Quasi-flag Manifolds and Moment Graphs

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Outline

1. Quasi-invariants of finite reflection groups
2. Topological Realization Problem
3. Rank 1 case
4. Higher Rank Case
5. Futher Directions

Invariants of Finite Reflection Groups

W finite reflection group (e.g., S_n)

V (complexified) representation

$\mathcal{H} = \{\alpha\}$ reflection hyperplanes $\alpha \in V^*$

$s_\alpha \in W$ reflection along α

Definition. A polynomial $f \in \mathbb{C}[V] = \mathrm{Sym}_{\mathbb{C}}(V^*)$ is <u>W-invariant</u> if

$$s_\alpha(f) = f \ , \ \forall \alpha \in \mathcal{H}$$

Theorem. (Chevalley-Shephard-Todd)

- $\mathbb{C}[V]^W$ is a polynomal algebra.
- $\mathbb{C}[V]$ is a free module over $\mathbb{C}[V]^W$ of rank $|W|$.

Quasi-invariants of Finite Reflection Groups

W finite reflection group

V (complexified) representation

$\mathcal{A} = \{\alpha\}$ reflection hyperplanes $\alpha \in V^*$

$s_\alpha \in W$ reflection along α

$m: \mathcal{A} \longrightarrow \mathbb{Z}_{\geq 0}$ W-invariant multiplicity

Definition. (Veselov–Chalykh) A polynomial $f \in \mathbb{C}[V] = \mathrm{Sym}_{\mathbb{C}}(V^*)$ is W-quasi-invariant of multiplicity m if

$$s_\alpha(f) \equiv f \mod (\alpha)^{2m_\alpha}, \ \forall \alpha \in \mathcal{H}$$

$Q_m(W)$ = the set of quasi-invariants of multiplicity m.

$$Q_0(W) = \mathbb{C}[V], \quad Q_\infty(W) := \bigcap_m Q_m(W) = \mathbb{C}[V]^W.$$

Example 1. (Rank 1) $W = \mathbb{Z}/2$ $\mathbb{C}[V] = \mathbb{C}[x]$ $\alpha = x$ $m \in \mathbb{Z}_{\geq 0}$

$$
\begin{array}{rcl}
\mathbb{C}[V] = Q_0(\mathbb{Z}/2) & = \mathbb{C}[x] & = \mathbb{C}[x^2] \oplus x \cdot \mathbb{C}[x^2] \\
\cup & & \\
Q_1(\mathbb{Z}/2) & = \mathbb{C}[x, x^3] & = \mathbb{C}[x^2] \oplus x^3 \cdot \mathbb{C}[x^2] \\
\cup & & \\
Q_2(\mathbb{Z}/2) & = \mathbb{C}[x^2, x^5] & = \mathbb{C}[x^2] \oplus x^5 \cdot \mathbb{C}[x^2] \\
\cup & & \\
\vdots & & \\
\cup & & \\
Q_m(\mathbb{Z}/2) & = \mathbb{C}[x^2, x^{2m+1}] & = \mathbb{C}[x^2] \oplus x^{2m+1} \cdot \mathbb{C}[x^2] \\
\cup & & \\
\vdots & & \\
\cup & & \\
\mathbb{C}[V]^W = Q_\infty(\mathbb{Z}/2) & = \bigcap_m Q_m(\mathbb{Z}/2) & = \mathbb{C}[x^2]
\end{array}
$$

Observation. (1) $Q_m(\mathbb{Z}/2)$ is an algebra.

(2) $Q_m(\mathbb{Z}/2)$ is a free module over $\mathbb{C}[V]^W$ of rank $|\mathbb{Z}/2| = 2$.

Example 2. (Rank 2) Consider $W = S_3$ as dihedral group acts on $\mathbb{C}[V] = \mathbb{C}[z, \bar{z}]$ with $\mathcal{A} = \{ z^3 - \bar{z}^3 = 0 \}$, the configuration of 3 lines in $\mathbb{R}^2$ in coordinates $z = x + iy$ and $\bar{z} = x - iy$.

- $\mathbb{C}[V]^{S_3} = \mathbb{C}[z\bar{z}, z^3 - \bar{z}^3]$
- $Q_1(S_3)$ is generated as $\mathbb{C}[V]^{S_3}$-module by $\{ 1, \; z^4 + 2\bar{z}^3 z, \; \bar{z}^4 + 2z^3\bar{z}, \; z^5 + 5\bar{z}^3 z^2, \; \bar{z}^5 + 5z^3\bar{z}^2, \; (z^3 - \bar{z}^3)^3 \}$.

Theorem (Feigin-Veselov, Berest-Etingof-Ginzburg, Berest-Chalykh) For any finite complex reflection group W, and any W-invariant multiplicity m, $Q_m(W)$ is a free graded module over $\mathbb{C}[V]^W$ of rank $|W|$.

Topological Realization of Quasi-invariants

- (Quillen) $Q_m(W)$ is topological realizable for any m.
- Diagram of spaces with natural structures.

Let $\mathcal{M}(W)$ be the poset of W-invariant multiplicities, where

$$m < m' \iff m_\alpha < m'_\alpha \ , \ \forall \alpha \in \mathcal{A}.$$

- $Q_m(W)$ is stable under W-action.
- $Q_m(W)$ assembles into a W-equivariant diagram of algebras

$$Q_\bullet(W) : \mathcal{M}(W)^{op} \longrightarrow \mathrm{CommAlg}_{\mathbb{C}}.$$

- $Q_0(W) = \mathbb{C}[V]$.
- $\lim\limits_{m} Q_m(W) = \mathbb{C}[V]^W$.
- $Q_m(W)^W = \mathbb{C}[V]^W$.

Topological Realization of Quasi-invariants

Let G be a compact connected Lie group, $T \subset G$ maximal torus

$W = N_G(T)/T$ the associated Weyl group.

$BG = EG/G$ classifying space of G

$BT = EG/T$ classifying space of T

$$G/T \xhookrightarrow{j} BT \xrightarrow{p} BG$$

Theorem (Borel 53') The map $p: BT \to BG$ induces a monomorphism on rational cohomology algebras $p^*: H^*(BG;\mathbb{Q}) \hookrightarrow H^*(BT;\mathbb{Q})$, so that

$$H^*(BG;\mathbb{Q}) \cong H^*(BT;\mathbb{Q})^W = \mathbb{Q}[V]^W$$

where $V = H_*(BT;\mathbb{Q})$ is a reflection representation of W.

Realization Problem $\quad T \subset G$

Construct a diagram of spaces $X_m(G,T)$ over the poset $\mathcal{M}(W)$:

$$\begin{tikzcd}
BT = X_0(G,T) \arrow[r] \arrow[drrrr, "p_0"'] & \cdots \arrow[r] & X_m(G,T) \arrow[r, "\pi_{m,m'}"] \arrow[drr, "p_m"] & X_{m'}(G,T) \arrow[r] \arrow[dr, "p_{m'}"] & \cdots \\
& & & & BG
\end{tikzcd}$$

satisfying:

(P1) $X_m(G,T)$ is a W-space, all maps are W-equivariant, and

$$p_0 = p : BT \longrightarrow BG, \qquad p_{m'} \circ \pi_{m,m'} = p_m.$$

(P2) $\operatorname{hocolim}_{\mathcal{M}(W)} X_m(G,T) \simeq BG$.

(P3) p_m factors through the htpy W-orbit $X_m(G,T)_{hW}$, inducing

$$H^*_W(X_m(G,T);\mathbb{Q}) \cong H^*(BG;\mathbb{Q}).$$

(P4) $\pi_{m,m'}$ induces an injective map on (even-dimensional) rational cohomology so that p^* factors into a $\mathcal{M}(W)^{op}$-diagram of algebra maps

$$H^*(BT;\mathbb{Q}) \hookleftarrow H^*(X_1(G,T);\mathbb{Q}) \hookleftarrow \cdots \hookleftarrow H^*(X_m(G,T);\mathbb{Q}) \hookleftarrow \cdots \hookleftarrow H^*(BG;\mathbb{Q}).$$

(P5) $H^*(X_m(G,T);\mathbb{Q}) \otimes \mathbb{C} \cong Q_m(W)$.

Realization Problem

Theorem (Berest-L-Ramadoss) For any compact connected Lie group G with maximal torus T, there exists a diagram of spaces $X_m(G,T)$ satisfying properties (P1)-(P3), and (P4), (P5) holds for even-dimensional cohomology. Moreover

$$X_m(G,T) = EG \times_G F_m^+(G,T)$$

where the G-spaces $F_m^+(G,T)$ form a diagram over $\mathcal{M}(W)$ with properties similar to (P1) - (P5).

Example (BR) Rank 1 case: $G = SU(2)$, $T = U(1)$, $W = \mathbb{Z}/2$

$$F_m(G,T) = G/T \overbrace{* \cdots *}^{m} G \simeq S^2 *^m S^3 = S^{4m+2}$$

$$X_m(G,T) = EG \times_G F_m(G,T) \qquad H^*(X_m;\mathbb{Q}) \otimes \mathbb{C} \simeq Q_m(\mathbb{Z}/2)$$

$$\begin{tikzcd}
G/T \arrow[r] \arrow[d, equal] & BT \arrow[r] \arrow[d, equal] & BG \arrow[d, equal] \\
F_0(G,T) \arrow[r] \arrow[d] & X_0(G,T) \arrow[r] & BG \arrow[d, equal] \\
F_1(G,T) \arrow[r] \arrow[d] & X_1(G,T) \arrow[r] \arrow[d] & BG \arrow[d, equal] \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d, equal] \\
F_m(G,T) \arrow[r] \arrow[d] & X_m(G,T) \arrow[r] \arrow[d] & BG \arrow[d, equal] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

iterated Ganea construction

Rmk. Direct application of Ganea construction does not provide realization for higher rank Lie groups.

Bruhat Moment Graph

- "decomposition" of flag manifold G/T over the face category $C(\Gamma)$ of Bruhat moment graph Γ of the root system R_W of W to get $Fo(G,T)$.
- replace $Fo(G,T)$ with an "m-deformed" diagram defined over a "m-thickened" category $C^{(m)}(\Gamma)$.

$\Gamma = (V_\Gamma, E_\Gamma)$ simple unoriented graph

- $V_\Gamma = W$
- $E_\Gamma = \{ e(s_\alpha, w),\ \alpha \in R_+,\ w \in W \}$

$$w \ \overset{e(s_\alpha, w)}{\underset{\alpha}{\rule[0.5ex]{3em}{0.4pt}}} \ s_\alpha w$$

label

$\Downarrow$

Poset category $C(\Gamma)$

- Objects $V_\Gamma \sqcup E_\Gamma$
- Morphisms non-identity morphisms

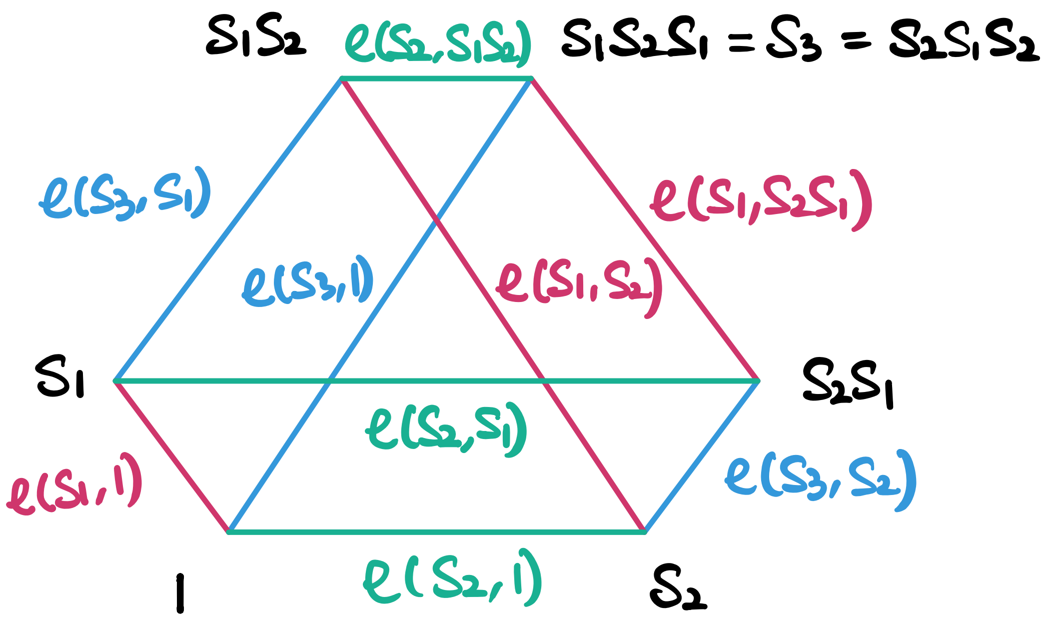
$$w \xleftarrow{} e(s_\alpha, w) \xrightarrow{} s_\alpha w$$

Example. Bruhat graph for $W = S_3$

$s_1 = (12)$

$s_2 = (23)$

$s_3 = (13)$



Geometric Interpretation of Γ

$G_{\mathbb{C}}$ complex connected reductive algebraic group containing G as a maximal compact subgroup (e.g. $GL_n(\mathbb{C}) \supset U(n)$)

$T_{\mathbb{C}}$ complex torus in $G_{\mathbb{C}}$ containing T (e.g. diagonal)

B Borel subgroup in $G_{\mathbb{C}}$ containing $T_{\mathbb{C}}$ (e.g. upper-triangular)

The inclusion $G \hookrightarrow G_{\mathbb{C}}$ induces a homeomorphism

$$\boxed{G/T \cong G_{\mathbb{C}}/B =: F}$$

where F is viewed as a space with analytic topology.

The Bruhat graph Γ describes the structure of 0- and 1-dimensional $T_{\mathbb{C}}$-orbits in F.

- $V_\Gamma \cong (G_{\mathbb{C}}/B)^{T_{\mathbb{C}}} \cong (G/T)^T$ fixed points
- $E_\Gamma \cong \{T_{\mathbb{C}}\text{-orbits of complex dimension 1 } \mathcal{O}_{e(s_\alpha, w)}\}$

"Homology Decomposition" of G/T

$\mathcal{C}(\Gamma)$ is equipped with a W-action induced from $N_G(T)$-action on G/T:

$$w \longmapsto gw, \quad e(s_\alpha, w) \longmapsto e(g s_\alpha g^{-1}, gw)$$

$$\mathcal{C}(\Gamma)_{hW} = \underline{W} \int \mathcal{C}(\Gamma)$$ Grothendieck construction $\underline{W} \to \mathcal{C}at$

There is a T-orbit function on $\mathcal{C}(\Gamma)$

$$\mathcal{F}_0 : \mathcal{C}(\Gamma) \longrightarrow \mathrm{Top}^T$$
$$w \longmapsto \{w\}$$
$$e(s_\alpha, w) \longmapsto \mathcal{O}_{e(s_\alpha, w)}$$ $\simeq T/T_\alpha \quad T_\alpha = \ker(\alpha) \subset T$

and it extends to a functor

$$\boxed{G \underset{T}{\times} \mathcal{F}_0 : \mathcal{C}(\Gamma)_{hW} \longrightarrow \mathrm{Top}^G}$$

"Homology Decomposition" of G/T

Theorem (BLR) For every prime $p \neq 2$ (including $p=0$), there is a G-equivariant mod-p cohomology isomorphism

$$F_0(G,T) := \operatorname{hocolim}_{C(\Gamma)_{hW}} \left[G \underset{T}{\times} \mathcal{F}_0 \right] \underset{p}{\simeq} G/T .$$

Moreover, there is a p-plus construction X^+ on X such that

$$F_0(G,T) \underset{p}{\simeq} F_0^+(G,T), \quad \forall p \neq 2$$

and $F_0^+(G,T)$ is simply connected.

Quasi-flag Manifold $Fm^{+}(G,T)$

- Replace $C^{0}(\Gamma) = C(\Gamma)$ by $\mathcal{C}^{(m)}(\Gamma) := C(\Gamma) \int \Gamma^{(m)}$, where

$$\Gamma^{(m)} : C(\Gamma) \longrightarrow \mathrm{Cat}$$
$$e(S_\alpha, w) \longmapsto sd(\Delta^{m_\alpha})$$

wher $sd(\Delta^{m_\alpha})$ is the barycentric subdivision of the m-simplex Δ^{m_α}

- $\mathcal{C}^{(m)}(\Gamma)$ is equipped with a W-action $\rightsquigarrow \mathcal{C}^{(m)}(\Gamma)_{hW}$
- Define $Fm : \mathcal{C}^{(m)}(\Gamma) \longrightarrow \mathrm{Top}^{T}$
- Extend to $G \underset{T}{\times} Fm : \mathcal{C}^{(m)}(\Gamma)_{hW} \longrightarrow \mathrm{Top}^{G}$
- Obtain $Fm(G,T) := \operatorname{hocolim}_{\mathcal{C}^{(m)}(\Gamma)_{hW}} G \underset{T}{\times} Fm$
- Apply universal p-plus construction to obtain simply connected $Fm^{+}(G,T)$

Quasi-flag Manifold $F_m^+(G,T)$

Definition. We call the G-spaces $F_m^+(G,T)$ the <u>m-quasi-flag manifold</u> associated to (G,T). In addition, we define <u>the spaces of m-quasi-invariants</u> by taking the Borel homotopy G-quotient

$$\boxed{X_m(G,T) := EG \times_G F_m^+(G,T).}$$

Theorem (BLR) For any compact connected Lie group G, the spaces $X_m(G,T)$ satisfy all axioms of our realization problem. Thus they provide a cohomological realization for the algebra of quasi-invariants $Q_m(W)$ of the Weyl group $W = W_G(T)$.

Remark. There is another natural W-action on Γ given by $w \mapsto w.g^{-1}$ and $e(S_\alpha, g) \longmapsto e(S_\alpha, wg^{-1})$ that induces a nontrivial W-action on F_m and X_m.

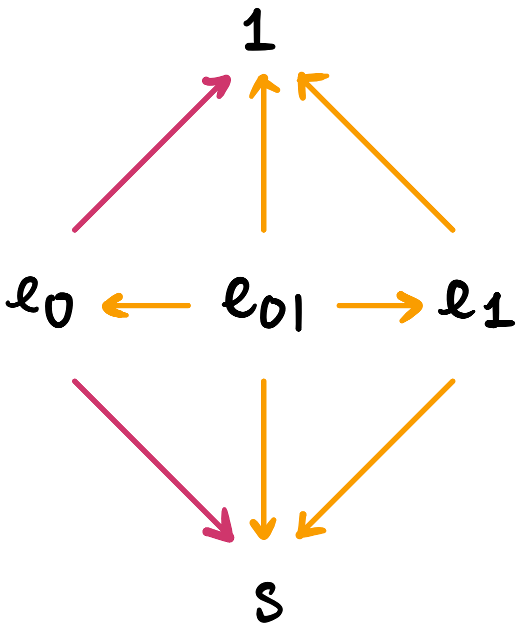
Work In Progress / Future Directions

- K-theory $\longleftrightarrow$ exponential quasi-invariants
- Elliptic cohomology $\longleftrightarrow$ elliptic quasi-invariants
- Partial flag manifolds and other GKM-spaces
- m-deformation/thickening in terms of strata

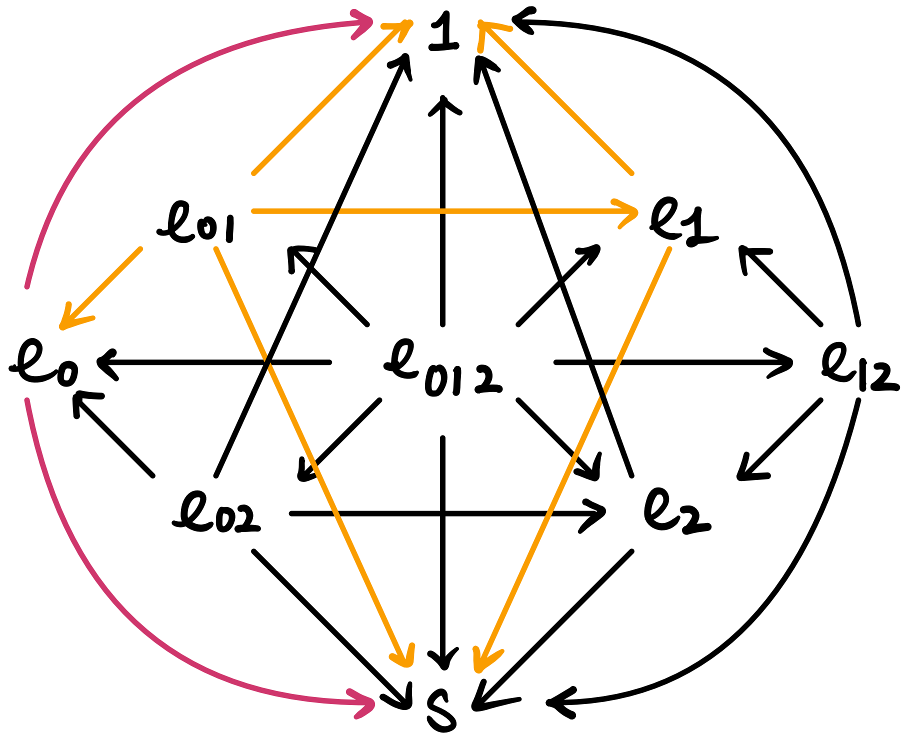
  $\rightsquigarrow$ T-invariant CW decomposition of flag manifold

Visualization of $C^{(m)}(\Gamma)$ for $W = \mathbb{Z}/2\mathbb{Z}$

$C^{0}(\Gamma)$



$C^{(1)}(\Gamma)$



$C^{(2)}(\Gamma)$

G-homotopy decomposition of $F_m(G,T)$

- $\mathcal{F}_m : C^{(m)}(\Gamma) \longrightarrow Top^T$

  $\omega \longmapsto \{\omega\}$

  $e_\sigma(S_\alpha,\omega) \longmapsto O^{\times\sigma}_{e(S_\alpha,\omega)}$ $\circlearrowleft$ T diagonal $\quad \sigma \subset [m_\alpha]$

- $F_m(G,T) := \mathrm{hocolim}_{C^{(m)}(\Gamma)_{hW}} G \times_T \mathcal{F}_m$

- $F_m(G,T) \simeq \mathrm{hocolim}_{C(\Gamma)_{hW}} (G \times_T F_m)^*$ for

$$(G \times_T F_m)^* : C(\Gamma)_{hW} \longrightarrow Top^G$$

$$\{\omega\} \longmapsto G \times_T \{\omega\}$$

$$e(S_\alpha,\omega) \longmapsto G \times_T O^{*(m_\alpha+1)}_{e(S_\alpha,\omega)}$$

$$\begin{tikzcd}
& G \times_T O^{*(m_\alpha+1)}_{e(S_\alpha,1)} \arrow[loop above, "S_\alpha"] \arrow[d, shift right=2, bend right, "f_\alpha"'] \arrow[d, shift left=2, bend left, "f_\alpha S_\alpha"] \arrow[ddl, dash, dotted, bend right] \arrow[ddr, dash, dashed, bend left] & \\
& G/T & \\
G \times_T O^{*(m_\gamma+1)}_{e(S_\gamma,1)} \arrow[loop below, "S_\gamma"] \arrow[ur, bend left, "f_\gamma"] \arrow[ur, bend right, "f_\gamma S_\gamma"'] & & G \times_T O^{*(m_\beta+1)}_{e(S_\beta,1)} \arrow[loop below, "S_\beta"] \arrow[ul, bend right, "f_\beta"'] \arrow[ul, bend left, "f_\beta S_\alpha"]
\end{tikzcd}$$

p-plus Construction

- Quillen's plus construction : X path connected
  - eliminate $\pi_1(X)$ ← perfect group $\pi_1(X) = [\pi_1(X), \pi_1(X)]$
  - does not change cohomology
- $\pi_1(\mathcal{F}_m(G,T)) = \underset{\alpha \in R^+}{*} W_\alpha$, $W_\alpha = \langle S_\alpha \rangle \cong \mathbb{Z}/2$ not perfect
- R-plus construction (Broto-Levi-Oliver, 2021)
  - R comm ring with 1
  - X connected CW complex
  - either $\mathrm{char}(R) \neq 0$ and $\pi_1(X)$ is R-perfect, i.e.

$$H_1(\pi_1(X), R) = 0,$$

or $\mathrm{char}(R) = 0$ and $\pi_1(X)$ is strongly perfect, i.e.

$$\mathrm{Tor}^1_{\mathbb{Z}}(R, H_1(\pi_1(X), R)) = 0.$$

- $R = \mathbb{F}_p$ $(p \neq 2)$ or $\mathbb{Q}$ $(p = 0)$

p-plus Construction

Theorem. The spaces $F_m(G,T)$ admits a universal functorial p-plus construction
$$q_m : F_m(G,T) \longrightarrow F_m^+(G,T)$$
that eliminates the fundamental groups for all $m \in \mathcal{M}(W)$ without changing mod p cohomology for all $p \neq 2$.

The spaces $F_m^+(G,T)$ admits an explicit homotopy decomposition
$$\boxed{F_m^+(G,T) \simeq \operatorname{hocolim}_{\mathcal{C}^{(m)}(\Gamma)^+_{hW}} \left(G \times_T F_m^+ \right)}$$
where $\mathcal{C}^{(m)}(\Gamma)^+_{hW} = \operatorname{hocof}\left\{ \mathcal{C}(\Gamma)_{hW} \xrightarrow{i} \mathcal{C}^{(m)}(\Gamma)_{hW} \right\}$.