Relative Join Construction and Towers of Borel Fibrations

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Motivation

$$\begin{array}{lcccc}
\text{Lie Groups} & \xrightarrow{B} & \text{Algebraic Topology} & \xrightarrow{H^*(-;\mathbb{Q})} & \mathbb{Q}\text{-algebras} \\
(\text{cpt, conn}) \; G & & BG = EG/G & & H^*(BG;\mathbb{Q}) \\
\uparrow & & \uparrow & & \downarrow \\
\text{max torus } \; T & & BT = EG/T & & H^*(BT;\mathbb{Q}) \\
\text{Weyl group } \; W = N_G(T)/T & & W \times BT \longrightarrow BT & & W \curvearrowright H^*(BT;\mathbb{Q}) \\
 & & (nT, [x]_T) \longmapsto [xn^{-1}]_T & &
\end{array}$$

Thm (Notbohm, 95') Two compact Lie groups G and H are isomorphic as Lie groups iff BG and BH are homotopy equivalent.

Thm (Borel 53') The natural inclusion $i: T \hookrightarrow G$ induces a monomorphism on rational cohomology rings

$$Bi^*: H^*(BG;\mathbb{Q}) \hookrightarrow H^*(BT;\mathbb{Q})$$

whose image is the subring of W-invariants

$$H^*(BG;\mathbb{Q}) \cong H^*(BT;\mathbb{Q})^W$$

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Example. 1) $G = SU(2)$, $T = U(1)$, $W = \mathbb{Z}/2\mathbb{Z}$

$$H^*(BG;\mathbb{Q}) = \mathbb{Q}[y] \hookrightarrow H^*(BT;\mathbb{Q}) = \mathbb{Q}[x] \circlearrowleft \mathbb{Z}/2\mathbb{Z} \quad (x \mapsto -x)$$

$$y \longmapsto x^2$$

2) $G = U(n)$, $T = U(1)^n$, $W = S_n$

$$H^*(BG;\mathbb{Q}) = \mathbb{Q}[c_1,\cdots,c_n] \hookrightarrow H^*(BT;\mathbb{Q}) = \mathbb{Q}[t_1,\cdots,t_n]$$

(Chern class) $c_i \longmapsto e_i(t_1,\cdots,t_n)$ i-th elementary symm poly

3) $G = Sp(n)$, $T = U(1)^n$, $W = B_n$ signed permutation

$$H^*(BG;\mathbb{Q}) = \mathbb{Q}[q_1,\cdots,q_n] \hookrightarrow H^*(BT;\mathbb{Q}) = \mathbb{Q}[t_1,\cdots,t_n]$$

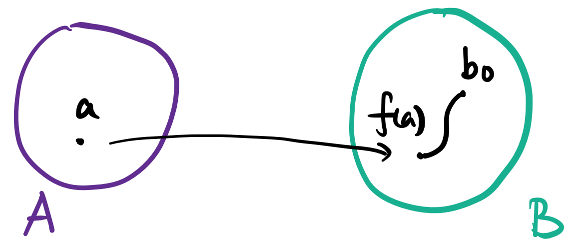
(Pontryagin class) $q_i \longmapsto e_i(t_1^2,\cdots,t_n^2)$ i-th signed symm poly

Ganea Construction (65', 67') Lusternik-Schineralmann cat

$f : (A, a_0) \to (B, b_0)$ pointed topological spaces

Homotopy fiber $\quad \mathrm{fib}_*(f) = \{ (a, \gamma) \in A \times B^I \mid \gamma(0) = f(a),\ \gamma(1) = b_0 \}$

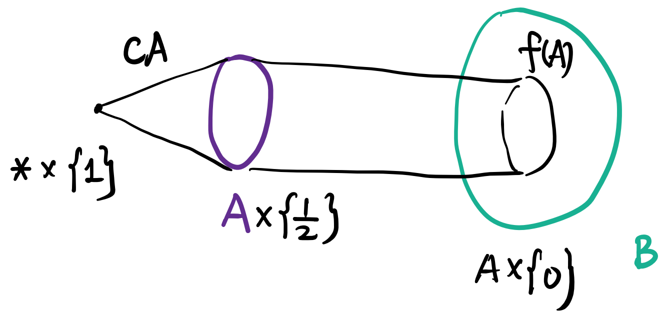
$$\begin{tikzcd} \mathrm{fib}_*(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E_f = \{ (a,\gamma) \in A \times B^I \mid \gamma(1) = b_0 \} \simeq A \arrow[d] \\ pt \arrow[r, "b_0"'] & B \end{tikzcd}$$



Homotopy cofiber $\quad \mathrm{cof}_*(f) = A \times I \sqcup B \big/ (f(a) \sim (a,0),\ (a,1) \sim (a',1),\ \forall a, a' \in A)$

$= \text{mapping cone of } f$

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & M_f = A \times I \underset{f}{\cup} B \simeq B \arrow[d] \\ pt \arrow[r] & \mathrm{cof}_*(f) \end{tikzcd}$$



Ganea Construction (Fiber-cofiber construction)

Thm (Ganea) Let $F \xrightarrow{j} E \xrightarrow{p} B$ be a fibration, $F = \mathrm{fib}(p)$.

Let $X_1 = \mathrm{cof}(j)$

$$\begin{tikzcd} E \arrow[r, "p"] \arrow[d] & B \\ X_1 \arrow[ur, dotted, "p_1"'] & \end{tikzcd}$$

$F_1 = \mathrm{fib}(p_1)$

then there is a homotopy commutative diagram

$$\begin{tikzcd} F \arrow[r, "j"] \arrow[d] & E \arrow[r, "p"] \arrow[d, "\pi_1"] & B \arrow[d, equal] \\ F * \Omega B = F_1 \arrow[r, "j_1"] & E_1 \arrow[r, "p_1"] & B \end{tikzcd}$$

where $A * B := A \times I \times B / \{(a,0,b) \sim (a',0,b),\ (a,1,b) \sim (a,1,b') \ \forall\ a, a' \in A,\ b, b' \in B\}$

Ganea tower

$$\begin{tikzcd} F \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & B \arrow[d, equal] \\ F * \Omega B \simeq F_1 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & B \arrow[d, equal] \\ F * \Omega B * \Omega B \simeq F_2 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & B \arrow[d, equal] \\ \cdots & \cdots & \cdots \end{tikzcd}$$

Let's consider the fibration $G/T \longrightarrow BT \longrightarrow BG$ for $G = SU(2)$

Example of Ganea Construction

$G = SU(2)$ $\quad T = U(1)$ $\quad W = \mathbb{Z}/2\mathbb{Z}$

G-spaces

$$\begin{tikzcd}
 & G/T \arrow[r, "j"] \arrow[d] & BT \arrow[r, "p"] \arrow[d, "\pi_1"] & BG \arrow[d, equal] \\
G/T * G \simeq G/T * \Omega BG \simeq & F_1 \arrow[r, "j_1"] \arrow[d] & X_1 \arrow[r, "p_1"] \arrow[d, "\pi_2"] & BG \arrow[d, equal] \\
G/T * G * G \simeq G/T * \Omega BG * \Omega BG \simeq & F_2 \arrow[r, "j_2"] \arrow[d] & X_2 \arrow[r, "p_2"] \arrow[d] & BG \arrow[d, equal] \\
 & \cdots & \cdots & \cdots
\end{tikzcd}$$

In step m, we have a fibration sequence

$$F_m = G/T * \underbrace{G * \cdots * G}_{m} \xrightarrow{j_m} X_m = (F_m)_{hG} = EG \times_G F_m \xrightarrow{p_m} BG$$

Thm [Berest, Ramadoss] The spaces X_m together with the whisker map $p_m : X_m \to BG$ satisfies the following properties:

1) X_m is a W-space, $\pi_m : X_m \to X_{m+1}$ is W-equivariant

2) The induced map $(X_m)_{hW} \xrightarrow{\overline{p_m}} BG$ gives an isomorphism on $H^*(-;\mathbb{Q})$

3) $\operatorname{hocolim}_m X_m \simeq BG$

4) π_m induces monomorphisms $H^*(X_{m+1};\mathbb{Q}) \hookrightarrow H^*(X_m;\mathbb{Q})$

5) $H^*(X_m;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = Q_m(W)$ the ring of W-quasiinvariants with multiplicity m.

Quasi-invariants in Rank 1 Case

$G = SU(2) \quad T = U(1) \quad W = \mathbb{Z}/2\mathbb{Z}$

$$H^*(BG;\mathbb{Q}) = \mathbb{Q}[x^2] = \mathbb{Q}[x]^{\mathbb{Z}/2\mathbb{Z}} \hookrightarrow \mathbb{Q}[x] = H^*(BT;\mathbb{Q})$$

Def. A polynomial $f \in \mathbb{C}[x]$ is called W-invariant if $f(-x) - f(x) \equiv 0$.

Def. A polynomial $f \in \mathbb{C}[x]$ is called W-quasi-invariant of multiplicity m if $\quad f(-x) - f(x) \in (x)^{2m}$

Let $Q_m(W)$ be the set of all W-quasi-invariants of multiplicity m.

$$Q_m(\mathbb{Z}/2\mathbb{Z}) = \mathbb{C}[x^2, x^{2m+1}]$$

$$\mathbb{C}[x^2] \hookrightarrow \mathbb{C}[x^2, x^3] \hookrightarrow \mathbb{C}[x^2, x^5] \hookrightarrow \cdots \hookrightarrow \mathbb{C}[x^2, x^{2m+1}] \hookrightarrow \cdots \hookrightarrow \mathbb{C}[x]$$

Properties
1) $Q_m(W)$ is a subalgebra of $\mathbb{C}[x]$ with W-action
2) There is a W-equivariant filtration.
3) $Q_m(W)$ is a finite module over $\mathbb{C}[x]^W$.
4)* $Q_m(W)$ is a finitely generated free module over $\mathbb{C}[x]^W$ of rank $|W|$.

Rmk. The above definition can be generalized to finite (and complex) reflection groups.
- * [Feigin - Veselov] dihedral group
- * [Etingof - Ginzburg, BEG] finite reflection group

Relative Join Construction

Homotopy pullback

$$\begin{tikzcd} A \times^h_B C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner\ \text{h.p.b}"] & C \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

Homotopy pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{h.p.o.}\ \ulcorner"] & C \arrow[d] \\ A \arrow[r] & A \vee^h_X C \end{tikzcd}$$

(Relative) Join

$$\begin{tikzcd} A \times^h_B C \arrow[r] \arrow[d] & C \arrow[d] \arrow[ddr, bend left, "g"] & \\ A \arrow[r] \arrow[drr, bend right, "f"'] & A \underset{B}{*} C \arrow[dr, dashed] & \\ & & B \end{tikzcd}$$

Example. 1) $B \simeq pt$ $\quad A \underset{B}{*} C \simeq A * C$

2) $X_1 = \mathrm{Cof}(BT \xrightarrow{p} BG) \simeq pt \underset{BG}{*} BT$

Thm (J-P. Doeraene) Given two fibrations $F \to E \to B$ and $F' \to E' \to B$, there is a fibration sequence

$$F * F' \longrightarrow E \underset{B}{*} E' \longrightarrow B.$$

Rewrite Ganea Construction

$\mathbb{C}P^1 \quad G/T \to BT \to BG$

$S^3 \quad G \to EG \simeq pt \to BG$

$$\left.\begin{array}{l} G/T \to BT \to BG \\ G \to EG \simeq pt \to BG \end{array}\right\} \Rightarrow \underset{\underset{G/T * \Omega BG}{\|\wr}}{G/T * G} \longrightarrow \underset{\underset{BT \underset{BG}{*} pt}{\|\wr}}{(G/T * G)_{hG}} \longrightarrow BG$$

Generalization for high rank G

Naive attempt: apply Ganea construction to $BT \to BG$

Issue: rational cohomology is not free as module over $H^*(BG;\mathbb{Q})$.

Modification:

1) choose a corank 1 Lie subgroup $H \subset G$ s.t. $G/H \simeq S^{2k-1}$

2) apply relative join construction to the following two fibrations

$$\left.\begin{array}{l} G/T \xrightarrow{j} BT \xrightarrow{p} BG \\ G/H \longrightarrow BH \longrightarrow BG \end{array}\right\} \Rightarrow \boxed{F_m \xrightarrow{j_m} X_m = (F_m)_{hG} \xrightarrow{p_m} BG}$$

$$F_m = G/T *^m G/H, \qquad X_m \simeq BT \underset{BG}{*^m} BH$$

Generalization for high rank G

Thm (Berest-L-Ramadoss) The spaces $X_m = BT \ast^{m}_{BG} BH$ together with the whisker maps $p_m: X_m \to BG$ and $\pi_m: X_m \to X_{m+1}$ satisfies the following properties:

1) X_m are W-spaces and all maps are W-equivariant
2) $(X_m)_{hW} \to BG$ induces isomorphisms on $H^*(-;\mathbb{Q})$
3) $\operatorname{hocolim}_m X_m \xrightarrow{\sim} BG$ is a weak homotopy equivalence
4) $H^*(X_m;\mathbb{Q})$ is a free module over $H^*(BG;\mathbb{Q})$ of rank $|W|$.
5) There is a W-equivariant filtration

$$H^*(BG;\mathbb{Q}) \hookrightarrow H^*(X_1;\mathbb{Q}) \hookrightarrow \cdots \hookrightarrow H^*(X_m;\mathbb{Q}) \hookrightarrow \cdots \hookrightarrow H^*(BT;\mathbb{Q}).$$

Remark. When $G = SU(2)$, $H = \{e\}$, this recovers the result of [BR] in the rank 1 case.

Examples.

For $G = U(n)$, $H = U(n-1)$,

$$H^*(BT;\mathbb{Q}) = \mathbb{Q}[t_1,\cdots,t_n]$$

$$H^*(BG;\mathbb{Q}) = \mathbb{Q}[c_1,\cdots,c_n] \qquad c_i = e_i(t_1,\cdots,t_n)$$

$$H^*(BH;\mathbb{Q}) = \mathbb{Q}[c_1,\cdots,c_{n-1}]$$

$$H^*(X_m;\mathbb{Q}) = \mathbb{Q}[c_1,\cdots,c_n] + \mathbb{Q}[t_1,\cdots,t_n]\,c_n^m$$

$$= \{ f \in \mathbb{Q}[t_1,\cdots,t_n] \mid \sigma_{ij}\cdot f \equiv f \mod (c_n)^m,\ \forall \text{ permutation } \sigma_{ij} \}$$

$$\sigma_{ij}\cdot f(t_1,\cdots,t_n) = f(t_1,\cdots,\underset{i}{t_j},\cdots,\underset{j}{t_i},\cdots)$$

Comparison With Classical Quasi-invariants

$$Q_m(S_n) = \{ f \in \mathbb{C}[t_1,\cdots,t_n] \mid \sigma_{ij}\cdot f \equiv f \mod (t_i - t_j)^{2m},\ \forall i,j \}$$

When $n=3$

$$\mathrm{Hilb}_{H^*(X_m;\mathbb{Q})}(t) = \frac{1+2t^{6m+2}+2t^{6m+4}+t^{6m+6}}{(1-t^2)(1-t^4)(1-t^6)}$$

$$\mathrm{Hilb}_{Q_m(S_3)}(t) = \frac{1+2t^{6m+2}+2t^{6m+4}+t^{12m+6}}{(1-t^2)(1-t^4)(1-t^6)}$$

More Results:

1) basis as free module
2) T-equivariant rational cohomology
3) G-equivariant K-theory
4) T-equivariant K-theory
5) other examples
6) topological realization of quasi-invariants (partial)

Ongoing Processes:

1) GKM (Goresky-Kottwitz-MacPherson) and moment graph description
2) (Complete) topological realization of quasi-invariants
3) Uniform description of everything above