_{K,X} : K \otimes F(X) \to F(K \otimes X)$ such that the previous two conditions are satisfied.*

Remark. Such $F : \mathcal{C} \to \mathcal{D}$ are called **continuous** functors.

Example 6.4. Suppose $F : \mathcal{M} \to \mathcal{N}$ with $\mathcal{M}, \mathcal{N}$ cocomplete.

Example 6.5 (Counterexample)**.** Let $\mathcal{C} = \mathcal{D} = \textbf{Top}$. **Top** is tensored over **sSet** with

$$\begin{aligned} \textbf{sSet} \times \textbf{Top} &\longrightarrow \textbf{Top} \\ (K, X) &\longmapsto |K| \times X \end{aligned}$$

and is a simplicial category with

$$\mathrm{Map}_n(X, Y) = \mathrm{Hom}_{\textbf{Top}}(X \times \Delta^n, Y).$$

Fix $\emptyset \neq A \in \textbf{Top}$, we define a functor $F_A : \textbf{Top} \to \textbf{Top}$ by

- $F_A(X) \coloneqq \coprod_{\alpha \in \mathrm{Hom}_{\textbf{Top}}(A,X)} A^{\alpha}$, and
- $F_A(f) : \coprod_{\alpha \in \mathrm{Hom}_{\textbf{Top}}(A,X)} A^{\alpha} \to \coprod_{\beta \in \mathrm{Hom}_{\textbf{Top}}(A,Y)} A^{\beta}$.

Then F_A cannot be extended to mapping spaces.

Let $X = A$, $Y = \Delta^1 \times A$, there exists 0-simplices $i_0, i_1 \in \mathrm{Map}(X, Y)_0$

$$\begin{aligned} i_0 : A &\to \Delta^1 \times A & \quad i_1 : A &\to \Delta^1 \times A \\ a &\longmapsto (0, a) & a &\longmapsto (1, a) \end{aligned}$$

which are connected by the (identity) 1-simplex in $\mathrm{Map}_1(X, Y)$.

Chekck that $F_A(i_0)$ and $F_A)i_1)$ are in different components in

$$\mathrm{Map}(F_A(A), F_A(\Delta^1 \times A))$$

and hence cannot be connected by 1-simplex.

Question 6.6. *Suppose the functor $F : \mathcal{C} \to \mathcal{D}$ admits a simplicial extension and has a right adjoint $G : \mathcal{C} \to \mathcal{D}$, when does*

$$\mathrm{Hom}_{\mathcal{D}}(FX, Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, GY)$$

extends to an isomorphism of simplcial sets

$$\mathrm{Map}_{\mathcal{D}}(FX, Y) \cong \mathrm{Map}_{\mathcal{C}}(X, GY)?$$

Corollary 6.7. *If $\sigma_{K,X}$ are all isomorphisms for all K and X, then the answer is yes.*

Indeed,

$$\begin{aligned} \mathrm{Map}_{\mathcal{D}}(FX, Y)_n \cong \mathrm{Hom}_{\mathcal{D}}(\Delta[n]_* \bigotimes FX, Y) &\cong \mathrm{Hom}_{\mathcal{D}}(F(\Delta[n]_* \bigotimes X), Y) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(\Delta[n]_* \bigotimes X, GY) \cong \mathrm{Map}_{\mathcal{C}}(X, GY)_n \end{aligned}$$

Detecting weak equivalences (Why do we want simplicial enrichment for model categories)

Proposition 6.8 ([H2], Proposition 9.7.1)**.** *If $\mathcal{C}$ is a simplciail model category and $f : X \to Y$ is a morphsims in $\mathcal{C}$, then f is a weak equivalence if and only if one of the following holds*

1. $f^* : \mathrm{Map}(Y, Z) \xrightarrow{\sim} \mathrm{Map}(X, Z)$ *is a weak equivalence for all fibrant Z in $\mathcal{C}$.*
2. $f_* : \mathrm{Map}(A, X) \xrightarrow{\sim} \mathrm{Map}(A, Y)$ *is a weak equivalence for all cofibrant A in $\mathcal{C}$.*

6.2 Simplicial Categories

Analogy Simplicial categories are "nonlinear" analogues of DG categories.

DG categories $\mathbf{DGCat}_k$ has Tabuada's model structure, $\mathbf{Hom}(X, Y) \in \mathbf{Ch}_k$ and weak equivalences are quasi-equivalences $F : \mathcal{C} \to \mathcal{D}$ such that

1. $F_{X,Y} : \mathbf{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{\sim} \mathbf{Hom}_{\mathcal{D}}(FX, FY)$ are quasi-isomorphisms, and
2. $\mathrm{Ho}(F) : \mathrm{Ho}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Ho}(\mathcal{D})$ is an equivalence of categories.

Simplicial categories Let $\mathcal{SC}$ be the category of all small simplicial categories with morphisms being simplicial functors. Recall a **DK equivalence** is a simplicial functor $F : \mathcal{C} \to \mathcal{D}$ such that

DK1 for any $X, Y \in \mathrm{Ob}(\mathcal{C})$, $F_{X,Y} : \mathrm{Map}_{\mathcal{C}}(X, Y) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{D}}(FX, FY)$ are weak equivalences of simplicial sets, and

DK2 $\pi_0(F) : \pi_0(\mathcal{C}) \xrightarrow{\sim} \pi_0(\mathcal{D})$ is an equivalence of categories.

There is a model structure on $\mathcal{SC}$ such that DK equivalences are exactly the weak equivalences.

Definition 6.9. 1. Given a simplicial set K, we can define a simplicial categotry UK by

- $\mathrm{Ob}(UK) = \{x, y\}$, with two objects, and
- morphisms are

$$\begin{aligned} &\mathrm{Map}_{UK}(x, x) = \mathrm{Map}_{UK}(y, y) = \mathrm{id} = \Delta[0]_* \\ &\mathrm{Map}_{UK}(x, y) = K \\ &\mathrm{Map}_{UK}(y, x) = \emptyset \end{aligned}$$

This construction gives a functor $U : \mathbf{sSet} \to \mathcal{SC}$.

2. A map $e : x \to y$ is called a **homotopy equivalence** if $\pi_0 e : x \xrightarrow{\cong} y$ is an isomorphism in $\pi_0\mathcal{C}$.

6.2.1 Bergner Model Structure

Theorem 6.10 ([B3]). *There is a cofibrantly generated (left and right) proper model structure on $\mathcal{SC}$ with*

- *WE = DK equivalences,*
- *Fib = the simplicial functors $F : \mathcal{C} \to \mathcal{D}$ such that*

  F1 $F_{c_1,c_2} : \mathrm{Map}_{\mathcal{C}}(c_1, c_2) \to \mathrm{Map}_{\mathcal{D}}(Fc_1, Fc_2)$ *Kan complexes,*

  F2 (homotopy equivalences lifting property) for any object c_1 in $\mathcal{C}$ and any homotopy equivalence $e : Fc_1 \to d$ in $\mathcal{D}$, there exists an object c_2 in $\mathcal{C}$ and a homotopy equivalence $\tilde{e} : c_1 \to c_2$ in $\mathcal{C}$ such that $F(\tilde{e}) = e$.

- *a set of generating cofibrations $I \subset \mathrm{Cof}$ is given by*

  C1 the maps $U(\partial\Delta[n]_*) \to U(\Delta[n]_*)$ for all $n \geq 1$.

  C2 the trivial maps $\emptyset \hookrightarrow *$.

- *a set of generating acyclic cofibrations $J \subset \mathrm{Cof} \cap WE$ is given by*

  A1 the maps $U(\Lambda_k[n]_*) \to U(\Delta[n]_*)$ for all $n \geq 1, 0 \leq k \leq n$.

  A2 the inclusions $* \xrightarrow[DK]{\sim} \mathcal{H}$ where $\mathcal{H}$ is a simplicial category with two objects $\{x, y\}$ and all mapping spaces are weakly contractible and having only countably many simplices, and the inclusion $\{x\} \sqcup \{y\} \hookrightarrow \mathcal{H}$ is a cofibration in $\mathcal{SC}_{\{x\}\sqcup\{y\}}$.

We will introduce the model structure on simplicial categories with a fixed set of objects in the next lecture.

6.3 Simplicial Categories with fixed set of objects

Fix a set $\mathcal{O}$ and consider the category $\mathcal{SC}_{\mathcal{O}}$ of all simplicial categories $\mathcal{C}$ with $\mathrm{Ob}(\mathcal{C}) = \mathcal{O}$, and morphisms in $\mathcal{SC}_{\mathcal{O}}$ are simplicial functors $F : \mathcal{C} \to \mathcal{D}$ such that $F = \mathrm{id} : \mathrm{Ob}(\mathcal{C}) = \mathcal{O} \to \mathrm{Ob}(\mathcal{D}) = \mathcal{O}$. The data encoded in F are only simplicial maps of simplicial sets

$$F_{X,Y} : \mathrm{Map}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Map}_{\mathcal{D}}(X, Y)$$

for all $X, Y \in \mathcal{O}$.

Our goal is to define a model structure on $\mathcal{SC}_{\mathcal{O}}$ with $F : \mathcal{C} \to \mathcal{D}$ being a weak equivalence if and only if $F_{X,Y}$ are weak equivalences in **sSet** for any $X, Y \in \mathcal{O}$.

(Co)limits Note that $\mathcal{SC}$ has all finite limits and colimits. Indeed, $\mathcal{SC}$ has all coproducts and coequalizers built from these in **Set** (on objects) and **sSet** (on morphisms).

For instance, for $\mathcal{C}, \mathcal{D} \in \mathcal{SC}$, $\mathcal{C} \coprod \mathcal{D}$ is defined by

- $\mathrm{Ob}(\mathcal{C} \coprod \mathcal{D}) = \mathrm{Ob}(\mathcal{C}) \coprod \mathrm{Ob}(\mathcal{D})$, and
- morphisms are defined by

$$\mathrm{Map}_{\mathcal{C} \coprod \mathcal{D}}(x, y) = \begin{cases} \mathrm{Map}_{\mathcal{C}}(X, Y), & X, Y \in \mathrm{Ob}(\mathcal{C}) \\ \mathrm{Map}_{\mathcal{D}}(X, Y), & X, Y \in \mathrm{Ob}(\mathcal{D}) \\ \emptyset, & \text{otherwise.} \end{cases}$$

Lemma 6.11. *$\mathcal{SC}_{\mathcal{O}}$ has all limits and colimits.*

Proof. Suppose $X : I \to \mathcal{SC}_{\mathcal{O}}$ is a small diagram in $\mathcal{SC}_{\mathcal{O}}$. Let's construct colimit of X as follows.

Consider $X : I \to \mathcal{SC}_{\mathcal{O}} \hookrightarrow \mathcal{SC}$ and take $\mathrm{colim}_I X$ in $\mathcal{SC}$. Consider $\mathrm{colim}_I(X)^{\delta}$ the discrete simplicial category with the same object set as $\mathrm{colim}_I X$ but no (nonidentity) maps. We have a canonical inclusion

$$\mathrm{colim}_I(X)^{\delta} \hookrightarrow \mathrm{colim}_I(X)$$

By universal property of colimits, there is a natural folding map

$$\nabla : \mathrm{colim}_I(X)^{\delta} \to \mathcal{O}^{\delta}.$$

Define $\mathrm{colim}^{\mathcal{O}}$ as the pushout in $\mathcal{SC}$

$$\mathrm{colim}_I^{\mathcal{O}}(X) \coloneqq \mathrm{colim}\left[\ \mathcal{O}^{\delta} \longleftarrow \mathrm{colim}_I(X)^{\delta} \xhookrightarrow{can} \mathrm{colim}_I(X)\ \right]$$

This defines the required colimit in $\mathcal{SC}_{\mathcal{O}}$. (Check the UMP of colimit.) □

Similarly we can define $\lim_I^{\mathcal{O}}(X)$ as the full subcategory of $\lim_I(X)$ in $\mathcal{SC}$ with objects being the images of diagonal map

$$\Delta : \mathcal{O}^{\delta} \longrightarrow \lim_I(X)^{\delta}.$$

Free products Recall a free category $\mathbb{F}(\mathcal{C})$ on a small category $\mathcal{C}$ is defined to be $F(\mathcal{C}) = \mathbb{F}U(\mathcal{C})$, where

$$\mathbb{F} : \mathbf{Graph} \rightleftarrows \mathbf{Cat} : U$$

is the adjunction between the category of small categories and the category of graphs.

- $\mathrm{Ob}(\mathbb{F}(\mathcal{C})) = \mathrm{Ob}(\mathcal{C})$,
- $\mathrm{Mor}(\mathbb{F}(\mathcal{C})$ are sequence of composable maps.

A **free product** of in $\mathcal{SC}_{\mathcal{O}}$ is defined by

$$\mathcal{C} * \mathcal{D} = \mathrm{colim}\{\ \mathcal{C} \hookleftarrow \mathcal{O}^{\delta} \hookrightarrow \mathcal{D}\ \}$$

A simplicial functor $F : \mathcal{C} \to \mathcal{D}$ is called a **(semi)free extension** (or **free map**) if

1. for any $X, Y \in \mathcal{O}$, $F_{X,Y} : \mathrm{Map}_{\mathcal{C}}(X,Y) \hookrightarrow \mathrm{Map}_{\mathcal{D}}(X,Y)$ are monomorphisms in **sSet**.
2. in each simplicial degree $k \geq 0$, $\mathcal{D}_k \cong f(\mathcal{C}_k) * \mathbb{F}(\mathcal{B}_k)$ for some small category $\mathcal{B}_k$.
3. for all k, $\mathcal{B}_* = \{\mathcal{B}_k\}_{k \geq 0}$ give a diagram

$$\begin{aligned} (\Delta^+)^{op} &\longrightarrow \mathbf{Cat} \\ [k] &\longrightarrow \mathcal{B}_k \end{aligned}$$

   i.e. $\mathcal{B}_k$ are closed under degeneracy maps.

Definition 6.12 ([DK2]). A functor $f : \mathcal{C} \to \mathcal{D}$ is called **strong retract** of $f' : \mathcal{C} \to \mathcal{D}'$ if there is a commutative diagram

$$\begin{tikzcd} & \mathcal{C} \arrow[dl, "f"'] \arrow[d, "f'"] \arrow[dr, "f"] & \\ & \mathcal{D}' \arrow[dr, "r"] & \\ \mathcal{D} \arrow[ur, "\eta"] \arrow[rr, "\mathrm{id}_{\mathcal{D}}"'] & & \mathcal{D} \end{tikzcd}$$

in $\mathcal{SC}_{\mathcal{O}}$.

Theorem 6.13 ([DK2]). *The category $\mathcal{SC}_{\mathcal{O}}$ has a natural proper simplicial model structure with*

- *WE = simplicial functors $F : \mathcal{C} \to \mathcal{D}$ giving weak equivalences of mapping complexes, i.e. $F_{X,Y} \in \mathrm{WE}(\mathbf{sSet})$ for any $X, Y \in \mathcal{O}$.*
- *Fib = simplicial functors $F : \mathcal{C} \to \mathcal{D}$ giving Kan fibrations of mapping complexes*
- *Cof = simplicial functors $F : \mathcal{C} \to \mathcal{D}$ which are strong retracts of free extensions.*

Remark. When $\mathcal{O} = \{*\}$, $\mathcal{SC}_{\{*\}} = \mathbf{sMon}$, and when $\mathcal{O} = \{x, y\}$, $\mathcal{SC}_{\mathcal{O}} = \mathcal{SC}_{\{x,y\}}$.

Remark. $\mathcal{SC}$ was defined by Quillen in 1967 to study equivariant homotopy theory. Boardman and Vogt [BV] also used it to study homotopy invariant structures of spaces.

Theorem 6.14 ([B3])**.** *1. Fibrant objects in $\mathcal{SC}$ are precisely the simplicial categories $\mathcal{C}$ with all $\mathrm{Map}_{\mathcal{C}}(-,-)$ being Kan complexes.*

2. Cofibrant objects are the so-called "simplicial computads".

Remark. Let $\mathcal{C} \in \mathbf{Cat}_{\mathcal{O}}$ be an ordinary category, it can be viewed as a (discrete) simplicial category in $\mathcal{SC}_{\mathcal{O}}$. $\mathcal{C}_*$ is not a cofibrant category but it has a canonical resolution. More explicitly, we apply comonad resolution $F_*(C_*) = \mathbb{F}U_{*+1}(\mathcal{C}_*) \in \mathbf{ssCat}$ and take diagonal $Q_*\mathcal{C} = \mathrm{diag}F_*(\mathcal{C}_*)$.

6.4 Properties of Simplicial Categories

Theorem 6.15 (Lurie)**.** *There is a Quillen equivalence*

$$\mathbb{C} : \mathbf{sSet} \rightleftarrows \mathcal{SC} : \mathbb{N}.$$

Analogue of $\mathcal{SC}$ and $\mathtt{dgCat}_k$

1. $\mathcal{SC}$ consists of small categories enriched over **sSet**,
   - mapping space $\mathrm{Map}_{\mathcal{C}}(x, y)$,
   - WE = DK equivalences
   - Bergner model structure (2007)
2. $\mathbf{dgCat}_k$ consists of small categories enriched over chain complexes.
   - chain complexes $\mathrm{Hom}_{\mathcal{C}}(x, y)$
   - WE = quasi-siomorphisms
   - Takana model structure (2005)

In case of DG categories, there is a (2-object) square-zero functor

$$E : \mathbf{Ch}_k \longrightarrow \mathbf{dgCat}_k$$

$$V \longmapsto EV = \begin{bmatrix} \underset{x}{\bullet} \to \underset{y}{\bullet} \end{bmatrix} \cong \begin{bmatrix} k & V \\ 0 & k \end{bmatrix}.$$

thus we have a short exact sequence

$$0 \longrightarrow V \longrightarrow EV \longrightarrow k \times k \longrightarrow 0$$

which gives a square-zero extension.

Fibrant Objects in $\mathcal{SC}$

Proposition 6.16 ([B4]). *The fibrant objects in $\mathcal{SC}$ are precisely the simplicial categories $\mathcal{C}$ will all $\mathrm{Map}_{\mathcal{C}}(-,-)$ being Kan complexes (i.e. fibrant in* **sSet***).*

Proof. Recall fibrations are the morphisms with right lift property with respect to generating acyclic cofibrations, and fibrant objects are the simplicial categories so that the unique map $\mathcal{C} \to *$ is a fibration.

Observe that the lift

$$\begin{tikzcd} U(\Lambda_k[n]) \arrow[r] \arrow[d, hook] & \mathcal{C} \\ U(\Delta[n]) \arrow[ur, dashed] & \end{tikzcd}$$

exists for all $n \geq 0$ if and only if the lift

$$\begin{tikzcd} \Lambda_k[n] \arrow[r] \arrow[d, hook] & \mathrm{Map}_{\mathcal{C}}(x,y) \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

exists in **sSet** for any $x, y \in \mathrm{Ob}(\mathcal{C})$, if and only if $\mathrm{Map}_{\mathcal{C}}(x,y)$ is fibrant for any $x, y \in \mathrm{Ob}(\mathcal{C})$.

Condition A2 holds automatically because $\{x\} \xrightarrow{\sim} \mathcal{H}$ has a (unique) projection $p : \mathcal{H} \to \{x\}$, so for any $f : \{x\} \to \mathcal{C}$ there is a lift as follows.

$$\begin{tikzcd} \{x\} \arrow[r, "f"] \arrow[d, hook, "\sim"', shift right] & \mathcal{C} \\ \mathcal{H} \arrow[u, bend left, "p"] \arrow[ur, "fp"'] & \end{tikzcd}$$

$\square$

Cofibrant Objects in $\mathcal{SC}$ Embed $\mathcal{SC} \hookrightarrow$ **sCat** so that every object $\mathcal{C}$ in $\mathcal{SC}$ is viewed as

$$\mathcal{C}_* = \{ \ \mathcal{C}_0 \leftleftarrows \mathcal{C}_1 \ \substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow} \ \mathcal{C}_2 \cdots \ \}$$

with

- constant set of objects $\mathrm{Ob}(\mathcal{C}_n) = \mathrm{Ob}(\mathcal{C})$.
- $\mathrm{Hom}_{\mathcal{C}_n}(x,y) \coloneqq \mathrm{Map}_{\mathcal{C}}(x,y)_n$.

We call $f \in \mathrm{Hom}_{\mathcal{C}_n}(x,y)$ an n**-arrow** in $\mathcal{C}$. We will think of $\mathcal{C}_*$ as a right module over Δ.

Recall a morphism in a category $\mathcal{D}$ is **atomic** if $f = f' \circ f''$ implies either $f' = \mathrm{id}$ or $f'' = \mathrm{id}$.

Lemma 6.17. *If $\mathcal{C}$ is a simplicial category, for any n-arrow $f \in \mathrm{Hom}_{\mathcal{C}_n}(x, y)$ can be factored uniquely as*

$$f = f' \circ \alpha$$

for some $\alpha : [n] \to [m] \in \Delta^+$ and f' an atomic m-arrow.

Definition 6.18. A **simplicial computad** is a simplicial category with properties

P1 each C_n is a free category generated by atomic n-arrows,

P2 for any $\alpha \in \Delta^+$, f is an atomic m-arrow if and only if $f \circ \alpha$ is an atomic n-arrow.

Proposition 6.19. *Cofibrant objects in $\mathcal{SC}$ are precisely simplicial computads.*

Key fact. Simplicial computads are closed under retracts.

Proposition 6.20 ([B3])**.** *$\mathcal{SC}$ is right proper.*

Proposition 6.21 (Lurie)**.** *$\mathcal{SC}$ is left proper.*

This property is important for localization.

Deficiency

1. $\mathcal{SC}$ is not (known to be) a simplicial model category, i.e. (MC7) fails.

2. $\mathcal{SC}$ is not a cartesian monoidal model category. For any cofbration $f : A \hookrightarrow A'$ and $g : B \hookrightarrow B'$,

$$A \times B' \coprod_{A \times B} A' \times B \hookrightarrow A' \times B'$$

is not necessarily a cofibration. For instance, $f : \varnothing \hookrightarrow [1]$ and $g : \varnothing \hookrightarrow [1]$ are cofibrant, but $\varnothing \to [1] \times [1]$ is not.

3. Colimits in $\mathcal{SC}$ are very difficult.

7 Segal Categories and Complete Segal Spaces

7.1 Segal Spaces

7.1.1 Category Γ and Γ-spaces and Δ-spaces

The main reference for this section is [S2].

Question 7.1. *How to characterize spaces equipped with homotopy invariant (algebraic) structures?*

Example 7.2. Fix a pointed (homotopy type of) space $A \in \mathbf{Sp}_*$. Consider

$$\begin{aligned} \mathrm{Map}_*(A,-) : \mathrm{Ho}(\mathbf{Sp}_*) &\longrightarrow \mathrm{Ho}(\mathbf{Sp}_*) \\ X &\longmapsto \mathrm{Map}_*(A,X) \end{aligned}$$

A Given $X \in \mathrm{Ho}(\mathbf{Sp}_*)$, does there exist $Y \in \mathrm{Ho}(\mathbf{Sp}_*)$ such that $X \simeq \mathrm{Map}_*(A,Y)$? For example, for $A = \mathbb{S}^n$, $\mathrm{Map}_*(\mathbb{S}^n, Y) = \Omega^n(Y)$ is the n-fold loop space.

B Given a criterion in (A), how can we recover Y from $X \simeq \mathrm{Map}_*(A,Y)$?

{eg:segal}

Example 7.3 (Segal)**.** How to characterize infinite loop space?
Recall a space X is an infinite loop space if $X \cong E_0$, where is an Ω-spectrum $\{E_i, \sigma_i : E_i \xrightarrow{\simeq} \Omega E_{i+1}\}_{i\geq 0}$ such that $X \simeq E_0$. The category of infinite loop spaces has morphisms the homotopy classes of maps $f_0 : E_0 \to D_0$ such that there exists $f_i : E_i \to D_i$ such that the following diagram commutes.

$$\begin{tikzcd} E_i \arrow[r, "\sigma_i"] \arrow[d, "f_i"'] & \Omega E_{i+1} \arrow[d, "\Omega f_{i+1}"] \\ D_i \arrow[r, "\sigma_i"] & \Omega D_{i+1} \end{tikzcd}$$

General Approach Find a small (simplicial) category $\mathcal{C}$ (usually moniodal, with a distinguished object 1) together with a functor

$$\underline{X} : \mathcal{C} \longrightarrow \mathbf{Sp}_*$$

with the property that X causes a desired structure if and only if $\underline{X}(1) \simeq X$.

Let's look at example 7.3.

Category Γ We define Γ^{op} as the full subcategory of the category $\mathbf{Fin}_*$ of pointed finite sets with

- objects $n_+ = \{0, 1, \cdots, n\}$ where 0 is the base point, and

- morphisms are maps $f : n_+ \to m_+$ of sets which preserve base points, i.e. $f(0) = 0$.

Segal's original definition of Γ is the category with objects representatives of iso-classes of all finite sets and maps

$$\mathrm{Hom}_\Gamma(S, T) = \{\Theta : S \to \mathcal{P}(T) \text{ such that } \Theta(\alpha) \cap \Theta(\beta) = \emptyset, \text{ whenever } \alpha \neq \beta\}$$

where $\mathcal{P}(T)$ is the power set of T.

Exercise 7.4. 1. Check that Γ is indeed the opposite of Γ^{op}.

2. There is a different description of Γ^{op}. Let $\mathbf{Fin}'$ be the category of finite sets with morphisms being partially defined maps $f : X \to Y$ in $\mathrm{Mor}(\mathbf{Fin}')$ is a given by a pair $(D(f), f : D(f) \to Y)$ where $D(f) \subseteq X$ and f is an honest map between sets.

3. Composition is given by

$$g \circ f = (D(gf) = f^{-1}(D(g)), g \circ f : D(gf) \to Z).$$

There is a natural functor

$$\begin{aligned} D : \Gamma^{op} &\longrightarrow \mathbf{Fin}' \\ n_+ &\longmapsto \underline{n} = \{1, 2, \cdots, n\} = n_+ \backslash \{0\} \\ (f : n_+ \to m_+) &\longmapsto (D(f), f : D(f) \to \underline{m}) \end{aligned}$$

where $D(f) = \{x \in \underline{n}, f(x) \neq 0\} \subseteq \underline{n}$.

Lemma 7.5 (Segal). *D is an equivalence of categories.*

Proof. The inverse functor is

$$\begin{aligned} (-)_+ : \mathbf{Fin}' &\longrightarrow \mathbf{Fin}_* \xleftarrow{\simeq} \Gamma^{op} \\ X &\longmapsto X_+ = X \cup \{*\} \\ (f : X \to Y) &\mapsto (f_+ : X_+ \to Y_+) \end{aligned}$$

where

$$f_+(x) = \begin{cases} f(x), & x \in D(f), \\ *, & x \notin D(f). \end{cases}$$

□

Consider in Γ^{op} the following map

$$\varphi_{n,k} : n_+ \longrightarrow 1_+ \qquad (1 \leq k \leq n)$$

where

$$\varphi_{n.k}(i) = \begin{cases} 0, & i = 0, \\ 1, & i = k, \\ 0, & i \neq 0, k. \end{cases}$$

In Γ this gives us maps

$$\varphi^o_{n,k} : 1^o_+ \longrightarrow n^o_+, \qquad (1 \leq k \leq n)$$

or equivalently,

$$\varphi^o_n \coloneqq \coprod_{k=1}^{n} \varphi^o_{n,k} : \coprod_{k=1}^{n} 1^o_+ \longrightarrow n^o_+$$

called **Segal map**.

Definition 7.6. A **Γ-space** is a functor $\underline{X} : \Gamma^{op} \to \mathbf{sSet}$ such that

S1 $\underline{X}(1) \cong \Delta[0]_*$ is an isomorphism in **sSet**.

S2 $\underline{X}(\varphi_n) \coloneqq \prod_{k=1}^{n} \underline{X}(\varphi_{n,k}) : \underline{X}(n) \xrightarrow[\sim]{\underline{X}(1)^n}$ are weak equivalences for all $n \geq 2$.

If $\underline{X}(\varphi_n)$ happen to be isomorphisms for all $n \geq 2$, then $\underline{X}$ is called a **strict Γ-space**.

Remark. In [S1], (S1) is a weak equivalence.

{thm:sAbMon}

Theorem 7.7. *The category of all strict Γ-spaces is equivalent to* **sAbMon** *the category of simplicial abelian monoids.*

Theorem 7.8 ([S1])**.** *The category of all Γ-spaces is equivalent to the category of infinite loop spaces.*

Question 7.9. *Take some algebraic category $\mathcal{A}$ (e.g. $\mathcal{A} =$* **Mon**, **Gr**, **Ab***), is there any analogue of Theorem 7.7?*

Analogy [Algebraic Goemtry] Let X be a smooth projective algebraic variety (e.g. $X =^{\mathbf{n}}_{\mathbf{k}}, \mathbf{n} \geq \mathbf{1}$), $\mathbf{Coh}(X)$ be the category of coherent sheaves on X and $\mathcal{D}^b(X) = \mathrm{Ho}(\mathbf{Ch}^b(\mathbf{Coh}(X)))$.

Theorem 7.10 (Serre). *Let* $A = \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X(n))$,

$$\begin{aligned} \mathbf{Coh}(X) &\xleftarrow{\sim} \mathbf{grMod}(A)/\mathbf{FDim}(A) =: \mathbf{Proj}(A) \\ \mathcal{M} = \tilde{M} &\longleftarrow\!\!\!\!\!\!\!\longmapsto M \end{aligned}$$

Theorem 7.11 ([B1]). *There is an equivalence of triangulated categories*

$$\mathcal{D}^b(\mathbf{n}) \xrightarrow{\sim} \mathcal{D}^b(\mathrm{Rep}^{fd}(Q_n^{op}))$$

where Q_n *is the Beilinson quiver*

$$Q_n = \{ \underset{0}{\bullet} \underset{x_{n+1}^{(0)}}{\overset{x_1^{(0)}}{\rightrightarrows}} \underset{1}{\bullet} \underset{x_{n+1}^{(1)}}{\overset{x_1^{(1)}}{\rightrightarrows}} \cdots \underset{x_{n+1}^{(n-2)}}{\overset{x_1^{(n-2)}}{\rightrightarrows}} \underset{n-1}{\bullet} \underset{x_{n+1}^{(n-1)}}{\overset{x_1^{(n-1)}}{\rightrightarrows}} \underset{n}{\bullet} \}$$

with the relations

$$x_i^{(l+1)} x_j^{(l)} = x_j^{(l+1)} x_i^{(l)}, \quad 1 \le i, j \le n, 0 \le l \le n-2.$$

Connection between Δ and Γ There is a natural functor defined by

$$\begin{aligned} \theta : \Delta &\longrightarrow \Gamma \\ [n] &\longmapsto \underline{n} = \{1, \cdots, n\} \\ (f : [m] \to [n]) &\mapsto (\theta(f) : \underline{m} \to \underline{n}) \end{aligned}$$

where

$$\theta(f)(i) = \{j \in \underline{n} | f(i-1) \le j \le f(i)\}.$$

Lemma 7.12. $\theta : \Delta \to \Gamma$ *maps the following Segal maps*

$$\begin{aligned} \alpha^{n,k} : [1] &\to [n], \qquad 0 \le k \le n-1 \\ 0 &\longmapsto k, \\ 1 &\longmapsto k+1. \end{aligned}$$

in Δ *to the Segal maps in* Γ,

$$\theta(\alpha^{n,k}) = \varphi^o_{n,k}.$$

Check.

7.2 Segal Spaces

Recall in Δ, we defined

$$\begin{aligned} \alpha^i : [1] &\to [n], \qquad 0 \le i \le n-1 \\ 0 &\longmapsto i, \\ 1 &\longmapsto i+1. \end{aligned}$$

These give us maps of simplicial sets

$$\begin{aligned} \alpha^i_* : \Delta[1]_* &\to \Delta[n], \qquad 0 \le i \le n-1 \\ f &\longmapsto \alpha^i \circ f. \end{aligned}$$

Define for all $n \geq 0$, the simplicial sets

$$G(n) = \bigcup_{i=0}^{n} \alpha^i_*[1] \subseteq \Delta[n]_*.$$

More conceptually,

$$G(n) = \Delta[1]_* \coprod_{\Delta[0]_*} \Delta[1]_* \coprod_{\Delta[1]_*} \cdots \coprod_{\Delta[0]_*} \Delta[1]_*$$

where RHS is the colimit of representables for

$$[1] \xleftarrow{d^0} [0] \xrightarrow{d^1} [1] \xleftarrow{d^0} \cdots \xrightarrow{d^1} [1]\,.$$

Example 7.13. $G(0) = \Delta[0]_*$ and $G(1) = \Delta[1]_*$.

Recall $\Delta[n]_k = \{(j_0, \cdots, j_k), 0 \le j_0 \le \cdots \le j_k \le n\}$. When $n = 2$,

$$\begin{aligned} \alpha^0 : [1] &\to [2] & \alpha^1 : [1] &\to [2] \\ 0 &\longmapsto 0 & 0 &\longmapsto 1 \\ 1 &\longmapsto 1 & 1 &\longmapsto 2 \end{aligned}$$

so

$$\begin{aligned} \alpha^0_* : \Delta[1]_* &\longrightarrow \Delta[2]_* & \alpha^1_* : \Delta[1]_* &\longrightarrow \Delta[2]_* \\ (0, \cdots, 0, 1, \cdots, 1) &\mapsto (0, \cdots, 0, 1, \cdots, 1) & (0, \cdots, 0, 1, \cdots, 1) &\mapsto (1, \cdots, 1, 2, \cdots, 2) \end{aligned}$$

and

$$G(2)_k = \{(0, \cdots, 0, 1, \cdots, 1)\} \bigcup \{(1, \cdots, 1, 2, \cdots, 2)\}$$

contains $2(k+2) - 1$ elements. In fact

$$G(2)_* \cong \Lambda_1[2]_*$$

but for $n \geq 2$, they are different.

To sum up, we have natural inclusions in **sSet**

$$S \coloneqq \{\alpha : G(n)_* \hookrightarrow \Delta[n]_*\}_{n\geq 2}.$$

For any $K \in$ **sSet**, these maps induce

$$K_n = \mathrm{Hom}_{\mathbf{sSet}}(\Delta[n]_*, K) \longrightarrow \mathrm{Hom}_{\mathbf{sSet}}(G(n)_*, K) \cong K_1 \times_{K_0} K_1 \times \cdots \times_{K_0} K_1.$$

Recall $\mathbf{sSp} = \mathbf{sSet}_\Delta$ is the category of simplicial spaces (i.e. bisimplicial sets). There are two natural embeddings

1. constant

$$\begin{aligned} c : \mathbf{sSet} &\to \mathbf{sSp} \\ X &\longmapsto c(X) \end{aligned} \tag{7.1}$$ {eq:tensor-hom}

where

$$c(X) = \{X \Lleftarrow X \Lleftarrow \cdots\}$$

are the **constant simplicial spaces**.

2. transpose

$$\begin{aligned} (-)^t : \mathbf{sSet} &\to \mathbf{sSp} \\ X &\longmapsto X^t \end{aligned} \tag{7.2}$$

where

$$(X)^t = \{c(X_0) \leftleftarrows c(X_1) \Lleftarrow \cdots\}$$

are called **discrete simplicial spaces**.

Definition 7.14. Given $W \in$ **sSp**, we have natural maps of simplicial sets,

$$\Phi_n : W_n = \mathrm{Map}(\Delta[n]^t_*, W) \longrightarrow \mathrm{Map}(G(n)^t_*, W) = W_1 \times_{W_0} \times \cdots \times_{W_0} W_1$$

called **Segal maps**.

Example 7.15. Recall there is a natural functor

$$\mathbf{s}\mathcal{N} : \mathcal{SC} \to \mathbf{sSp}$$

called the simplicial nerve. This functor factors through

$$\begin{aligned} \mathcal{SC} &\hookrightarrow \mathbf{sCat} \\ C &\longrightarrow C_* \end{aligned}$$

where

$$C_* = \{C_0 \leftleftarrows C_1 \Lleftarrow \cdots\}$$

is a simplicial category with $\mathrm{Ob}(C_n) = \mathrm{Ob}(C)$ and $\mathrm{Mor}(C_n) = \mathrm{Map}(-,-)_n$.
Define $\mathbf{s}\mathcal{N}(\mathcal{C})_{*n} = \mathcal{N}(C_n)_*$ for $\mathcal{C} \in \mathcal{SC}$. That is,

$$\begin{aligned} \mathbf{s}\mathcal{N}(\mathcal{C})_n &= \underbrace{\mathrm{Mor}(C) \times_{\mathrm{Ob}(C)} \times \cdots \times_{\mathrm{Ob}(C)} \mathrm{Mor}(C)}_{n} \\ &\cong \underbrace{\mathbf{s}\mathcal{N}(\mathcal{C})_1 \times_{\mathbf{s}\mathcal{N}(\mathcal{C})_0} \cdots \times_{\mathbf{s}\mathcal{N}(\mathcal{C})_0} \mathbf{s}\mathcal{N}(\mathcal{C})_1}_{n}. \end{aligned}$$

For $n \geq 2$, the Segal maps are isomorphisms.

Definition 7.16 (Rezk)**.** A simplicial space W is called a **Segal space** if

1. all $\varphi_n : W_n \xrightarrow{\sim} W_1 \times_{W_0} \cdots \times_{W_0} W_1$ are weak equivalences in **sSet** for $n \geq 2$.

2. W is Reedy fibrant. [20]

Remark. 1. A discrete simplicial space $W = X^t$ is Segal if $X = \mathcal{N}(\mathcal{C})$ for some small category $\mathcal{C}$.

2. A Segal space W is **strict** if all Φ_n are isomorphisms. Strict Segal spaces are exactly simplicial nerves of small simplicial categories, i.e. $W \cong \mathbf{s}\mathcal{N}(\mathcal{C})$ for some $\mathcal{C} \in \mathrm{Ob}(\mathcal{SC})$.

3. Rezk nerves (classifying diagrams of small categories)

$$\mathcal{N}^R : \mathbf{Cat} \longrightarrow \mathbf{sSp}$$

$$\mathcal{C} \longrightarrow \mathcal{N}(\mathrm{Iso}(\mathcal{C}^{[n]}))$$

are Segal spaces.

Theorem 7.17 (Rezk)**.** *On* **sSp** *there is a simplicial left proper combinatorial model structure* **SeSp** *with*

- *WE = maps* $f : X \to Y$ *such that* $f^* : \mathrm{Map}(Y, W) \xrightarrow{\sim} \mathrm{Map}(X, W)$ *are weak equivalences of simplicial sets for every Segal space* W.

- *Cof = monomorphisms* $X \hookrightarrow Y$. [21]

Proof. We define $\mathbf{SeSp} = \mathcal{L}_S(\mathbf{sSp}^{\mathrm{Reedy}})$ to be the left Bousfield localization of the Reedy model structure with respect to the set of maps $S = \{G(n)^t_* \hookrightarrow \Delta[n]^t_*\}$. ☐

[20] This ensures some natural constructions are homotopy invariant.
[21] Thus **SeSp** is cofibrant model structure.

7.2.1 Segal Spaces As Categories Up To Homotopy

We want to mimic in Segal spaces the structure of simplicial categories.

Idea Segal categories are "categories up to homotopy".

Definition 7.18 (Rezk)**.** Let W be a Segal space,

1. $\mathrm{Ob}(W) = W_{0,0} = (W_0)_0$ the 0-simplices in W_0.
2. Given $x, y \in \mathrm{Ob}(W)$, define the mapping space $\mathbf{map}_W(x, y)$ to be the pullback in **sSet**

$$\begin{tikzcd} \mathbf{map}_W(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & W_1 \arrow[d, "d_0 \times d_1"] \\ * = \{(x,y)\} \arrow[r, hook] & W_0 \times W_0. \end{tikzcd}$$

   or equivalently, this is the homotopy fiber product
3. for any $x \in \mathrm{Ob}(W)$, define $\mathrm{id}_x = s_0(x) \in \mathbf{map}_w(x, x)_0$, where $s_0 : W_0 \to W_1$.
4. next, we should define composition map.

Remark. W is Reedy fibrant implies that $d_0 \times d_1 : W_1 \to W_0 \times W_0$ is a fibration in **sSet**. This implies that $\mathbf{map}_W(x, y)$ is homotopy invariant, and therefore the homotopy type $[\mathbf{map}_W(x, y)]$ of the simplicial set $\mathbf{map}_W(x, y)$ depends only on the homotopy types $[x]$, $[y]$ in $\pi_0(W)$. So we can write

$$\mathbf{map}_W(x, y) \simeq \mathrm{hofib}_{x,y}(p_2).$$

Notation. We can also write $\mathbf{map}_w(x, y) = \{x\} \times^h_{W_0} W_1 \times^h_{W_0} \{y\}$.

Recall the coskeleton tower for $N \geq 0$, $i_N : \Delta_{\leq N} \hookrightarrow \Delta$ induces

$$\mathbf{sSp} = \mathrm{Fun}(\Delta^{op}, \mathbf{sSet}) \xrightarrow{(i_N)^*} \mathbf{s}_{\leq N}\mathbf{Sp} = \mathrm{Fun}(\Delta^{op}_{\leq N}, \mathbf{sSet})$$

which has both left adjoint $(i_N)_!$ and right adjoint $(i_N)_*$,

$$\begin{tikzcd} \mathbf{sSp} \arrow[d, "(i_N)^*" description] \\ \mathbf{s}_{\leq N}\mathbf{Sp} \arrow[u, bend left=50, "(i_N)_!"] \arrow[u, bend right=50, "(i_N)_*"'] \end{tikzcd}$$

and $\mathrm{cosk}_N(W) = (i_N)_*(i_N)^*(W)$ and we have a canonical map $W \to \mathrm{cosk}_N(W)$.

Take $N = 0$, we have $\mathrm{cosk}_0(W)_n = W_0^{n+1}$. Define $p_n : W_n \to \mathrm{cosk}_0(W)_n$. In particular, for $n = 1$, $p_1 = d_1 \times d_1 : W_1 \to W_0 \times W_0$.

Definition 7.19. For $(x_0, \cdots, x_n) \in \mathrm{Ob}(W)$, define

$$\mathbf{map}_W(x_0, \cdots, x_n) = \mathrm{hofib}_{(x_0,\cdots,x_n)}(p_n)$$

$$\begin{tikzcd}
\mathbf{map}_W(x_0, \cdots, x_n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & W_n \arrow[d, "p_n"] \\
\{(x_0, \cdots, x_n)\} \arrow[r, hook] & W_0^{n+1}
\end{tikzcd}$$

Again, W is Reedy fibrant and p_n is a Kan fibration.

Observe that p_n factors as

$$\begin{tikzcd}
W_n \arrow[rr, "p_n"] \arrow[dr, "\varphi_n"', "\sim"] & & W_0^{n+1} \\
& W_1 \times_{W_0} \cdots \times_{W_0} W_1 \arrow[ur, "\overline{p_n}"'] &
\end{tikzcd}$$

which gives us map of fibration sequences

$$\begin{tikzcd}
\mathbf{map}_W((x_0, \cdots, x_n) \arrow[r] \arrow[d, "\overline{\varphi_n}"', "\sim"] & W_n \arrow[r, "p_n"] \arrow[d, "\varphi_n", "\sim"'] & W_0^{n+1} \arrow[d, equal] \\
\mathbf{map}_W(x_0, x_1) \times \cdots \times \mathbf{map}_W(x_{n-1}, x_n) \arrow[r] & W_1 \times_{W_0} \cdots \times_{W_0} W_1 \arrow[r, "\overline{p_n}"] & W_0^{n+1}
\end{tikzcd}$$

thus for all $n \geq 2$, we have natural maps

$$\overline{\varphi_n} : \mathbf{map}_W((x_0, \cdots, x_n) \xrightarrow{\sim} \mathbf{map}_W(x_0, x_1) \times \cdots \times \mathbf{map}_W(x_{n-1}, x_n).$$

In particular, for $n = 2$ and any $x, y, z \in \mathrm{Ob}(W)$, we have maps

$$\overline{\varphi_2} : \mathbf{map}_W((x, y, z) \xrightarrow{\sim} \mathbf{map}_W(x, y) \times \mathbf{map}_W(y, z).$$

In addition, we have projection map

$$\begin{tikzcd}
\mathrm{map}_W((x, y, z) \arrow[r] \arrow[d, "d_1"', "\sim"] & W_2 \arrow[r, "p_2"] \arrow[d, "d_1", "\sim"'] & W_0^3 \arrow[d, "pr_1 \times pr_2"] \\
\mathrm{map}_W(x, z) \arrow[r] & W_1 \arrow[r, "\overline{p_1}"] & W_0 \times W_0.
\end{tikzcd}$$

Definition 7.20. Given $f \in \mathbf{map}_W(x, y)$ and $g \in \mathbf{map}_W(x, y)$, define

$$g \circ f = d_1(\varphi_2^{-1}(f, g)).$$

Note since φ_2 is a weak equivalence, not an isomorphism, this composition is only unique up to homotopy.

Definition 7.21. Given $f, g \in \mathbf{map}_W(x, y)$ we say $f \sim g$ in $\mathbf{map}_W(x, y)$ if $[f] = [g] \in \pi_0 \mathbf{map}_W(x, y)$, i.e. f, g are in the same component of $\mathbf{map}_W(x, y)$.

{lem:htpy}

Lemma 7.22. *Given $f \in \mathbf{map}_W(x, y)$, $g \in \mathbf{map}_W(y, z)$, and $h \in \mathbf{map}_W(z, w)$, then the composition map satisfies*

1. *homotopy associativity* $h \circ (g \circ f) \sim (h \circ g) \circ f$.
2. *homotopy identity* $f \sim f \circ \mathrm{id}_x \sim \mathrm{id}_y \circ f$.

Definition 7.23. The **homotopy category** $\mathrm{Ho}(W)$ is defined by

- $\mathrm{Ob}(\mathrm{Ho}(W)) = \mathrm{Ob}(W)$ and
- $\mathrm{Hom}_{\mathrm{Ho}(W)}(x, y) = \pi_0(\mathbf{map}_W(x, y))$.

We call $f \in \mathbf{map}_W(x, y)$ a **homotopy equivalence** in W if $[f]$ is an isomorphisms in $\mathrm{Ho}(W)$.

Notice f is a homotopy equivalence if there exists $h, g \in \mathbf{map}_W(y, x)_0$ such that $f \circ g \sim \mathrm{id}_y$ and $h \circ f \sim \mathrm{id}_x$. Furthermore, lemma 7.22 implies

$$g \sim \mathrm{id}_x \circ g \sim (h \circ f) \circ g \sim h \circ (f \circ g) \sim h \circ \mathrm{id}_y \sim f.$$

Lemma 7.24. *If $[f] = [g]$ in $\mathrm{Ho}(W)$, then f is a homotopy equivalence if and only if g is.*

Question 7.25. *How to characterize homotopy equivalences?*

Take a simplicial set

$$Z[3] = \mathcal{N}\left\{ 0 \longrightarrow 2 \longleftarrow 1 \longrightarrow 3 \right\}.$$

Notice that $Z(3) \hookrightarrow \mathcal{N}(\overrightarrow{3}) = \Delta[3]_*$, so $Z[3]^t \hookrightarrow \Delta[3]^t$ in $\mathbf{sSp}$.

$$\begin{tikzcd}
\mathrm{Map}(\Delta[3]^t, W) \arrow[r, two heads, "\pi"] \arrow[d, "\cong"'] & \mathrm{Map}(Z[3]^t, W) \arrow[d, "\cong"] \\
W_3 & \lim\{W_1 \xrightarrow{d_1} W_0 \xleftarrow{d_1} W_1 \xrightarrow{d_1} W_0 \xleftarrow{d_1} W_1\}
\end{tikzcd}$$

Answer: (check) f is a homotopy equivalence if and only if $(\mathrm{id}_x, f, \mathrm{id}_y)$ lifts along π to W_3.

7.3 Complete Segal Spaces

Definition 7.26. The **space of homotopy equivalences** in W is $W_{\mathrm{heq}} \subseteq W_1$ consisting of components with 0-simplices being homotopy equivalences

$$W_{\mathrm{heq}} = \{f \in W_1 : f_0 \in \mathbf{map}_W(x,y)_0 \text{ such that } [f_0] \in \mathrm{Iso}(\mathrm{Ho}(W))\}.$$

Notice that for any $x \in \mathrm{Ob}(W)$, $\mathrm{id}_x = s_0(x) \in W_{\mathrm{heq}} \subseteq W_1$.

Definition 7.27. W is called a **complete Segal space** if $s_0 : W_0 \xrightarrow{\sim} W_{\mathrm{heq}}$ is a weak equivalence in **sSet**.

The map s_0 can be equivalently defined as follows.
Let $I = \{0 \rightleftarrows 1\}$, then the natural inclusion $\overrightarrow{1} \hookrightarrow I$ gives us

$$\Delta[1]_* \hookrightarrow \mathcal{N}(I) = E.$$

Lemma 7.28. *1. The image of*

$$\mathrm{Map}(E^t, W) \longrightarrow \mathrm{Map}(\Delta[1]^t, W) = W_1$$

is in W_{heq}, and we have a weak equivalence

$$\mathrm{Map}(E^t, W) \xrightarrow{\sim} W_{\mathrm{heq}} \hookrightarrow W_1.$$

2. $p^* : W_0 \to \mathrm{Map}(E^t, W)$ *is a weak equivalence if and only if* $s_0 : W_0 \to W_{\mathrm{heq}}$ *is a weak equivalence.*

Definition 7.29. Let $S_0 = \{\Delta[1]^t \hookrightarrow E^t\}$ and $\tilde{S} = S_0 \cup S$, then the model category of Complete Segal Spaces is

$$\mathcal{CSS} = \mathcal{L}_{S_0}[\mathbf{SeSp}] = \mathcal{L}_{\tilde{S}}[\mathbf{sSp}^{\mathrm{Reedy}}].$$

Theorem 7.30. *$\mathcal{CSS}$ is a left proper combinatorial simplicial model category with*

1. $\mathrm{Cof}(\mathcal{CSS}) = \mathrm{Cof}(\mathbf{SeSp}) = \mathrm{Cof}(\mathbf{sSp}^{\mathrm{Reedy}})$ = *monomorphisms in* **sSp**.

2. *Fibrant objects are complete Segal Spaces.*

3. *A map $f : X \to Y$ is a weak equivalence if and only if*

$$\mathrm{Map}(f, W) : \mathrm{Map}(Y, W) \xrightarrow{\sim} \mathrm{Map}(X, W)$$

are weak equivalences for any complete Segal spaces.

8 Bousfield Localization

8.1 Bousfield Localization

Recall if $\mathcal{M}$ is a simplicial model category with mapping space

$$\mathrm{Map}_{\mathcal{M}}(-,-) : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \longrightarrow \mathbf{sSet}$$

If $\mathcal{M}$ is a model category, we can apply simplicial localization to $(\mathcal{M}, \mathcal{W})$

$$L(\mathcal{M}, \mathcal{W}) = F_*\mathcal{M}[(F_*\mathcal{W})^{-1}]$$

and Hammock localization

$$L^H(\mathcal{M}, \mathcal{W})$$

Fact 8.1.

1. For a simplicial model category, there is a DK equivalence

$$\mathcal{M} \xrightarrow{\sim} L^H(\mathcal{M}, \mathcal{W}).$$

2. For a model category, there is a DK equivalence [22]

$$L(\mathcal{M}, \mathcal{W}) \simeq L^H(\mathcal{M}, \mathcal{W}).$$

Definition 8.2. A **homotopy mapping space** in a model category $\mathcal{M}$ is given by

$$\mathrm{Map}^h_{\mathcal{M}}(-,-) = \mathrm{Map}_{L(\mathcal{M},\mathcal{W})}(-,-).$$

8.2 Left Bousfield Localization

Definition 8.3 (Bousfield). Let $\mathcal{M}$ be a model category, and $\mathcal{S} \subset \mathrm{Mor}(\mathcal{M})$ be a class of morphisms in $\mathcal{M}$,

1. $W \in \mathrm{Ob}(\mathcal{M})$ is called **$\mathcal{S}$-local** if
   - W is fibrant in $\mathcal{M}$,
   - for any $f : A \to B$ in $\mathcal{S}$,

   $$f^* : \mathrm{Map}^h_{\mathcal{M}}(B, W) \xrightarrow{\sim} \mathrm{Map}^h_{\mathcal{M}}(A, W)$$

   is a weak equivalence of simplicial sets.

2. A morphisms $g : X \to Y$ is called **$\mathcal{S}$-local equivalence** if for any $\mathcal{S}$-local object W,

$$g^* : \mathrm{Map}^h_{\mathcal{M}}(Y, W) \xrightarrow{\sim} \mathrm{Map}^h_{\mathcal{M}}(X, W)$$

is a weak equivalence of simplicial sets.

[22] There is no explicit map $L(\mathcal{M}, \mathcal{W}) \to L^H(\mathcal{M}, \mathcal{W})$ but zigzags of weak equivalences.

Remark.

1. Recall in a model category, a map $g : X \to Y$ is a weak equivalence if and only if

$$\mathrm{Map}^h_{\mathcal{M}}(g, W) : \mathrm{Map}^h_{\mathcal{M}}(Y, W) \xrightarrow{\sim} \mathrm{Map}^h_{\mathcal{M}}(X, W)$$

is a weak equivalence for any fibrant object W in $\mathcal{M}$. And there is a similar behavior in the localized case, where $\mathcal{S}$-local objects should be viewed as fibrant objects and $\mathcal{S}$-local equivalences are the weak equivalences which can be detected by $\mathcal{S}$-local objects.

2. When $\mathcal{S} = \{f : A \to B\}$ contains a single morphism, we use f-local instead of $\mathcal{S}$-local.

3. When $f : \varnothing \to A$, we say A-null to imply f-local.

4. Dually, one can define $\mathcal{S}$-colocal objects and $\mathcal{S}$-colocal equivalences.

Definition 8.4. The **left Bousfield localization** $\mathcal{L}_{\mathcal{S}}(\mathcal{M})$ to be the model category with the same underlying category $\mathcal{M}$ with

- $WE(\mathcal{L}_{\mathcal{S}}(\mathcal{M})) = \mathcal{S}$-local equivalences.
- $\mathtt{Cof}(\mathcal{L}_{\mathcal{S}}(\mathcal{M})) = \mathtt{Cob}(\mathcal{M})$.
- $\mathtt{Fib}(\mathcal{L}_{\mathcal{S}}(\mathcal{M})) = \mathrm{RLP}(\mathtt{Cof}(\mathcal{L}_{\mathcal{S}}(\mathcal{M})) \cap WE(\mathcal{L}_{\mathcal{S}}(\mathcal{M}))$.
- fibrant objects are $\mathcal{S}$-local objects.

Remark. This idea provides a very general setting for localization.

Cohn Localization (universal localization for (non)commutative rings). In commutative algebra, let A be a commutative algebra and S be a multiplicative closed subset of A, we can define localization $A[S^{-1}]$.

In noncommutative algebra, we start with a noncommutative ring A and a class of morphisms

$$S = \{f_i : P_i \to Q_i\}_{i \in I}$$

where P_i, Q_i are finitely generated projective (left) A-modules.

1. Let $S = \{s_i \in A\}_{i \in I}$,

$$\mathcal{S} = \{f_i : A \to A, a \mapsto as_i\}_{i \in I}$$

2. $A = kQ$ where Q is a quiver. Let $P_i = Ae_i$, any arrow $a : i \to j$ gives a map

$$\begin{aligned} a : P_j &\longrightarrow P_i \\ e_j &\longmapsto e_j a = a e_i \end{aligned}$$

and localization is equivalent to formally adding an inverse of the arrow a.

$$i \underset{a^*}{\overset{a}{\rightleftarrows}} j \longrightarrow k \circlearrowleft$$

Example 8.5 (Chain complexes)**.** Let $\mathcal{M} = \mathbf{Ch}(A)$ be the category of (unbounded) chain complexes of left A-modules. Let $\mathcal{S} \subset \mathbf{Mod}(A) \subset \mathrm{Mor}(\mathbf{Ch}(A))$ be a set of objects. $\mathcal{M}$ is equipped with mapping spaces given by morphism complexes.

$$\mathrm{Map}^h_{\mathcal{M}}(-,-) = \mathbf{Hom}_A(-,-)$$

1. W is a S-local complex if for any $f : P \to Q$ in S,

$$\mathbf{Hom}_A(Q, W) \xrightarrow{\sim} \mathbf{Hom}_A(P, W)$$

   is a weak equivalence.

2. $f : P \to Q$ is an S-local weak equivalence if

$$\mathbf{Hom}_A(Q, W) \xrightarrow{\sim} \mathbf{Hom}_A(P, W)$$

   is a weak equivalence for all S-local complex W.

Definition 8.6. A **Cohn localization** of A with respect to S is a map of algebras $\rho : A \to A[S^{-1}]$ such that

1. $\rho^*(f) : A[S^{-1}] \otimes_A P \xrightarrow{\sim} A[S^{-1}] \otimes_A Q$ is a weak equivalence for any $f : P \to Q$ in S, and

2. ρ is universal among all such maps.

Exercise 8.7. Show

1. for any $X \in \mathcal{M} = \mathbf{Ch}(A)$, there is a morphism $\rho_S(X) : X \to L_S(X)$ such that
   - $\rho_S(X)$ is an S-local equivalence.
   - $L_S(X)$ is an S-local object.

   Moreover, $L_S(X)$ is unique up to quasi-isomorphism.

2. For $X = A$ concentrated in degree 0, $L_S(A)$ has a structure of DGA such that

$$H_i(L_S(A)) = 0, \forall i < 0.$$

3. $H_0(L_S(A)) \cong A[S^{-1}]$ the Cohn localization.

Theorem 8.8 (must be)**.** $H_i(L_S(A)) = 0, \forall i > 0$ *if and only if Cohn localization is stably flat, i.e.*

$$\mathrm{Tor}^A_i(A[S^{-1}], A[S^{-1}]) = 0, \forall i > 0.$$

8.3 Basic Properties

Theorem 8.9. *For any left Bousfield localization,*

$$j = \mathrm{id} : \mathcal{M} \rightleftarrows \mathcal{L}_{\mathcal{S}}(\mathcal{M}) : \mathrm{id} = q$$

is a Quillen pair with the left adjoint being homotopical. This implies

1. $WE(\mathcal{M}) \subseteq WE(\mathcal{L}_{\mathcal{S}}(\mathcal{M}))$.
2. $\mathtt{Cof}(\mathcal{M}) = \mathtt{Cob}(\mathcal{L}_{\mathcal{S}}(\mathcal{M}))$.
3. $\mathtt{Cob}(\mathcal{M}) \cap WE(\mathcal{M}) \subseteq \mathtt{Cof}(\mathcal{L}_{\mathcal{S}}(\mathcal{M})) \cap WE(\mathcal{L}_{\mathcal{S}}(\mathcal{M}))$.
4. $\mathtt{Fib}(\mathcal{M}) \supseteq \mathtt{Fib}(\mathcal{L}_{\mathcal{S}}(\mathcal{M}))$.
5. $\mathtt{Fib}(\mathcal{M}) \cap WE(\mathcal{M}) \supseteq \mathtt{Fib}(\mathcal{L}_{\mathcal{S}}(\mathcal{M})) \cap WE(\mathcal{L}_{\mathcal{S}}(\mathcal{M}))$.

The derived counit is

$$\bar{j} \circ \mathbb{R}q \xrightarrow{\sim} \mathrm{id}_{\mathrm{Ho}(\mathcal{L}_{\mathcal{S}}(\mathcal{M}))}.$$

The derived unit is

$$\eta : \mathrm{id}_{\mathrm{Ho}(\mathcal{M})} \to \mathbb{R}q \circ \bar{j} =: \mathcal{L}_{\mathcal{S}},$$

and for any $X \in \mathrm{Ob}(\mathcal{M})$,

$$\eta_X : X \to \mathcal{L}_{\mathcal{S}}(X) := R_{\mathtt{loc}}(X)$$

is called the **localization** map, where $\mathcal{L}_{\mathcal{S}}(X)$ is fibrant replacement in $\mathcal{L}_{\mathcal{S}}(\mathcal{M})$.

Proposition 8.10 (Weak Whitehead)**.** *If X, Y are $\mathcal{S}$-local objects, then $f : X \to Y$ is $\mathcal{S}$-local if and only if f is a weak equivalence in $\mathcal{M}$.*

Corollary 8.11. *This gives a characterization (unique up to weak equivalences in $\mathcal{M}$) for $X \to \mathcal{L}_{c\mathcal{S}}(X)$ as a local $\mathcal{S}$-equivalence into a $\mathcal{S}$-local object.*

Take an $\mathcal{S}$-local equivalence $f : X \xrightarrow[\mathcal{S}-\mathtt{loc}]{\sim} W$

$$\begin{tikzcd}
X \arrow[r, "\mathcal{S}-\mathtt{loc}", "\sim"'] \arrow[d, hook, "\mathcal{S}-\mathtt{loc}"', "\sim"] & W \arrow[d, two heads] \\
\mathcal{L}_{\mathcal{S}}(X) \arrow[r] \arrow[ur, dashed, "\alpha"'] & *
\end{tikzcd}$$

When f is $\mathcal{S}$-local, so is α, thus α is a global weak equivalence.

Theorem 8.12 (Left Bousfield localization is left localization)**.** *The functor* $j : \mathcal{M} \to \mathcal{L}_{\mathcal{S}}(\mathcal{M})$ *is the left* S*-localization in the sense, given any left Quillen functor* $F : \mathcal{M} \to \mathcal{N}$ *such that*

$$\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$$

maps $\overline{\beta} \in \overline{\mathcal{S}}$ *to isomorphisms in* $\mathrm{Ho}(\mathcal{N})$*, there exists a unique factorization*

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "F"] \arrow[dr, "j"'] & & \mathcal{N} \\ & \mathcal{L}_{\mathcal{S}}(\mathcal{M}) \arrow[ur, "\overline{F}"'] & \end{tikzcd}$$

where $\overline{F}$ *is left Quillen.*

Theorem 8.13 (Localization of Quillen pairs)**.** *Let* $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ *be a Quillen pair, and let* $\mathcal{S} \subset \mathrm{Mor}(\mathcal{M})$*. Assume* $\mathcal{L}_{\mathcal{S}}(\mathcal{M})$ *exists, then* (F, G) *localizes at* $\mathcal{S}$*, i.e. induces Quillen pair*

$$\mathcal{L}_{\mathcal{S}}F : \mathcal{L}_{\mathcal{S}}\mathcal{M} \rightleftarrows \mathcal{N} : \mathcal{L}_{\mathcal{S}}G$$

if and only if

$$\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$$

takes $\overline{f} \in \overline{\mathcal{S}}$ *into isomorphisms in* $\mathrm{Ho}(\mathcal{N})$*.*

In particular, Quillen equivalences localize to $\mathcal{S}$*-local Quillen equivalences.*

Theorem 8.14 (Existence)**.** *Let* $\mathcal{M}$ *be a left proper cofibrantly generated model category. Assume that* $\mathcal{M}$ *is either*

- *cellular* [H2]*, or*
- *combinatorial* [D2]

then $\mathcal{L}_{\mathcal{S}}(\mathcal{M})$ *exists for all* $\mathcal{S} \subset \mathrm{Mor}(\mathcal{M})$*.*

Definition 8.15. A cofibrantly generated model category $\mathcal{M}$ is called **combinatorial** if $\mathcal{M}$ is **locally presentable**, i.e.

1. all small colimits in $\mathcal{M}$ exist,
2. there exists a set P of small objects[23] in $\mathcal{M}$ such that every object X in $\mathcal{M}$ is the colimit of a small diagram $\mathcal{I} \xrightarrow{\tilde{X}} \mathcal{M}$ such that $\tilde{X}(i) \in P$

[23] an object X is small if h_X commutes with filtered colimits.

Theorem 8.16 ([D2]). *Combinatorial model category has a small presentation, i.e. there is a small category* $\mathcal{C}$ *and* $\mathcal{S} \subset \mathrm{Mor}(\mathcal{UC})$ *the universal model category generated by* $\mathcal{C}$ *such that any functor* $\gamma : \mathcal{C} \to \mathcal{M}$ *where* $\mathcal{M}$ *is a model category factors through* $\mathcal{UC}$ *and there is a adjunction* $(\mathrm{Res}_\gamma, \mathrm{Sing}_\gamma)$

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\gamma"] \arrow[dr, "r"'] & & \mathcal{M} \arrow[dl, shift left, "\mathrm{Sing}_\gamma"] \\ & \mathcal{UC} \arrow[ur, shift left, "\mathrm{Res}_\gamma"] & \end{tikzcd}$$

(which is unique to "homotopy") such that $(\mathrm{Res}, \mathrm{Sing})$ *localizes at* $\mathcal{S}$ *gives a Quillen equivalence*

$$\mathcal{UC}/\mathcal{S} \rightleftarrows \mathcal{M}$$

where $\mathcal{UC}/\mathcal{S} \coloneqq \mathcal{L}_{\mathcal{S}}(\mathcal{UC})$.

8.4 Complete Segal Spaces v.s. Segal Categories

Theorem 8.17 ([R1]). $\mathcal{S}$*-local equivalences in* $\mathcal{CSS}$ *are precisely the DK equivalences, namely, a map* $f : W_1 \to W_2$ *is an* $\mathcal{S}$*-local weak equivalence if and only if*

DK1 *for any objects* x, y *in* W_1,

$$f : \mathbf{map}_{W_1}(x, y) \xrightarrow{\sim} \mathbf{map}_{W_2}(f(x), f(y))$$

is a weak equivalence, and

DK2 $\mathrm{Ho}(f) : \mathrm{Ho}(W_1) \to \mathrm{Ho}(W_2)$ *is an equivalence of categories.*

8.5 Higher Segal Spaces

Remark (Models for (∞, n)-Categories). Complete Segal spaces have a natural generalizations by replacing $\Delta = \Theta_1$ by Θ_n for $n \geq 2$. For detailed discussions about Θ-construction (Joyal), please see [R2] and [B2].

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