MATH6540 Homotopy Theory: Spectra

Taught by Inna Zakharevich

Notes by Yun Liu

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Part I

Spectrum

1 Homotopy Groups and Stable Homotopy Groups

1.1 Homotopy Groups

Question. Where does spectra come from?

In the study of homotopy groups, we define the n-th homotopy group of a topological space X as the set of homotopy classes

$$\pi_n(X) = [\mathbb{S}^n, X]_*, n \geq 0$$

which is a group when $n \geq 1$ and abelian when $n \geq 2$.

The group structure follows from

$$[\mathbb{S}^n, X]_* \times [\mathbb{S}^n, X]_* \cong [\mathbb{S}^n \vee \mathbb{S}^n, X]_*$$

and the map

$$\mathbb{S}^n \to \mathbb{S}^n \vee \mathbb{S}^n$$

given by collapsing equator.

Note that even if we have $\pi_n(\mathbb{S}^n) = \mathbb{Z} = H_n(\mathbb{S}^n)$, homotopy groups and homology groups are not the same, for instance, $\pi_3(\mathbb{S}^2) = \mathbb{Z}$ but $H_3(\mathbb{S}^2) = 0$.

Nevertheless, there are still similarities between them.

In homology theory, if we have a good sequence (excision)

$$\begin{tikzcd} A \arrow[r, hook] & X \arrow[r, two heads] & X/A \end{tikzcd}$$

where X/A is the pushout

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & X \arrow[d] \\ \bullet \arrow[r] & X/A \end{tikzcd}$$

then it induces a long exact sequence in homology groups H_*. The proof, roughly speaking, contains three steps.

1. construct the relative homology group $H_*(X, A)$ and prove the long exact sequence for relative homology group.

2. prove that $H_*(X, A) \cong H_*(X/A)$.

3. show excision, if $X = A \bigcup B$ then $H_*(A, A \cap B) \cong H_*(X, B)$.

Similarly, in homotopy groups, if we have a fibre sequence

$$F \longrightarrow E \longrightarrow B$$

then it induces a long exact sequence in homotopy groups π_*.

Definition 1.1. A topological space X is called *n-connected* if $\pi_i(X) = 0, \forall i \leq n$.

If a CW pair (X, A) is n-connected, then $\pi_i(X, A) = 0, \forall i \leq n$, consider the long exact sequence

$$\cdots \longrightarrow \pi_i(X) \longrightarrow \pi_i(X, A) \longrightarrow \pi_{i-1}(A) \longrightarrow \cdots$$

we have the following result.

Theorem 1.1. *[Blakers-Massey] [BM51] Let X be a CW complex such that $X = A \bigcup B$ and $A \cap B$ is nonempty and connected. If $(A, A \cap B)$ is m-connected and $(B, A \cap B)$ is n-connected, then*

$$\pi_i(A, A \cap B) \longrightarrow \pi_i(X, B)$$

is an isomorphism for $i < m + n$ and surjective for $i = m + n$.

1.2 Stable Homotopy Group

In general homotopy groups are difficult to compute.

Moral. Suspension makes it better.

Conclusion. Suspend infinitely many times.

Naive Stable Homotopy Group

Definition 1.2. The *naive stable homotopy group* of a topological space$X$ is $\pi_i^S(X) \coloneqq \operatorname{colim}_n \left(\mathbb{S}^{n+i}, \Sigma^n X\right)$.

$$\pi_i(X) \longrightarrow \pi_{i+1}(\Sigma X) \longrightarrow \pi_{i+2}\left(\Sigma^2 X\right) \longrightarrow \cdots$$

We have the following result for CW complexes.

Theorem 1.2. *If X is a finite CW complex, then this colimit stabilizes ($n \geq 2\dim X + 2$, which is not a minimal bound).*

Open question [Ker]. The *Kervaire invariant* is an invariant of a framed $(4k+2)$-dimensional manifold that measures whether the manifold could be surgically converted into a sphere. This invariant evaluates to 0 if the manifold can be converted to a sphere, and 1 otherwise. The Kervaire invariant problem is the problem of determining in which dimensions the Kervaire invariant can be nonzero. For differentiable manifolds, this can happen in dimensions $2, 6, 14, 30, 62$, and possibly 126, and in no other dimensions. The final case of dimension 126 remains open.

Stable Homotopy Group

The intuition comes from the following theorem.

Theorem 1.3. *(Brown Representability) For every cohomology theory $\{h^n : \mathbf{Top}^{op} \to \mathbf{Ab}\}_{n\in\mathbb{Z}}$ there exists a sequence of spaces*

$$E \coloneqq \{\cdots, E_{-2}, E_{-1}, E_0, E_1, E_2, \cdots\}$$

such that $h^n(X) = [X, E_n]$ and

$$E_n \simeq \Omega E_{n+1}, \quad n \geq 0$$
$$E_{-n} = \Omega^n E_0, \quad n > 0$$

A map $f : E \to F$ is a sequence of maps $\{f_n : E_n \to F_n\}_{n\in\mathbb{Z}}$ such that

$$\begin{tikzcd} E_n \arrow[r, "f_n"] \arrow[d] & F_n \arrow[d] \\ \Omega E_{n+1} \arrow[r, "\Omega f_{n+1}"] & F_{n+1} \end{tikzcd}$$

Warning. There exists phantom maps $E \xrightarrow{f} F$ such that $[-,E] \xrightarrow{f_*} [-,F]$ is zero, but $f \not\simeq \text{const}$, i.e. $h^* : \mathbf{Top}^{op} \to \mathbf{Ch}_{\mathbb{Z}}$ is not a faithful functor, but it is full.

Note that $X \to \Omega Y$ corresponds to $\Sigma X \to Y$, so given a sequence of spaces $\{X_0, X_1, \cdots, X_n, \cdots\}$ and maps $\Sigma X_n \to X_{n+1}$, we can define the *stable homotopy groups* as the colimit of the following diagram

$$\begin{tikzcd} \Sigma S^m \arrow[r, "\Sigma f"] \arrow[d] & \Sigma X_n \arrow[d] \\ S^{m+1} \arrow[r, "f"] & X_{n+1} \end{tikzcd}$$

Definition 1.3. A *spectrum* is a sequence of spaces $X = \{X_0, X_1, \cdots, X_n, \cdots\}$ together with maps $\sigma = \{\sigma_n : \Sigma X_n \to X_{n+1}\}_{n \geq 0}$. A *map between spectra* is a sequence of maps $f = \{f_n : X_n \to Y_n\}_{n \geq 0}$ such that

$$\begin{tikzcd} \Sigma X_n \arrow[r, "\Sigma f_n"] \arrow[d, "\sigma_n"'] & \Sigma Y_n \arrow[d, "\sigma'_n"] \\ X_{n+1} \arrow[r, "f_{n+1}"] & Y_{n+1} \end{tikzcd}$$

commutes. The *n-th stable homotopy* of a spectrum X is $\pi_n(X) \coloneqq \operatorname{colim}_k \left[\mathbb{S}^{n+k}, X_k\right], n \in \mathbb{Z}$.

Note when $n < 0$, the colimit is taking from $k = -n$.

Write $\mathbf{Sp}$ the category of spectra.

A map $f : X \to Y$ is a *stable equivalence* if $\pi_n f$ is an isomorphism for any $n \in \mathbb{Z}$.

A spectrum is a *suspension spectrum* if $\sigma_n : \Sigma X_n \xrightarrow{\sim} X_{n+1}$ is a weak equivalence for any $n \in \mathbb{Z}$.

An Ω-*spectrum* is a spectrum X such that $\sigma'_n : X_n \to \Omega X_{n+1}$ is a weak equivalence for any $n \in \mathbb{Z}$.

Observation. There is a Quillen pair

$$\Sigma^\infty : \mathbf{Top}_* \rightleftarrows \mathbf{Sp} : \Omega^\infty$$

where

$$\begin{aligned} \Sigma^\infty X &= \left\{X, \Sigma X, \Sigma^2 X, \cdots\right\} \\ \Omega^\infty Y &= \operatorname{colim}_n \Omega^n Y_n \end{aligned}$$

2 Symmetric Spectra

The main reference for this part ARE [Sch12] and [EKMM97], the second of which is quite technical.

The space are always compactly generated weak Hausdorff spaces.

2.1 Smash Product

Observation. In the category of pointed spaces, we have the functors $\pi_k : \mathbf{Top}_* \to \mathbf{Gr}$, and we work with the reduced (co)homology $\tilde{H}^*$ and $\tilde{H}_*$.

In **Top**, we have the adjoint pair $K\times \dashv (-)^K$.

In $\mathbf{Top}_*$, we need to adjust this pair a little bit so that it works.

Definition 2.1. The smash product of two pointed spaces $(X, *)$ and $(Y, \bullet)$ is

$$X \wedge Y \coloneqq \frac{X \times Y}{X \times \{\bullet\} \cup \{*\} \times Y}$$

a pointed space.

Theorem 2.1. *The reduced homology of $X \wedge Y$ satisfies*

$$\tilde{H}^* (X \wedge Y; R) \cong \tilde{H}^* (X; R) \otimes \tilde{H}^* (Y; R) .$$

Example 2.1.

1. $\mathbb{S}^n \wedge \mathbb{S}^m = \mathbb{S}^{n+m}$.

2. $\mathbb{S}^1 \wedge X = \Sigma X$.

2.2 Symmetric Spectra

Definition 2.2. A *symmetric spectrum* consists of the following data:

- a sequence of pointed spaces $X = \{X_0, X_1, \cdots, X_n, \cdots\}$

- a basepoint preserving continuous left action of the symmetric group Σ_n on X_n, for each $n \geq 0$

- based maps $\sigma_n : X_n \wedge \mathbb{S}^1 \to X_{n+1}$ which is equivariant with respect to Σ_n-action, for $n \geq 0$.

satisfying that for any $n, m \geq 0$, the composite

$$X_n \wedge \mathbb{S}^m \xrightarrow{\sigma_n \wedge 1} X_{n+1} \wedge \mathbb{S}^{m-1} \xrightarrow{\sigma_{n+1} \wedge 1} \cdots \xrightarrow{\sigma_{n+m-2} \wedge 1} X_{n+m-1} \wedge \mathbb{S}^1 \xrightarrow{\sigma_{n+m-1}} X_{n+m}$$

is $\Sigma_n \times \Sigma_m$-equivariant.

Here the symmetric group Σ_m acts by permuting the coordinates of $\mathbb{S}^m$, and $\Sigma_n \times \Sigma_m$ acts by restriction of the Σ_{n+m}-action.

We write $\sigma_n^m : X_n \wedge \mathbb{S}^m \to X_{n+m}$ for the map that applies the structure map $\sigma_n : R_n \wedge \mathbb{S}^1 \to R_{n+1}$ m times.

A morphism $f : X \to Y$ between symmetric spectra consists of Σ_n-equivariant based maps $\{f_n : X_n \to Y_n\}_{n \geq 0}$ which are compatible with the structure maps in the sense that the following diagram

$$\begin{tikzcd}
X_n \wedge \mathbb{S}^1 \arrow[r, "f_n \wedge 1"] \arrow[d, "\sigma_n"'] & Y_n \wedge \mathbb{S}^1 \arrow[d, "\sigma_n"] \\
X_{n+1} \arrow[r, "f_{n+1}"] & Y_{n+1}
\end{tikzcd}$$

commutes.

Definition 2.3. A *symmetric ring spectrum* R is a symmetric spectrum with

- $\Sigma_n \times \Sigma_m$-equivariant multiplication maps $\mu_{n,m} : R_n \wedge R_m \to R_{n+m}$ for $n, m \geq 0$, and
- unit maps $\iota_0 : \mathbb{S}^0 \to R_0$ and $\iota_1 : \mathbb{S}^1 \to R_1$.

satisfying

- associativity: the square

$$\begin{tikzcd}
R_n \wedge R_m \wedge R_p \arrow[r, "1 \wedge \mu_{m,p}"] \arrow[d, "\mu_{n,m} \wedge 1"'] & R_n \wedge R_{m+p} \arrow[d, "\mu_{n,m+p}"] \\
R_{n+m} \wedge R_p \arrow[r, "\mu_{n+m,p}"] & R_{n+m+p}
\end{tikzcd}$$

  commutes for all $n, m, p \geq 0$.

- unit: the two compositions

$$R_n \xrightarrow{\cong} R_n \wedge \mathbb{S}^0 \xrightarrow{\mathrm{Id}\wedge\iota_0} R_n \wedge R_0 \xrightarrow{\mu_{n,0}} R_n$$

$$R_n \xrightarrow{\cong} \mathbb{S}^0 \wedge R_n \xrightarrow{\iota_0\wedge 1} R_0 \wedge R_0 \xrightarrow{\mu_{0,n}} R_n$$

are identities for all $n \geq 0$.

- centrality: the diagram

$$\begin{tikzcd} R_n \wedge \mathbb{S}^1 \arrow[r, "1\wedge\iota_1"] \arrow[d, "\mathrm{twist}"'] & R_n \wedge R_1 \arrow[r, "\mu_{n,1}"] & R_{n+1} \arrow[d, "\chi_{n,1}"] \\ \mathbb{S}^1 \wedge R_n \arrow[r, "\iota_1\wedge 1"] & R_1 \wedge R_n \arrow[r, "\mu_{1,n}"] & R_{1+n} \end{tikzcd}$$

commutes for all $n, m \geq 0$. Here $\chi_{n,m} \in \Sigma_{n+m}$ denotes the shuffle permutation which moves the first n elements past the last m elements, keeping each of the two blocks in order.

A morphism $f : R \to S$ of symmetric ring spectra consists of Σ_n-equivariant based maps $\{f_n : R_n \to S_n\}_{n\geq 0}$, which are compatible with the multiplication and unit maps in the sense that

$$\begin{tikzcd} R_n \wedge R_m \arrow[r, "f_n\wedge f_m"] \arrow[d, "\mu_{n,m}"'] & S_n \wedge S_m \arrow[d, "\mu_{n,m}"] \\ R_{n+m} \arrow[r, "f_{n+m}"] & S_{n+m} \end{tikzcd}$$

for any $n, m \geq 0$, and the two diagrams

$$\begin{tikzcd} \mathbb{S}^0 \arrow[r, "\iota_0"] \arrow[dr, "\iota_0"'] & R_0 \arrow[d, "f_0"] \\ & S_0 \end{tikzcd} \qquad \begin{tikzcd} \mathbb{S}^1 \arrow[r, "\iota_1"] \arrow[dr, "\iota_1"'] & R_1 \arrow[d, "f_1"] \\ & S_1 \end{tikzcd}$$

commute.

Definition 2.4. A *right module* M over a symmetric ring spectrum R is a spectrum $M = \{M_0, M_1, \cdots, M_n, \cdots\}$

with $\Sigma_n \times \Sigma_m$-equivariant action maps $\alpha_{n,m} : M_n \wedge R_m \to M_{n+m}$ such that the diagrams

$$\begin{tikzcd} M_n \wedge R_m \wedge R_p \arrow[r, "1\wedge\mu_{m,p}"] \arrow[d, "\alpha_{n,m}\wedge \mathrm{Id}"'] & M_n \wedge R_{m+p} \arrow[d, "\alpha_{n,m+p}"] \\ M_{n+m} \wedge R_p \arrow[r, "\alpha_{n+m,p}"] & M_{n+m+p} \end{tikzcd} \qquad \begin{tikzcd} M_n \cong M_n \wedge \mathbb{S}^0 \arrow[r, "1\wedge\iota_0"] \arrow[dr, equal] & M_n \wedge R_0 \arrow[d, "\alpha_{n,0}"] \\ & M_n \end{tikzcd}$$

commute.

We will give an important example of a ring spectrum now.

Definition 2.5. The ring spectrum $\mathbb{S}$ is $\mathbb{S} = \left\{\mathbb{S}^0, \mathbb{S}^1, \mathbb{S}^2, \cdots\right\}$ with $\mu_{m,1} : \mathbb{S}^m \wedge \mathbb{S}^1 \xrightarrow{\cong} \mathbb{S}^{m+1}$.

We will show in next lecture that all symmetric spectra are $\mathbb{S}$-modules.

Naive Homotopy Group

Definition 2.6. The n-naive homotopy group of a symmetric spectrum X is

$$\hat{\pi}_n (X) \coloneqq \mathrm{colim}_k \pi_{n+k} X_k, n \in \mathbb{Z}.$$

Note when $n < 0$, the colimit is taken from $k > -n$.

Example 2.2. The n-naive homotopy group of spheres is $\hat{\pi}_n \mathbb{S} \mathrm{colim}_k \left(\pi_{k+n} \mathbb{S}^k\right)$, and is denoted π_n^s.

$$\begin{aligned} \hat{\pi}_0^s &= \mathrm{colim}_k \left(\pi_k \mathbb{S}^k\right) = \mathbb{Z} \\ \hat{\pi}_n^s &= \mathrm{colim}_k \left(\pi_{k+n} \mathbb{S}^k\right) = 0, \quad n < 0. \end{aligned}$$

Since $\mathbb{S}^n$ is $(n-1)$-connected, the group π_n^s is trivial for negative values of n. The degree of a self-map of a sphere provides an isomorphism $\pi_0^s = \mathbb{Z}$. For $n \geq 1$, the homotopy group π_n^s is finite. This is a direct consequence of Serre's calculation of the homotopy groups of spheres modulo torsion, and Freudenthal's suspension theorem.

Theorem 2.2. *[Serre] For $m > n \geq 1$,*

$$\hat{\pi}_m \mathbb{S}^n = \begin{cases} P_{m,n} \oplus \mathbb{Z} & n = 2k, m = 2n-1, \\ P_{m,n} & o.w. \end{cases}$$

where $P_{m,n}$ is some finite group for any fixed m, n.

Exercise 2.1. $\pi_1^s = \mathbb{Z}/2$.

Proof. See [Sch12] P12-13. □

Example 2.3. (Suspension spectra). Every pointed space K gives rise to a suspension spectrum $\Sigma^\infty K$ via

$$(\Sigma^\infty K)_n = K \wedge \mathbb{S}^n$$

with structure maps given by the canonical homeomorphism

$$(K \wedge \mathbb{S}^n) \wedge \mathbb{S}^1 \xrightarrow{\cong} K \wedge \mathbb{S}^{n+1} .$$

For example, the sphere spectrum $\mathbb{S}$ is isomorphic to the suspension spectrum $\Sigma^\infty \mathbb{S}^0$.

The naive homotopy group

$$\pi_k^s K = \hat{\pi}_k \left(\Sigma^\infty K\right) = \mathrm{colim}_n \left(\pi_{k+m} K \wedge \mathbb{S}^n\right)$$

is called the *k-th stable homotopy group* of K. Since $K \wedge \mathbb{S}^n$ is $(n-1)$-connected, the suspension spectrum $\Sigma^\infty K$ is connective, i.e., all homotopy groups in negative dimensions are trivial. The Freudenthal suspension theorem implies that for every suspension spectrum, the colimit system for a specific homotopy group always stabilizes. A symmetric spectrum X is isomorphic to a suspension spectrum (necessarily that of its zeroth space X0) if and only if every structure map $\sigma_n : X_n \wedge \mathbb{S}^1 \longrightarrow X_{n+1}$ is a homeomorphism.

Example 2.4. (Eilenberg-Mac Lane spectra). Let A be discrete abelian group, the Eilenberg-Mac Lane spectrum HA is defined as

$$HA \coloneqq \{A, K(A, 1), K(A, 2), \cdots\}$$

where

$$\pi_k\left(K\left(A, n\right)\right) = \begin{cases} \pi_n\left(A\right) & k = n, \\ 0 & \text{o.w.} \end{cases}$$

We have a weak equivalence

$$K(A, n-1) \xrightarrow{\simeq} \Omega K(A, n)$$

taking adjoint of this map to be the structure map, then

$$\hat{\pi}_n(HA) = \begin{cases} \operatorname{colim}_k \pi_{n+k}(K(A,k)) = 0 & n \neq 0, \\ \operatorname{colim}_k \pi_k(K(A,k)) = A & n = 0 \end{cases}$$

Any symmetric spectrum is an $\mathbb{S}$-module

In order to show that any symmetric spectrum is an $\mathbb{S}$-module, we need to show that $\mathbb{S}$ is a ring spectrum first.

Remark 2.1. We want to define model structure on spectrum such that weak equivalences are the stable homotopy equivalences. The difficulty is, even if we define homotopy, two homotopic spectra does not necessarily remain homotopic after smashing with other (arbitrary) spectrum (unless that spectrum is cofibrant). This, however, works for the sphere spectrum $\mathbb{S}$ because it is cofibrant. In simplicial sets, this also works because everything is cofibrant.

Claim 2.1. $\mathbb{S}$ is a ring spectrum.

Proof. The symmetric group Σ_n acts on $\mathbb{S}^n$ via permutation by identifying $\mathbb{S}^n \cong (\mathbb{S}^1)^{\wedge n}$, so the order does not matter. The structure map is is the isomorphism $\sigma_n : \mathbb{S}^n \wedge \mathbb{S}^1 \to \mathbb{S}^{n+1}$.

To see that $\mathbb{S}$ is a ring spectrum, we consider the multiplication $\mu_{n,m} : \mathbb{S}^n \wedge \mathbb{S}^m \to \mathbb{S}^{n+m}$ on $\mathbb{S}$. It is associative, and compatible with the structure maps. □

Claim 2.2. The naive homotopy group $\hat{\pi}_* X$ is a π_*^s-module, so is the (true) homotopy group $\pi_* X$.

Proof. We first define the action of π_*^s on the naive homotopy groups $\hat{\pi}_*(X)$ of a symmetric spectrum X.

Suppose $f : \mathbb{S}^{k+n} \to X_n$ and $g : \mathbb{S}^{l+m} \to \mathbb{S}^m$ represents classes in $\hat{\pi}_k(X)$ respectively π_l^s. We denote by $f \cdot g$ the composition

$$f \cdot g : \mathbb{S}^{k+n+l+m} \xrightarrow{\cong} \mathbb{S}^{k+n} \wedge \mathbb{S}^{l+m} \xrightarrow{f \wedge g} X_n \wedge \mathbb{S}^m \xrightarrow{\sigma_n^m} X_{n+m}$$

then define

$$[f]\cdot[g] := (-1)^{nl}\,[f\cdot g]\,. \tag{1}$$

The sign can be explained by the principle that all natural number must occur in the 'natural order'. In $f\cdot g$ the dimension of the sphere of origin is naturally $(k+n)+(l+m)$, but in order to represent an element of $\hat{\pi}_{k+l}X$ the numbers should occur in the order $(k+l)+(n+m)$. Hence a shuffle permutation is to be expected, and it enters in the disguise of the sign $(-1)^{nl}$.

We need to check that the multiplication is well-defined. Note this is well-defined up to homotopy class of f and g.

If we replace g by $g\wedge 1_{\mathbb{S}^1}$,

$$f\cdot(g\wedge 1_{\mathbb{S}^1}) = \sigma_n^{m+1}\,(f\wedge g\wedge 1_{\mathbb{S}^1}) = \sigma_{n+m}\circ(\sigma_n^m\wedge 1_{\mathbb{S}^1})\circ(f\wedge g\wedge 1_{\mathbb{S}^1}) = \sigma_{n+m}\circ((f\cdot g)\wedge 1_{\mathbb{S}^1})\,.$$

$$\begin{tikzcd}
\mathbb{S}^{k+n+l+m+1} \arrow[r, "\cong"] \arrow[d, equal, "\cong"'] & \mathbb{S}^{k+n}\wedge\mathbb{S}^{l+m+1} \arrow[r, "f\wedge g\wedge 1_{\mathbb{S}^1}"] \arrow[d, equal, "\cong"'] & X_n\wedge\mathbb{S}^{m+1} \arrow[r, "\sigma_n^{m+1}"] & X_{n+m+1} \\
\mathbb{S}^{k+n+l+m}\wedge\mathbb{S}^1 \arrow[r, "\cong"] & \mathbb{S}^{k+n}\wedge\mathbb{S}^{l+m}\wedge\mathbb{S}^1 \arrow[r, "f\wedge g\wedge 1_{\mathbb{S}^1}"] & X_n\wedge\mathbb{S}^m\wedge\mathbb{S}^1 \arrow[r, "\sigma_n^m\wedge 1_{\mathbb{S}^1}"] \arrow[u, "\cong"'] & X_{n+m}\wedge\mathbb{S}^1 \arrow[u, "\sigma_{n+m}"']
\end{tikzcd}$$

Since the sign in the formula 1 does not change, the resulting stable class is independent of the representative g of the stable class in π_l^s.

If we replace f by $\sigma_n\circ(f\wedge 1_{\mathbb{S}^1}) : \mathbb{S}^{k+n+1}\to X_{n+1}$, then we have

$$\begin{tikzcd}
\mathbb{S}^{k+n+l+m+1} \arrow[r, "\cong"] \arrow[d, "1_{\mathbb{S}^{k+n}}\wedge\chi_{l+m,1}"'] & \mathbb{S}^{k+n}\wedge\mathbb{S}^{l+m+1} \arrow[r, "f\wedge g\wedge 1_{\mathbb{S}^1}"] \arrow[d, equal, "\cong"'] & X_n\wedge\mathbb{S}^{m+1} \arrow[r, "\sigma_n^{m+1}"] \arrow[d, "1_{X_n}\wedge\chi_{m,1}"] & X_{n+m+1} \arrow[d, "1\wedge\chi_{m,1}"] \\
\mathbb{S}^{k+n+1+l+m} \arrow[r, "\cong"] & \mathbb{S}^{k+n}\wedge\mathbb{S}^1\wedge\mathbb{S}^{l+m} \arrow[r, "f\wedge 1_{\mathbb{S}^1}\wedge g"] & X_n\wedge\mathbb{S}^{1+m} \arrow[r, "\sigma_n^{1+m}"] & X_{n+1+m}
\end{tikzcd}$$

By lemma 2.1, the map $1\wedge\chi_{m,1}$ induces multiplication by $(-1)^m$ on homotopy groups after one suspension. This cancels part of the sign $(-1)^{l+m}$ that is the effect of precomposition with the shuffle permutation $\chi_{l+m,1}$ on the left. So in the colimit $\hat{\pi}_{k+1}X$ we have

$$[\sigma_n\circ(f\wedge 1_{\mathbb{S}^1})\cdot g] = (-1)^l\left[\sigma_n^{m+1}\circ(f\wedge g\wedge 1_{\mathbb{S}^1})\right] = (-1)^l\,[f\cdot g]\,.$$

Hence the multiplication is independent of the representative of the stable class $[f]$.

Now we verify biadditivity. We only show the relation $(x+x')\cdot y = x\cdot y + x'\cdot y$, and additivity in

y is similar. Suppose $f, f' : \mathbb{S}^{k+n} \to X_n$ and $g : \mathbb{S}^{l+m} \to \mathbb{S}^m$ represents classes in $\hat{\pi}_k(X)$ respectively π_l^s, then the sum $f + f'$ in $\pi_{k+n}(X_n)$ is given by the composite

$$f + f' : \mathbb{S}^{k+n} \xrightarrow{\text{pinch}} \mathbb{S}^{k+n} \vee \mathbb{S}^{k+n} \xrightarrow{f \vee f'} X_n$$

Then we have

$$\begin{tikzcd}
\mathbb{S}^{k+l+n+m} \arrow[r, "1\wedge\chi_{n,1}\wedge 1"] \arrow[dd, "\text{pinch}"'] & \mathbb{S}^{k+n+l+m} \arrow[d, "\text{pinch}\wedge 1"'] \arrow[dr, "(f+f')\wedge g"] & & \\
& \left(\mathbb{S}^{k+n} \vee \mathbb{S}^{k+n}\right) \wedge \mathbb{S}^{l+m} \arrow[r, "(f\vee f')\wedge g"] & X_n \wedge \mathbb{S}^m \arrow[r, "\sigma_n^m"] & X_{n+m} \\
\mathbb{S}^{k+l+n+m} \wedge \mathbb{S}^{k+l+n+m} \arrow[r, "(1\wedge\chi_{n,1}\wedge 1)\wedge(1\wedge\chi_{n,1}\wedge 1)"] & \mathbb{S}^{k+n+l+m} \wedge \mathbb{S}^{k+n+l+m} \arrow[u, "\cong"] \arrow[ur, "(f\vee g)\wedge(f'\vee g)"'] & &
\end{tikzcd}$$

where the right part commutes and the left square commutes up to homotopy. The composite around the top of the diagram gives $(f + f') \cdot g$, whereas the composite around the bottom represents $f \cdot g + f' \cdot g$. This proves additivity of the dot product in the left variable.

If we specialize to $X = \mathbb{S}$ then the product provides a biadditive graded pairing

$$\cdot : \pi_k^s \times \pi_l^s \to \pi_{k+l}^s$$

of the stable homotopy groups of spheres. We claim that for every symmetric spectrum X the diagram

$$\begin{tikzcd}
\hat{\pi}_k X \times \pi_l^s \times \pi_j^s \arrow[r, "\cdot\times 1"] \arrow[d, "1\times\cdot"'] & \hat{\pi}_{k+l} X \times \pi_j^s \arrow[d, "\cdot"] \\
\hat{\pi}_k X \times \pi_{l+j}^s \arrow[r, "\cdot"] & \hat{\pi}_{k+l+j} X
\end{tikzcd}$$

commutes, so the product on the stable stems and the action on the homotopy groups of a symmetric spectrum are associative. After unraveling all the definitions, this associativity ultimately boils down to the equality

$$(-1)^{ln} \cdot (-1)^{j(n+m)} = (-1)^{jm} \cdot (-1)^{(l+j)n}$$

and commutativity of the square

$$\begin{tikzcd}
X_n \wedge \mathbb{S}^m \wedge \mathbb{S}^q \arrow[r, "\sigma_n^m \wedge 1"] \arrow[d, "1\wedge\cong"'] & X_{n+m} \wedge \mathbb{S}^q \arrow[d, "\sigma_{n+m}^q"] \\
X_n \wedge \mathbb{S}^{m+q} \arrow[r, "\sigma_n^{m+q}"] & X_{n+m+q}
\end{tikzcd}$$

Finally, the multiplication in the homotopy groups of spheres is commutative in the graded sense. Indeed, for representing maps $f : \mathbb{S}^{k+n} \to \mathbb{S}^n$ and $g : \mathbb{S}^{l+m} \to \mathbb{S}^m$ the square

$$\begin{tikzcd}
\mathbb{S}^{k+n+l+m} \arrow[r, "f\wedge g"] \arrow[d, "\chi_{k+n,l+m}"'] & \mathbb{S}^{n+m} \arrow[d, "\chi_{n,m}"] \\
\mathbb{S}^{l+m+k+n} \arrow[r, "g\wedge f"] & \mathbb{S}^{m+n}
\end{tikzcd}$$

commutes. The two vertical coordinate permutations induce the signs $(-1)^{(k+n)(l+m)}$ respectively $(-1)^{nm}$ (after one suspension) on homotopy groups. So in the stable group we have

$$[f] \cdot [g] = (-1)^{nl} [f \cdot g] = (-1)^{kl+km} (g \cdot f) = (-1)^{kl} [g] \cdot [f] .$$

$\square$

Lemma 2.1. *Let Y be a pointed space, $m \geq 0$ and $f : \mathbb{S}^m \to \mathbb{S}^m$ a based map of degree k. Then for every homotopy class $x \in \pi_n (Y \wedge \mathbb{S}^m)$ the classes $(Id_Y \wedge f)_* (x)$ and $k \cdot x$ become equal in $\pi_{n+1} \left(Y \wedge \mathbb{S}^{m+1}\right)$ after one suspension.*

The conclusion of this lemma does not in general hold without the extra suspension, i.e., $(\mathrm{Id}_Y \wedge f)_* (x)$ need not equal $k \cdot x$ in $\pi_n (Y \wedge \mathbb{S}^m)$: consider the Hopf map

$$\eta : \mathbb{S}^3 \subseteq \mathbb{C}^2 \backslash \{0\} \xrightarrow{\text{proj}} \mathbb{C}P^1 \cong \mathbb{S}^2$$

which is a locally trivial fiber bundle with fiber $\mathbb{S}^1$. It gives rise to a long exact sequence of homotopy groups. Since the fiber $\mathbb{S}^1$ has no homotopy above dimension one, the group $\pi_3 \left(\mathbb{S}^2\right)$ is free abelian of rank one, generated by the class of η, $(-1) \circ \eta$ is homotopic to η, which is not homotopic to $-\eta$ since η generates the infinite cyclic group $\pi_3 \left(\mathbb{S}^2\right)$.

Exercise 2.2. Given representing maps $f : \mathbb{S}^{k+n} \to \mathbb{S}^n$ and $g : \mathbb{S}^{l+m} \to \mathbb{S}^m$ where $n = l + m$, show that $[f \cdot g] = [g \circ f]$ if $X = \mathbb{S}$.

Proof. The composite

$$\mathbb{S}^{l+m+k} \xrightarrow{f} \mathbb{S}^{l+m} \xrightarrow{g} \mathbb{S}^m$$

is homotopic to (after one suspension)

$$\mathbb{S}^{k+l+m+l+m} \xrightarrow{f\wedge 1} \mathbb{S}^{l+m+l+m} \xrightarrow{g\wedge 1} \mathbb{S}^{m+m+l} \xrightarrow{\chi_{m+m,l}} \mathbb{S}^{l+m+m}$$

because the last shuffle has sign $(-1)^{2ml} = 1$, while $f \cdot g$ can be decomposed as

$$\mathbb{S}^{k+l+m+l+m} \xrightarrow{f\wedge 1} \mathbb{S}^{l+m+l+m} \xrightarrow{1\wedge g} \mathbb{S}^{l+m+m}$$

which is homotopic to

$$\mathbb{S}^{k+l+m+l+m} \xrightarrow{f\wedge 1} \mathbb{S}^{l+m+l+m} \xrightarrow{g\wedge 1} \mathbb{S}^{m+m+l} .$$

□

2.3 Mapping cone and homotopy fiber

The (reduced) mapping cone $C(f)$ of a morphism of based spaces $f : A \to B$ is defined by

$$C(f) = ([0,1] \wedge A) \cup_f B$$

where the unit interval $[0,1]$ is pointed by $0 \in [0,1]$, so that $[0,1] \wedge A$ is the reduced cone of A. The mapping cone comes with an inclusion

$$i : B \hookrightarrow C(f)$$

and a projection

$$p : C(f) \twoheadrightarrow \mathbb{S}^1 \wedge A$$

the projection sends B to the basepoint and is given on $[0,1] \wedge A$ by $p(x \wedge a) = t(x) \wedge a$ where we define

$$\begin{array}{rccc} t: & [0,1] & \to & \mathbb{S}^1 \\ & x & \mapsto & \frac{2x-1}{x(1-x)} \end{array}$$

What is relevant about the map t is not the precise formula, but that it passes to a homeomorphism between the quotient space $[0,1] / \{0,1\}$ and the circle $\mathbb{S}^1$, and that it satisfies $t(1-x) = -t(x)$.

We observe that an iteration of the mapping cone construction yields the suspension of A, up to homotopy, which induces a long exact sequence in homotopy groups.

We want to get a similar result in spectra.

Definition 2.7. The *mapping cone* $C(f)$ of a morphism of symmetric spectra $f : X \to Y$ is defined by the reduced mapping cone of $f_n : X_n \to Y_n$,

$$C(f)_n = C(f_n) = ([0,1] \wedge X_n) \cup_f Y_n$$

where the symmetric group Σ_n acts on $C(f)_n$ through the given action on X_n and Y_n and trivially on the interval.

The inclusions $i_n : Y_n \to C(f)_n$ and projections $p_n : C(f)_n \to \mathbb{S}^1 \wedge X_n$ assemble into morphisms of symmetric spectra $i : Y \to C(f)$ and $p : C(f) \to \mathbb{S}^1 \wedge X$.

We define the connecting homomorphism $\delta : \hat{\pi}_{k+1} C(f) \to \hat{\pi}_k X$ as the composite

$$\hat{\pi}_{k+1} C(f) \xrightarrow{p_*} \hat{\pi}_{k+1}\left(\mathbb{S}^1 \wedge X\right) \xrightarrow{\mathbb{S}^{-1} \wedge -} \hat{\pi}_k(X)$$

where the second map is the inverse of the suspension isomorphism

$$\mathbb{S}^1 \wedge - : \hat{\pi}_k X \longrightarrow \hat{\pi}_{k+1}\left(\mathbb{S}^1 \wedge X\right).$$

The map δ sends the class represented by a based map $\varphi : \mathbb{S}^{1+k+n} \to C(f)_n$ to $(-1)^{k+n}$ times the class of the composite

$$\mathbb{S}^{1+k+n} \xrightarrow{\varphi} C(f)_n \xrightarrow{p_n} \mathbb{S}^1 \wedge X_n \xrightarrow{\text{twist}} X_n \wedge \mathbb{S}^1 \xrightarrow{\sigma_n} X_{n+1}.$$

Proposition 2.1. *For every morphism $f : X \to Y$ of symmetric spectra the long sequence of abelian groups*

$$\cdots \longrightarrow \hat{\pi}_k X \xrightarrow{f_*} \hat{\pi}_k Y \xrightarrow{i_*} \hat{\pi}_k C(f) \xrightarrow{\delta} \hat{\pi}_{k-1} X \longrightarrow \cdots$$

is exact.

A continuous map $f : A \to B$ spaces is an h-cofibration if it has the homotopy extension property, i.e., given a continuous map $\varphi : B \to X$ and a homotopy $H : [0,1] \times A \to X$ such that $H(0,-) = \varphi f$, there is a homotopy $\tilde{H} : [0,1] \times B \to X$ such that $\tilde{H} \circ ([0,1] \times f) = H$and $\tilde{H}(0,-) = \varphi$.

An equivalent condition is that the map $([0,1] \times A) \cup_{0 \times f} B \to [0,1] \times B$ has a retraction.

Proposition 2.2. *For every h-cofibration the map $C(f) \to B/A$ which collapses the cone of A to a point is a based homotopy equivalence.*

Proof. See Corollary 2.2 of Appendix A in [Sch12]. □

Let $f : X \to Y$ be a morphism of symmetric spectra that is level-wise an h-cofibration. Then by the above proposition, the morphism $c : C(f) \to Y/X$ that collapses the cone of X is a level equivalence, and so it induces an isomorphism of homotopy groups. We can thus define another connecting map

$$\delta : \hat{\pi}_k(Y/X) \longrightarrow \hat{\pi}_{k-1} X$$

as the composite of the inverse of the isomorphism $c_* : \hat{\pi}_k C(f) \to \hat{\pi}_k(Y/X)$ and the connecting homomorphism $\hat{\pi}_k C(f) \to \hat{\pi}_{k-1} X$ as we defined before.

Corollary 2.1. *Let $f : X \to Y$ be a morphism of symmetric spectra that is level-wise an h-cofibration and denote by $q : Y \to Y/X$ the quotient map. Then the long sequence of naive homotopy groups*

$$\cdots \longrightarrow \hat{\pi}_k X \xrightarrow{f_*} \hat{\pi}_k Y \xrightarrow{q_*} \hat{\pi}_k(Y/X) \xrightarrow{\delta} \hat{\pi}_{k-1} X \longrightarrow \cdots$$

is exact.

As a consequence, we have

Proposition 2.3. *For any two spectra X, Y, we have a natural map $X \bigvee Y \to X \times Y$ where*

$$\begin{aligned} (X \bigvee Y)_n &= X_n \bigvee Y_n \\ (X \times Y)_n &= X_n \times Y_n \end{aligned}$$

for every $n \geq 0$.

1. *For every integer k the canonical map*

$$\hat{\pi}_k X \bigoplus \hat{\pi}_k Y \longrightarrow \hat{\pi}_k (X \bigvee Y)$$

   is an isomorphism of abelian groups.

2. *For every integer k the canonical map*

$$\hat{\pi}_k (X \times Y) \longrightarrow \hat{\pi}_k (X) \times \hat{\pi}_k (Y)$$

   is an isomorphism of abelian groups.

3. *The canonical morphism from the wedge to the weak product is a $\hat{\pi}_*$-isomorphism. In particular, for every finite family of symmetric spectra the canonical morphism from the wedge to the product is a $\hat{\pi}_*$-isomorphism.*

Proof.

1. the wedge inclusion $i_A : A \to A \bigvee B$ has a retraction. So the associated long exact homotopy group sequence of splits into short exact sequences

$$0 \longrightarrow \hat{\pi}_k A \xrightarrow{i_{A*}} \hat{\pi}_k (A \bigvee B) \xrightarrow{i_*} \hat{\pi}_k C(i_A) \xrightarrow{\delta} 0$$

   The mapping cone $C(i_A)$ is isomorphic to $(CA) \bigvee B$ and thus homotopy equivalent to B. So we can replace $\hat{\pi}_k C(i_A)$ with $\hat{\pi}_k B$ and conclude that $\hat{\pi}_k (A \bigvee B)$ splits as the sum of $\hat{\pi}_k A$ and $\hat{\pi}_k B$, via the canonical map.

2. Unstable homotopy groups commute with products, which for finite indexing sets are also sums, which commute with filtered colimits.

3. This is a direct consequence of 1 and 2. More precisely, the composite map

$$\begin{tikzcd}
\hat{\pi}_k X \bigoplus \hat{\pi}_k Y \arrow[r] \arrow[d, "\cong"'] & \hat{\pi}_k \left(X \bigvee Y\right) \arrow[d] \\
\hat{\pi}_k \left(X\right) \times \hat{\pi}_k \left(Y\right) & \hat{\pi}_k \left(X \times Y\right) \arrow[l]
\end{tikzcd}$$

is an isomorphism as coproducts and products are the same in abelian groups. The upper and lower maps are canonical isomorphism, hence so is the right map.

□

2.4 Smash product

Construction of the Smash Product

One of the main features which distinguishes symmetric spectra from the more classical spectra without symmetric group actions is the internal smash product. The smash product of symmetric spectra is very much like the tensor product of a monoidal category.

Recall in any monoidal category (e.g. R-**Mod** where R is a commutative ring), if we have $X \hookrightarrow X'$ and $Y \hookrightarrow Y'$ then we have a commutative diagram

$$\begin{tikzcd}
X \bigotimes X' \arrow[rr, hook] \arrow[dd, hook] & & X \bigotimes Y' \arrow[dl, dotted] \arrow[dd, hook] \\
& Z \arrow[dr, hook] & \\
X' \bigotimes Y \arrow[ur, dotted] \arrow[rr, hook] & & X' \bigotimes Y'
\end{tikzcd}$$

where Z is the pushout, and this should be compatible with respect to the monoidal structure.

To stress that analogy, we recall three different ways to look at the classical tensor product and then give analogies involving the smash product of symmetric spectra.

In the following, R is a commutative ring and $M, N, W \in Ob\left(R\text{-}\mathbf{Mod}\right)$.

Via bilinear maps. A bilinear map $b : M \times N \to W$ is R-linear in both M and N. The tensor product $M \otimes_R N$ is an R-module that represents the functor

$$\begin{array}{ccc} R\text{-}\mathbf{Mod} & \longrightarrow & \mathbf{Set} \\ W & \longmapsto & \mathrm{Bil}_R(M \times N, W) \end{array}$$

so there is a unique bilinear map $i : M \times N \to M \otimes_R N$ such that

$$\begin{array}{ccc} \mathrm{Hom}_R(M \otimes_R N, W) & \longrightarrow & \mathrm{Bil}_R(M \times N, W) \\ f & \mapsto & f \circ i \end{array}$$

is bijective.

Definition 2.8. A bimorphism $b : (X, Y) \to Z$ is a collection of morphisms $\Sigma_p \times \Sigma_q$-equivariant maps (of pointed spaces or simplicial sets)

$$b_{p,q} : X_p \wedge Y_q \longrightarrow Z_{p+q}$$

for $p, q \geq 0$ such that the bilinearity diagram

$$\begin{tikzcd} & X_p \wedge Y_q \wedge \mathbb{S}^1 \arrow[r, "X_p \wedge \tau"] \arrow[dl, "1 \wedge \sigma_q"'] \arrow[d, "b_{p,q} \wedge 1"] & X_p \wedge \mathbb{S}^1 \wedge Y_q \arrow[d, "\sigma_p \wedge 1"] \\ X_p \wedge Y_{q+1} \arrow[dr, "b_{p,q+1}"'] & Z_{p+q} \wedge \mathbb{S}^1 \arrow[d, "\sigma_{p+q}"] & X_{p+1} \wedge Y_q \arrow[d, "b_{p+1,q}"] \\ & X_{p+q+1} & X_{p+1+q} \arrow[l, "1 \wedge \chi_{1,q}"] \end{tikzcd}$$

commutes for all $p, q \geq 0$, where τ is the twist map.

We can then define a smash product of X and Y as a pair $(X \wedge Y, i)$ consisting of a symmetric spectrum $X \wedge Y$ and a universal bimorphism $i : (X, Y) \to X \wedge Y$, i.e., a bimorphism such that for every symmetric spectrum Z the map

$$\begin{array}{ccc} \mathrm{Sp}(X \wedge Y, Z) & \longrightarrow & \mathrm{Bimor}((X, Y), Z) \\ f & \longmapsto & fi = \{f_{p+q} \circ i_{p.q}\}_{p,q \geq 0} \end{array}$$

is bijective.

Adjoint to internal Hom. In $\mathbf{Mod}(R)$ we define the internal hom

$$W^N := \mathrm{Hom}_R(N, W)$$

which has a natural R-module structure, thus we have a functor

$$(-)^N = \mathrm{Hom}_R(N, -) : \mathbf{Mod}(R) \to \mathbf{Mod}(R).$$

We have an adjoint pair

$$- \otimes_R N : \mathbf{Mod}(R) \rightleftarrows \mathbf{Mod}(R) : (-)^N$$

which gives a bijective map

$$\mathrm{Hom}_R(M \otimes_R N, W) \xrightarrow{\cong} \mathrm{Hom}_R\left(M, W^N\right).$$

For any two symmetric spectra Y, Z, the internal hom spectra Z^Y is defined as

$$\left(Z^Y\right)_n = \mathbf{Map}(Y, \mathrm{sh}^n Z)$$

where $(\mathrm{sh}^n Z)_k = Z_{n+k}$ and the Σ_k-action is induced by the inclusion $\Sigma_k \hookrightarrow \Sigma_{n+k}$. The structure map

$$\sigma_n : \left(Z^Y\right)_n \wedge \mathbb{S}^1 \to \left(Z^Y\right)_{1+n}$$

is the composite

$$\mathbf{Map}(Y, \mathrm{sh}^n Z) \wedge \mathbb{S}^1 \xrightarrow{a} \mathbf{Map}\left(Y, \mathbb{S}^1 \wedge \mathrm{sh}^n Z\right) \xrightarrow{\lambda_*} \mathbf{Map}\left(Y, \mathrm{sh}^{1+n} Z\right)$$

where

$$\begin{array}{rccc} a: & \mathbf{Map}\,(Y, \mathrm{sh}^n Z) \wedge \mathbb{S}^1 & \longrightarrow & \mathbf{Map}\left(Y, \mathbb{S}^1 \wedge \mathrm{sh}^n Z\right) \\ & f \wedge t & \longmapsto & \left(\begin{array}{ccc} Y & \to & \mathbb{S}^1 \wedge \mathrm{sh}^n Z \\ y & \mapsto & t \wedge f(y) \end{array}\right) \end{array}$$

and $\lambda : \mathbb{S}^1 \wedge \mathrm{sh}^n Z \to \mathrm{sh}^{1+n} Z$ is defined level-wise as $\lambda_k : \mathbb{S}^1 \wedge Z_{n+k} \to Z_{1+n+k}$ by the composite

$$\mathbb{S}^1 \wedge Z_{n+k} \xrightarrow[\tau]{\cong} Z_{n+k} \wedge \mathbb{S}^1 \xrightarrow{\sigma_n} Z_{n+k+1} \xrightarrow{\chi_{n+k,1}} Z_{1+n+k} \ ,$$

and $\lambda_* : \mathbf{Map}\left(X, \mathbb{S}^1 \wedge \mathrm{sh}^n Y\right) \to \mathbf{Map}\left(X, \mathrm{sh}^{1+n} Y\right)$ is given by postcomposition with λ.

Then the morphism from Y to Z are (in natural bijection with) the points (respectively vertices) of the 0th level of Y^Z.

We claim that for fixed symmetric spectra X and Y , the set-valued functor $\mathrm{Sp}\left(X, (-)^Y\right)$ is representable; the smash product $X \wedge Y$ can then be defined as a representing symmetric spectrum. This point of view can be reduced to the first perspective since the sets $\mathrm{Sp}\left(X, Z^Y\right)$ and $\mathrm{Bimor}\left(\left(X, Y\right), Z\right)$ are in natural bijection. In particular, since the functor $\mathrm{Bimor}\left(\left(X, Y\right), -\right)$ is representable, so is the functor $\mathrm{Sp}\left(X, (-)^Y\right)$.

Smash product as a construction. In $\mathbf{Ch}_R$ we can define the tensor product of two chain complexes as the chain complex

$$\left(C_\bullet \otimes D_\bullet\right)_n \coloneqq \bigoplus_{p+q=n} C_p \otimes C_q$$

with $d_{C \otimes D} = d_C \otimes \mathrm{Id} + \mathrm{Id} \otimes d_D$.

This construction is given by the composite of functors

$$\mathbf{Ch}_R \times \mathbf{Ch}_R \longrightarrow \mathbf{DblCh}_R \xrightarrow{\text{total}} \mathbf{Ch}_R$$

where the first functor is given by $\left(C_\bullet, D_\bullet\right) \mapsto \left(C_p \otimes D_q\right)_{\bullet\bullet}$.

We can extend this idea to symmetric spectra.

We have a functor

$$\mathbf{SymSp} \times \mathbf{SymSp} \longrightarrow \mathbf{DblSymSp}$$

given by $(X,Y) \mapsto \{X_p \wedge Y_q\}_{p,q\geq 0}$. The question is how to define the "total" functor, i.e. how to internalize it?

We observe that $(X \wedge Y)_n \not\coloneqq \bigvee_{p+q=n} (X_p \wedge Y_q)$ because of the twist. More explicitly, if we consider $X = Y = \mathbb{S}$, we would like $\mathbb{S}$ to be a unit, however, we can check level-wise to see that in degree 0: $\mathbb{S}^0 \wedge \mathbb{S}^0 \cong \mathbb{S}^0$; in degree 1: $\left(\mathbb{S}^0 \wedge \mathbb{S}^1\right) \bigvee \left(\mathbb{S}^1 \wedge \mathbb{S}^0\right) \cong \mathbb{S}^1 \bigvee \mathbb{S}^1$ two copies of $\mathbb{S}^1$; and the higher degrees, this becomes a mess because of the twist. Hence we want to define a proper Σ_n-action such that the twists get killed off.

First, we construct

$$\bigvee_{p+q=n} (\Sigma_n)_+ \bigwedge_{\Sigma_p\times\Sigma_q} (X_p \wedge Y_q)$$

which has a natural Σ_n-action on the first component if we consider $(\Sigma_n)_+$ as a discrete pointed space or simplicial set in proper settings.

Second, there are two maps

$$\begin{tikzcd}
\bigvee_{p+q=n} (\Sigma_n)_+ \bigwedge_{\Sigma_p\times\Sigma_q} \left(X_p \wedge \mathbb{S}^1 \wedge Y_q\right) \arrow[d, shift right, "\alpha_X"'] \arrow[d, shift left, "\alpha_Y"] \\
\bigvee_{p+q=n} (\Sigma_n)_+ \bigwedge_{\Sigma_p\times\Sigma_q} (X_p \wedge Y_q)
\end{tikzcd}$$

where α_X is induced by the structure $X_p \wedge \mathbb{S}^1 \xrightarrow{\sigma_p} X_{p+1}$ on X

$$X_p \wedge \mathbb{S}^1 \wedge Y_q \xrightarrow{\sigma_p\wedge 1} X_{p+1} \wedge Y_q ,$$

and α_Y is induced by a twist of $\mathbb{S}^1$ and Y_q and the structure map $Y_q \wedge \mathbb{S}^1 \xrightarrow{\sigma_p} Y_{q+1}$ on Y

$$X_p \wedge \mathbb{S}^1 \wedge Y_q \xrightarrow{1\wedge\tau} X_p \wedge Y_q \wedge \mathbb{S}^1 \xrightarrow{1\wedge\sigma_q} X_p \wedge Y_{q+1} \xrightarrow{1\wedge\chi_{q,1}} X_p \wedge Y_{1+q} .$$

We want to identify these two maps, so we define

$$(X \wedge Y)_n \coloneqq \operatorname{coeq}\left\{ \bigvee_{p+q=n} (\Sigma_n)_+ \bigwedge_{\Sigma_p\times\Sigma_q} \left(X_p \wedge \mathbb{S}^1 \wedge Y_q\right) \underset{\alpha_Y}{\overset{\alpha_X}{\rightrightarrows}} \bigvee_{p+q=n} (\Sigma_n)_+ \bigwedge_{\Sigma_p\times\Sigma_q} (X_p \wedge Y_q) \right\}$$

The structure map $(X \wedge Y)_n \wedge \mathbb{S}^1 \to (X \wedge Y)_{n+1}$ is induced on coequalizers by the wedge of the maps

$$(\Sigma_n)_+ \bigwedge_{\Sigma_p \times \Sigma_q} X_p \wedge Y_q \wedge \mathbb{S}^1 \longrightarrow (\Sigma_n)_+ \textstyle\bigwedge_{\Sigma_p \times \Sigma_q} X_p \wedge Y_{q+1}$$

induced by $1 \wedge \sigma_q : X_p \wedge Y_q \wedge \mathbb{S}^1 \to X_p \wedge Y_{q+1}$.

Equivalently we could have defined the structure map by moving the circle past Y_q, using the structure map of X and then shuffling back with the permutation $\chi_{1,q}$; the definition of $(X \wedge Y)_{n+1}$ as a coequalizer precisely ensures that these two possible structure maps coincide, and that the collection of maps

$$X_p \wedge Y_q \longrightarrow \bigvee_{p+q=n} (\Sigma_n)_+ \bigwedge_{\Sigma_p \times \Sigma_q} (X_p \wedge Y_q) \xrightarrow{\text{proj}} (X \wedge Y)_{p+q}$$

forms a bimorphism, and in fact a universal one.

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