MATH-M321 Intuitive Topology

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1 Overview

1.1 What is topology?

If you are taking this class, most likely you've seen the following bus transit map in Bloomington in Figure 1.

Question 1.1. *Think about some routine bus trips that you usually take, what are the most essential things you need to keep in mind to reach your destination?*

The answers are most likely the connections you need to make – getting off at the right stop and changing to the correct new bus. You will not remember/care about the actual distance between stops (unless you are in a hurry and you do care about time), or the angles to make a turn at crossings. Therefore we can redraw the bus transit map near campus in Figure 2.

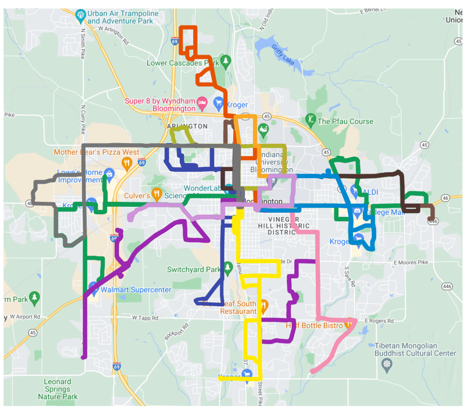

Figure 1: Bus Transit in Bloomington

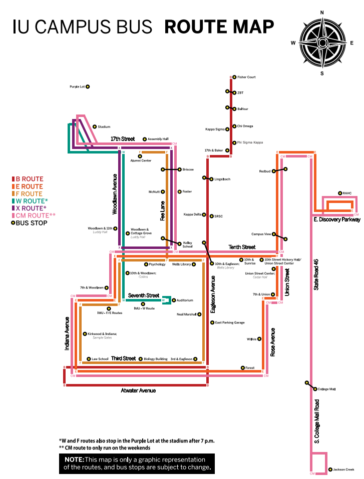

Figure 2: IU Campus Bus Route

Question 1.2. *Can you identify the bus routes in the two different maps?*

In fact, modern metro systems usually use the second type, which we call *topological* maps, to show important information to travelers – for example, the connections between lines, and the number of stops between stations. This demonstrates the differences between topology – which focuses more on shapes, connections, relative positions – and geometry, which cares more about distances, angles and area.

Subject *Topology* - meaning ’the study of space’ – is the branch of mathematics concerned with the properties of a geometric object that are preserved under continuous deformations, such as stretching, twisting, crumpling, and bending; that is, without closing holes, opening holes, tearing, gluing, or passing through itself.

History The ideas underlying topology go back to Gottfried Wilhelm Leibniz, who in the 17th century envisioned the geometria situs and analysis situs. Leonhard Euler’s **Seven Bridges of Königsberg** problem and **polyhedron formula** are arguably the field’s first theorems. The term topology was introduced by Johann Benedict Listing in the 19th century, although, it was not until the first decades of the 20th century that the idea of a topological space was developed.

Remark. The rigorous definition of *topological spaces* is the abstraction and generalization of what we know in geometry. Instead of working with distances (which falls under the category of metric topology) which captures how far two objects (called *points* in a space) are from each other, the topological way of separating points is to check if they have neighborhoods (by defining open and closed subsets of a space) which tells the difference between them. We don’t need to know the actual distance, which is the thing that geometers care about, the existence of such neighborhoods suffices.

You might find the description here does not agree the intuitive definition we mentioned above, but they are in fact the same – the definition of continuity depends on the definition of topology, in particular on the open and closed subsets!

1.2 What shall we expect from this course?

Talking about point set topology requires too much abstract definitions and sometimes deprives people’s interests and intuition on interesting and intuitive questions, so in this course, we are not going to spend time on those abstract nonsense – even thought they are not even the most abstract nonsense subjects in mathematics.

My goal for this class is not to train your ability to do rigorous and formal math, as they will most likely (and hopefully) to be useless for the rest of your life. In fact, apart from the formal definitions, there are in fact many other fascinating aspects of mathematics that we can explore.

In this course, I want to provide an opportunity for you to

- explore topological thinking, and
- experience solving open-ended problems by
  - guessing,

  - drawing pictures,
  - constructing models,
  - looking for patterns,
  - formulating conjectures,
  - finding counter-examples,
  - making arguments,
  - asking questions,
  - using analogies,
  - and making generalizations;
- observe new mathematical techniques springing from solutions to problems;
- see the connectivity of mathematical ideas—the solution to one problem helps solve other problems; the answer to one problem leads to new questions.

All of the above cannot be achieved without your participation, exploration, frustration, and reflection. Yes, frustration and reflection are very important.

1.3 What are the responsibilities of you as a student?

Participation Attending lectures is required for this course as we will ask you to participate in group activities in class. It is not only a great chance for you to play with your classmates, have discussions, and raise questions, but also to make sure you are on track for the class. I understand there are exceptional cases where people cannot show up and that's fine, but please send me an email if you are not able to attend the class or if you need extra help catching up with the class.

Assignments We will ask you to work on problems in class and additional exercises outside class, and you're required to turn them in on a regular basis. Usually in math classes we expect you to get the right answers as fast as possible and as accurate as possible, but in reality, doing mathematical research, or other scientific/sociological research, or living your life is not working that way. In fact, many time we are either learning from making mistakes ourselves or seeing other people making mistakes, and we are correcting ourselves step by step. Mistakes are frustrated, but they are inspiring and help us reflect upon our observations/thinking process. This is what we are going to do in this course. I will ask you to keep track of the mistakes you are making, and think about why you make those mistakes once you figure out the right answer, or not (sometimes there is no right answer, we just do not know).

Portfolio The portfolio is a relatively free-form activity. Think of it as an artists portfolio, a writers journal, a scientists lab notebook, a diary, or a mathematical scrapbook. Write summaries of the mathematics we learned. Write about any frustrations or confusions you have, any ideas or questions that spring to mind, anything you find yourself wondering about. Include reflections on your thinking: What were your 'ah-ha' moments? What is most confusing? What were the struggles? How is your understanding developing? Make connections to things outside of class such as newspaper articles, blog posts, your other classes, movies, literature, art, architecture, design etc.. You could write the entries immediately after class, or wait a few hours to let the ideas sink in. Get creative! You may also include any sketches, photographs, diagrams, collages, doodles etc. that might help you remember and convey your ideas more clearly.

Project There are no written exams for this course. Instead, we will ask you to work on a project (individual, no collaboration) and give final presentations during the last two weeks. More detailed will be updated during the semester.

Peer Review Every student need to be present for the presentation weeks at least three times and write peer review for three other students. You review should be comprehensive and constructive. More instructions on peer review will be posted later.

1.4 What are the responsibilities of me as an instructor?

As your instructor for this course, I will try my best to accommodate every students, and make sure you are able to keep up with the course, and hopefully enjoy the classes. To achieve this goal, I need your feedback and advice on the course regularly. I also welcome suggestions for future topics that you find relevant and interesting.

1.5 What are we going to learn in this course?

We will explore a range of interesting topics realted to topology, adapting our focus based on available time and student interests. Below is an overview of the key areas we will cover:

Discrete Mathematics We begin with the historical roots of topology linked to Euler's work:

- **Map Coloring**: Strategies for coloring maps with a minimal number of colors.

- **Seven Bridges of Königsberg**: Exploring Euler's problem of crossing each bridge once in the city of Königsberg.
- **Euler's polyhedron formula**: Understanding the relationship between vertices, edges, and faces in polyhedra.
- **Sprouts and Brussels Sprouts Game**: Play this two-player game that involves drawing curves and dots to strategize winning moves.

Geometry Moving towards rigid geometric properties and their applications:

- **Platonic solids**: Discover how Euler's formula helps classify all platonic solids.
  Definition: The Platonic solids, also called the regular solids or regular polyhedra, are convex polyhedra with equivalent faces composed of congruent convex regular polygons.
  Classification: There are exactly five such solids (Steinhaus 1999, pp. 252-256): the cube, dodecahedron, icosahedron, octahedron, and tetrahedron, as was proved by Euclid in the last proposition of the Elements.
- **Tessellations**: Create and analyze patterns that fit perfectly on a plane without gaps or overlaps.
- **Straight-cut Origami**. Experiment with the fold-and-cut technique to see what shapes can be created from a single cut process.
- **Cut and Paste**. Explore the problem of the dissection of polyhedra with equal volumes and the analogous question about polygons in 2 dimensions.
  Polyhedra: The third of Hilbert's list of mathematical problems, presented in 1900, was the first to be solved. The problem is related to the following question: given any two polyhedra of equal volume, is it always possible to cut the first into finitely many polyhedral pieces which can be reassembled to yield the second? *Solutions*: Based on earlier writings by Carl Friedrich Gauss, David Hilbert conjectured that this is not always possible. This was confirmed within the year by his student Max Dehn, who proved that the answer in general is "no" by producing a counterexample.
  Polygons: The answer for the analogous question about polygons in 2 dimensions is "yes" and had been known for a long time; this is the Wallace–Bolyai–Gerwien theorem.

Topology By the middle of the nineteenth century mathematicians had a much better understanding of how Euler's formula applied to polyhedra. It was during this time that they began to ask whether it applied to other objects.

What if the figure was not a polyhedron made of flat faces, but instead was a curved surface like a sphere or a torus? If so, what must the partitions look like?

These discussions illustrate the ongoing transition from a geometric to a topological way of thinking about shapes. In geometry it is crucial that the objects of study be rigid. Measurements of angles and lengths, proofs of congruences of figures, and computations of areas and volumes all rely on precise and unmoving geometric structure. However, in some cases the rigid, unbending features of geometric figures are not needed, and worse, they often obscure the underlying mathematics. Think about the bus transit map!

We begin our investigation of topology by looking at surfaces. Examples of surfaces are a 2-dimensional plane, a sphere, a torus, a disk, and a cylinder. A surface is any object that looks locally like a plane. If an ant were to sit on a large surface, it would think that it was sitting on a 2-dimensional expanse. This is not out of our realm of experience—the earth is a spherical globe, but it is so large that to its inhabitants it is indistinguishable from a flat plane.

- **Cylindrical Tic-Tac-Toe**: Play tic-tac-toe on a cylindrical surface.
  The goal of tictactoe is to align 3 or more of your symbols (X O, noughts and crosses) in a board.

- **Torus**: Construction and tic-tac-toe on torus.
  Construction: An example of a torus can be constructed by taking a rectangular strip of flexible material such as rubber, and joining the top edge to the bottom edge, and the left edge to the right edge, without any half-twists.
  Examples: The surface of a coffee cup and a doughnut are both topological tori with genus one.

- **3-Torus**
  Construction: It can be obtained by "gluing" the three pairs of opposite faces of a cube, where being "gluing" can be intuitively understood to mean that when a particle moving in the interior of the cube reaches a point on a face, it goes through it and appears to come forth from the corresponding point on the opposite face, producing periodic boundary conditions. Gluing only one pair of opposite faces produces a solid torus while gluing two of these pairs produces the solid space between two nested tori.

- **Möbius strips**
  Construction: A surface that can be formed by attaching the ends of a strip of paper together with a half-twist.
  Properties: The Möbius strip is a non-orientable surface, meaning that within it one cannot consistently distinguish clockwise from counterclockwise turns.

- **The space of 2-chords**
  Question: What is the relation between the space of 2-chords and surfaces?

- **Klein bottle**
  Construction: To construct the Klein bottle, glue the red arrows of the square together (left and right sides), resulting in a cylinder. To glue the ends of the cylinder together so that the arrows on the circles match, one would pass one end through the side of the cylinder. This creates a curve of self-intersection;
  Map Coloring: Six colors suffice to color any map on the surface of a Klein bottle.

- **Surfaces**
  Construction: A connected sum of two m-dimensional surfaces is a surface formed by deleting a disk inside each surface and gluing together the resulting boundary circles.
  Classification: How to classify all 2 surfaces?

- **Jordan curve theorem**
  Theorem: Every Jordan curve (a plane simple closed curve) divides the plane into an "interior" region bounded by the curve (not to be confused with the interior of a set) and an "exterior" region containing all of the nearby and far away exterior points. Every continuous path connecting a point of one region to a point of the other intersects with the curve somewhere.

- **Brouwer fixed-point theorem**
  In the plane: Every continuous function from a closed disk to itself has at least one fixed point.
  In Euclidean space: Every continuous function from a closed ball of a Euclidean space into itself has a fixed point.
  Convex compact set: Every continuous function from a nonempty convex compact subset K of a Euclidean space to K itself has a fixed point.
  Illustration: Take an ordinary map of a country, and suppose that that map is laid out on a table inside that country. There will always be a "You are Here" point on the map which represents that same point in the country.

- **Manifolds and embedding**
  Definition: A manifold is a topological space that locally resembles Euclidean space near each point.

- **Hairy ball theorem**
  Theorem: There is no nonvanishing continuous tangent vector field on even-dimensional n-spheres.

Corollary: Any continuous function that maps an even-dimensional sphere into itself has either a fixed point or a point that maps onto its own antipodal point.
Illustration 1: You can't comb a hairy ball flat without creating a cowlick.
Illustration 2: You can't comb the hair on a coconut.

- **Deformation**
  Question: Why Coffee Cups and Doughnuts are topologically equivalent?

- **Knots**
  Construction: A knot is created by beginning with a one-dimensional line segment, wrapping it around itself arbitrarily, and then fusing its two free ends together to form a closed loop.
  Concept: The idea of knot equivalence is to give a precise definition of when two knots should be considered the same even when positioned quite differently in space. This turns out to be an extremely difficult problem.
  Knot invariants: Topologist try to come up with different quantities that remains the same for equivalent quantities, called "knot invariants". We will spend some time investigating some of them.

- **Fractions and Knots** *Question*: What do adding positive and negative fractions have to do with tying knots? You will use ropes to explore and identify mathematical operations that untangle knots and lead to new thinking. Simple operations of twists and rotations circle back to practicing the addition of positive and negative fractions.

1.6 What if you want to read more?

To deliver a course on topology without defining topology rigorously is both fun and challenging, and to do so we have to sacrifice a lot of details. I tried my best to collect some useful references and books to prepare for this course, but there are definitely much more than what I found and what I will present.

Here is a (incomplete) list of references I may use to prepare for the course.

1. "Art Meets Mathematics in the Fourth Dimension" by Stephen Lipscomb.
2. Discovering the Art of Mathematics: Arts and Sculpture.
3. Discovering the Art of Mathematics: Geometry.
4. Discovering the Art of Mathematics: Knot Theory.
5. Discovering the Art of Mathematics: Music.
6. "Discrete Mathematics: Elementary and Beyond" by László Lovász, József Pelikán, and Katalin Vesztergombi.

7. "Experiments in topology" by Stephen Barr.
8. "Explorations in Topology: Map Coloring, Surfaces and Knots" by David Gay.
9. "Knots and surfaces: a guide to discovering mathematics", by David W. Farmer and Theodore B. Stanford.
10. "Knots, Molecules, and the Universe : An Introduction to Topology" by Erica Flapan.
11. "The Möbius Strip : Dr. August Möbius's Marvelous Band in Mathematics, Games, Literature, Art, Technology, and Cosmology" by
12. "Topology - A Very Short Introduction" by Richard Earl.
13. "Topology of Surfaces, Knots, and Manifolds : A First Undergraduate Course" by Stephen Carlson.

I must clarify that I claim no originality of the notes here. Due to time limit, it's merely a selection of topics from the above references and potential online references that I will add later, so do not be surprise if you find things similar (or even exactly the same as what you see on the reference books).

2 Map Coloring

2.1 How to color a map with minimal number of colors?

The first question we are going to encounter is map coloring. We will ask you to play with coloring maps of different regions, see you can color them with as minimal number of colors as possible. Think about the cases that will cause you trouble.

Rules for Coloring a Map To color a map properly:

1. Each country must be completely colored by one color.
2. Two countries sharing a common border must be colored in different colors.

Goal Color a map properly with the fewest number of colors, that is, a minimal coloring.

Question 2.1. *Do this for the map of Western Europe.*

1. *How many colors did you use?*
2. *Will three colors work? There are two choices:*
   (a) *Color the map properly in three colors.*
   (b) *Give an argument that explains why the map cannot be colored properly in three colors.*
3. *What about South America?*

Then we will move to more complicated maps and ask you to consider again the question of minimal number of colors needed in this case.

Question 2.2. *Color the map of the 48 lower states of the United States properly and minimally. How many colors do you need? Do you encounter anything unusual?*

Look at the Four Corners, where Utah, Arizona, New Mexico, and Colorado meet.The four states all touch at the Four Corners so each one should be a different color. But four colors could really push the number of colors needed way up. What if a map had a “Five Corners” or a “Six Corners”?

New Rule for Coloring Maps Properly A border between two countries must consist of a positive length; it cannot consist of a single point or even two or three points or any finite number of points.

Here is some language (we call them definitions in math) that will help in discussing maps and that you will see used in sources outside this class.

Definition 2.3. The ingredients of a *map* – on a sphere, a rectangle, or an island – are countries, borders, and vertices.

- A *country* is the interior of a polygon or distorted polygon. The countries of a map do not overlap.
- A *border* of the map is an edge of one or more of the country polygons. Two countries may meet along one of these borders.
- A *vertex* of the map is where two or more borders meet.
- The map is the union of all these elements.

Problem 2.4. *Given any map of countries, how many colors would you need to color it properly and minimally? Is there a fixed number of colors that would be sufficient for all these maps? Or could you find more and more complicated maps that would need more and more colors?*

Answer. The answer is four, and I hope you will have some trouble trying to figure out an optimal strategy – it should be, as this is a really hard problem which took mathematicians more than 100 hundred years to find the first correct proof (and the first proof was done by proof of contradiction with the aid of computer programs plus over 400 pages of human-verifiable portion) and I have to confess that I never read the proof. Also I am not aware of any constructive proof of it up to now, so our goal is not to torture you with the almost impossible goal of finding a "easy" strategy that works in any cases, even though you are very welcome to think about it. Before doing that, I have to warn you that people have made many wrong attempts, so be careful if you think you figure out a solution.

2.2 Story about maps: How long is the coast of Britain?

Coastline paradox (from Wikipedia) You probably have seen maps of the same places in different styles, some are more "accurate" than others. In 1967, Benoit Mandelbrot published a paper called "How long is the coast of Britain?" This may sound like a straightforward question which may make you wonder why people even can publish a paper about it. The answer is, however, we do not know.

The measured length of the coastline depends on the method used to measure it and the degree of cartographic generalization. Since a landmass has features at all scales, from hundreds of kilometers in size to tiny fractions of a millimeter and below, there is no obvious size of the smallest feature that should be taken into consideration when measuring, and hence no single well-defined perimeter to the landmass. Various approximations exist when specific assumptions are made about minimum feature size.

For instance, if the coastline of Great Britain is measured using units 100 km (62 mi) long, then the length of the coastline is approximately 2,800 km (1,700 mi). With 50 km (31 mi) units, the total length is approximately 3,400 km (2,100 mi), approximately 600 km (370 mi) longer.

The coastline paradox is the counterintuitive observation that the coastline of a landmass does not have a well-defined length. This results from the fractal curve-like properties of coastlines; i.e., the fact that a coastline typically has a fractal dimension.

The problem is fundamentally different from the measurement of other, simpler edges. It is possible, for example, to accurately measure the length of a straight, idealized metal bar by using a measurement device to determine that the length is less than a certain amount and greater than another amount—that is, to measure it within a certain degree of uncertainty. The more precise the measurement device, the closer results will be to the true length of the edge. With a coastline, however, measuring in finer and finer detail does not improve the accuracy; it merely adds to the total. Unlike with the metal bar, it is impossible even in theory to obtain an exact value for the length of a coastline.

2.3 Maps of Simpler Shapes

If you think about the map coloring problems we did in the previous class, you probably noticed one thing, the actual shape (angles, lengths and etc.) of each piece that we need to color does not really matter so much. In this class, we will try to revisit the maps we saw in the previous class and see if we can transform the "real" maps into maps of simpler shapes.

Fact 2.5. Any map can be transformed to a map on a rectangle where borders can be made up of straight lines. Solving the coloring problem for any map is reduced to solving the coloring problem for the transformed map.

2.3.1 Maps Cut by Lines

If we want to solve a problem, usually we start with some simple cases and see what we can find.

Now consider the case when the map is being cut only by lines. Work on some examples, did you find anything?

Theorem 2.6. *On a rectangle, form a map by drawing lines from one side of the rectangle to another. Then this map can be colored properly with two colors.*

Moreover, we can describe exactly how to carry out the coloring.

Question 2.7. *Do you have a more efficient method of coloring such a map in two colors? Could you explain why your method works? What we did has two parts:*

1. *a proof that, given such a map, there exists a way to color it using two colors, and*
2. *a method for coloring such maps that always works.*

You could assume part 1 and argue from there.

2.3.2 Maps Cut by Closed Curves

Lines case solved! Now let's think about some variations.

Question 2.8.

1. *Draw a bunch of circles on a piece of paper. These create regions, including the region outside of all the circles. How many colors will you need?*
2. *Next, draw a bunch of distorted circles. Make sure they intersect like circles intersect. Now color the regions. How many colors will you need?*

Remark. You get a distorted circle if you take a pencil, and starting at a point on a piece of paper, you draw a curve without letting the pencil leave the paper and continue until you end up at the point where you started. Don't allow the curve to intersect itself. (Such a curve is called a simple closed curve.)

Now allow such a curve to intersect itself. (Such a curve is called a closed curve.) Make sure the curve intersects itself only in isolated points.

Question 2.9. *Replace the simple closed curves of in Question 2.8 with closed curves. What would happen? How many colors will you need?*

A bit of history The problem of coloring maps goes back to October 23, 1852, when Francis Guthrie (then a graduate student at the University of London) posed it to his teacher Augustus de Morgan (of de Morgan's Law fame), who in turn wrote to the Irish mathematician, William Rowan Hamilton:

> A student of mine asked me today to give him a reason for a fact which I did not know was a fact, and do not yet. He says that if a figure be anyhow divided, and the compartments differently colored, so that figures with any portion of common boundary line

> are differently colored—four colors may be wanted, but no more... What do you say? And has it, if true, been noticed? My pupil says he guessed it in coloring a map of [the counties] of England. The more I think of it, the more evident it seems.

We will see more of this problem in this course.

2.4 Aside: How do we reason?

Solving the special case didn't solve the whole problem, but it's still pretty powerful. One of these maps—formed by millions of lines—can be colored properly with only two colors! Solving the special case was kind of an investigation.

How to Carry Out an Investigation

- Create a bunch of examples of items satisfying a certain property. (Drew lots of the special type of map; colored them.)
- Look for patterns in the examples. (Noticed that all examples considered in Section 2.3.1 could be colored in two colors.)
- Make a conjecture about all items satisfying a certain property. ("All the special type maps—even ones you didn't draw—can be colored in two colors.")
- Do one of two things:
  1. Give an argument explaining why the conjecture is true, or
  2. Find a counterexample to the conjecture.

If we want to prove a statement, what can we do? Think about Theorem 2.6.

2.4.1 Induction

Mathematical induction is a method for proving that a statement $P(n)$ is true for every natural number n (or natural numbers greater than or equal to n_0 for some natural number n_0).

Process of Proof by Induction There are two types of induction: regular and strong. The steps start the same but vary at the end. Here are the steps.

In mathematics, we start with a statement of our assumptions and intent:
Let $p(n)$ be a statement for natural numbers n.
We would show that $p(n)$ is true for all natural numbers $n \geq n_0$.

1. Show that $p(n)$ is true for the smallest possible value of n : In our case $p(n_0)$. AND
2. For Regular Induction: Assume that the statement is true for $n = k$, for some integer $k \geq n_0$. Show that the statement is true for $n = k+1$.
   OR
   For Strong Induction: Assume that the statement $p(r)$ is true for all integers r, where $n_0 \leq r \leq k$ for some $k \geq n_0$. Show that $p(k+1)$ is true.

If these steps are completed and the statement holds, we are saying that, by mathematical induction, we can conclude that the statement is true for all values of $n \geq n_0$.

So far we only need to use regular induction, so I will give you some examples to show how it works.

Example 2.10. Prove $2^n > n+4$ for $n \geq 3, n \in \mathbb{N}$.

Proof by induction. **Base Case**: Let $n = 3$. Then $2^3 > 3+4$ is true since clearly $8 > 7$. Thus the statement is true for $n = 3$.
Induction Assumption: Assume that $2^n > n+4$ is true for some $n = k$.
Induction goal: We will show that $2^{k+1} > (k+1)+4$.

Consider $2^{k+1} = 2 \cdot 2^k > 2 \cdot (k+4) = 2k+8$. Since $2k > k+1$ and $8 > 4$, we have $2k+8 > (k+1)+4$.

Thus the statement is true for all $n = k$.
Conclusion: By induction, $2^n > n+4$ for all $n \geq 3, n \in \mathbb{Z}$. $\square$

Example 2.11. Prove that $1+2+\ldots+n = \frac{n(n+1)}{2}$, for any natural number n.

Proof by induction. **Base Case**: Choose $n = 1$. Then the left hand side (L.H.S) $= 1$. and the right hand side (R.H.S) $= \frac{(1)(1+1)}{2} = 1$.
Induction Assumption: Assume that $1+2+\ldots+k = \frac{k(k+1)}{2}$, for $k \in \mathbb{N}$.
Induction goal: We shall show that

$$1+2+\ldots+k+(k+1) = \frac{(k+1)[(k+1)+1]}{2} = \frac{(k+1)(k+2)}{2}.$$

Consider

$$\begin{aligned} 1+2+\ldots+k+(k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= (k+1)\left(\frac{k}{2} + \frac{1}{1}\right) \\ &= (k+1)\left(\frac{k+2}{2}\right) \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

Conclusion: Thus, by induction we have $1+2+\ldots+n = \frac{n(n+1)}{2}$, for any natural number n. $\square$

2.5 Coloring Regions with Two Colors

Theorem 2.12. *The rectangle cut by n lines can be colored with two colors (e.g., red and blue) in such a way that any two regions that share a common border (of positive length) will be colored differently.*

Let us first see why a direct approach to proving this theorem does not work. We could start with coloring one region, say blue; then we have to color its neighbors red. Could it happen that two neighbors are at the same time neighbors of each other? Perhaps drawing some pictures and then arguing carefully about them, you can design a proof that this cannot happen. But then we have to color the neighbors of the neighbors blue again, and we would have to prove that no two of these are neighbors of each other. This could get quite complicated! And then we would have to repeat this for the neighbors of the neighbors of the neighbors...

We should find a better way to prove this, and fortunately, there is a better way!

Proof. We prove the assertion by induction on n, the number of lines.
Base case: If $n = 1$, then we get only two regions, and we can color one of them red, the other one blue.

So let $n > 1$. Select any of the lines, say L, and forget about it for the time being.
Induction hypothesis: We assume that the regions formed by the remaining $n-1$ lines can be colored with red and blue so that regions that share a common boundary get different colors.
Induction step: Now we put back the remaining line L, and change the coloring as follows: On one side of L (for simplicity we call it LHS), we leave the coloring as it was; on the other side of L (for simplicity we call it RHS), we interchange red and blue.

It is easy to see that the coloring obtained this way satisfies what we wanted. In fact, look at any small piece of line segment of any of the lines.

- If this line segment is on the LHS of L, then the two regions on its two sides were different and their colors did not change.
- If the line segment is on the RHS of L, then again, the two regions on its two sides were differently colored, and even though their colors were switched, they are still different.
- Finally, the line segment could be on L itself. Then the two regions on both sides of the arc were one and the same before we put L back, and so they had the same color. Now, one of them is one the RHS of L and this switched its color; the other is on the LHS, and this did not. So after the recoloring, their colors will be different.

Thus we have proved that the regions formed by n lines can be colored with two colors, provided that the regions formed by $n-1$ lines can be colored with 2 colors.

Conclusion: By the Principle of Induction, this proves the theorem. □

2.6 Counting Regions

Question 2.13.

1. *Look at the the example of maps cut by lines, how many lines do you have in each case? How many regions do you obtain when the plane is divided by those lines?*
2. *Draw some other examples and see if you observe anything.*
3. *What are the differences if*
   (a) *You have parallel lines.*
   (b) *You have more than two lines intersecting at the same point.*
4. *A set of lines in the plane such that no two are parallel and no three go through the same point is said to be in general position.*
   Let us draw n lines in the plane in the general position. These lines divide the plane into some number of regions. How many regions do we get?

Remark. If we choose the lines "randomly," then accidents like two being parallel or three going through the same point will be very unlikely, so our assumption that the lines are in general position is quite natural.

Question 2.14. *Use the same method to answer the question for the case when the map is cut by n circles.*

2.7 References

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3 Graphs

3.1 Tours

The Seven Bridges of Königsberg is a historically notable problem in mathematics. The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands – Kneiphof and Lomse –which were connected to each other, and to the two mainland portions of the city, by seven bridges. The problem was to devise a path through the city that would cross each of those bridges once and only once.

Note reaching an island or mainland bank other than via one of the bridges, or accessing any bridge without crossing to its other end are explicitly unacceptable.

Euler proved that the problem has no solution. Euler first pointed out that the choice of route inside each land mass is irrelevant and that the only important feature of a route is the sequence of bridges crossed. This allowed him to reformulate the problem in abstract terms, eliminating all features except the list of land masses and the bridges connecting them. In modern terms, one replaces each land masses with an abstract "vertex" or node, and each bridge with an abstract connection, an "edge", which only serves to record which pair of vertices (land masses) is connected by that bridge. The resulting mathematical structure is a graph.

Since only the connection information is relevant, the shape of pictorial representations of a graph may be distorted in any way, without changing the graph itself. Only the number of edges (possibly zero) between each pair of vertices is significant. It does not, for instance, matter whether the edges drawn are straight or curved, or whether one node is to the left or right of another.

Informally we call such a valid path a "nice tour".

Ingredients of a Nice Tour With a pencil, trace a path on the graph that

- has the pencil touching the paper at all times;
- covers each edge of the graph exactly once;
- returns to the vertex where the path started.

The Seven Bridges of Königsberg Euler observed that (except at the endpoints of the path), whenever one enters a vertex by a bridge, one leaves the vertex by a bridge. In other words, during any path in the graph, the number of times one enters a non-terminal vertex equals the number of times one leaves it. Now, if every bridge has been traversed exactly once, it follows that, for each land mass (except for the ones chosen for the start and finish), the number of bridges touching that land mass must be even (half of them, in the particular traversal, will be traversed "toward" the landmass; the other half, "away" from it). However, all four of the land masses in the original problem are touched by an odd number of bridges (one is touched by 5 bridges, and each of the other three is touched by 3). Since, at most, two land masses can serve as the endpoints of a path, the proposition of a path traversing each bridge once leads to a contradiction.

When is a Nice Tour Possible? If a nice tour of a graph is possible, then the number of lines attached to every vertex must be even. In other words, if the graph has a vertex such that the number of edges connected to it is odd, then no nice tour is possible.

Example 3.1. Flight schedule.

Example 3.2. Door inspection tour.

Our conjecture is that if every vertex in a connected network is even, then there is a nice trip starting at a vertex and returning to the same vertex. Part of the investigation has already taken place: we've taken nice trips on a lot of networks; we have a lot of examples for which we know it can be done. What we want is a surefire, guaranteed method for taking a nice tour on any network of the prescribed kind. This would prove our conjecture.

Of course, it's possible that something really nasty could happen: the presence of a counterexample to the conjecture—a connected network having every vertex even for which there is no nice tour. Of course, then there would be no surefire, guaranteed method. So what is it?

- Can you show that our conjecture is true? Can you find a surefire method?
- Or can you find a counterexample?

3.2 Graphs

To answer the question we proposed above, we need to spend some time on basic graph theory.

Definition 3.3. A (multi)graph $G = (V, E, \phi)$ consists of

1. a set V of vertices (also called nodes or points), and
2. a set E of edges (also called links or lines), and
3. an incidence function ϕ mapping every edge to an unordered pair of vertices (that is, an edge is associated with two distinct vertices).

Note our definition of graph is probably more general than what you've seen before, and one of the reason is we want to allow parallel edges (multiple edges with the same end points), which is the case in the example of Seven Bridges of Königsberg.

3.2.1 Degrees

Definition 3.4. The *degree* of a vertex of a graph is the number of edges that are incident to the vertex.

Theorem 3.5. *In every graph, the number of vertices with odd degree is even.*

Proof. This is a proof by induction on the number of edges. We will build up the graph one edge at a time, and observe how the parities of the degrees change.

Now if we connect two vertices by a new edge, we change the parity of the degrees at these vertices. In particular,

- if both endpoints of the new edge had even degree, we increase the number of vertices with odd degree by 2;
- if both endpoints of the new edge had odd degree, we decrease the number of vertices with odd degree by 2;
- if one endpoint of the new edge had even degree and the other had odd degree, then we don't change the number of vertices with odd degree.

Thus if the number of vertices with odd degree was even before adding the new edge, it remained even after this step. This proves the theorem. □

This will hinge on the answer to the following question: How many edges does a graph have? This can be answered easily if we think back to the problem of counting handshakes: For each vertex, we count the edges that leave that vertex (this is the degree of the vertex). If we sum these numbers, we count every edge twice. So dividing the sum by two, we get the number of edges. Let us formulate this observation as a theorem:

Theorem 3.6. *The sum of degrees of all vertices in a graph is twice the number of edges.*

In particular, we see that the sum of degrees in any graph is an even number. If we omit the even terms from this sum, we still get an even number. So the sum of odd degrees is even. But this is possible only if the number of odd degrees is even (since the sum of an odd number of odd numbers is odd). Thus we have obtained a new proof of Theorem 3.5.

Corollary 3.7. *If a graph has an even number of vertices, then the number of vertices with even degree is even.*

Corollary 3.8. *If a graph has an odd number of vertices, then the number of vertices with even degree is odd.*

Corollary 3.9. *If a graph has an odd number of vertices, then it has a vertex with even degree.*

Exercise 3.10. Prove that at a party with 51 people, there is always a person who knows an even number of others in the following two steps.

1. Translate this question into a graph problem.
2. Use the results we provided above to prove the statement related to graphs.

3.2.2 Paths, Cycles and Connectivity

Let us get acquainted with some special kinds of graphs.

Edgeless graphs The simplest graphs are the edgeless graphs, having any number of vertices but no edges.

Complete graphs We get another very simple kind of graphs if we take n vertices and connect any two of them by an edge. Such a graph is called a *complete graph* (or a *clique*). A complete graph with n vertices is denoted by K_n.

Exercise 3.11. A complete graph with n vertices has $\frac{n(n-1)}{2}$ edges.

Proof. Proof by induction on the number of vertices. □

Paths Let us draw n vertices in a row and connect the consecutive ones by an edge. This way we obtain a graph with $n-1$ edges, which is called a *path*.

Cycles The first and last vertices in the row are called the endpoints of the path. If we also connect the last vertex to the first, we obtain a *cycle* (or *circuit*). The number of edges in a path or cycle is called its *length*. A cycle of length k is often called a k-cycle.

3.3 Eulerian path

Definition 3.12. An *Eulerian path* is a path that goes through every edge exactly once. An Eulerian path that starts and ends at the same vertex is called *Eulerian cycle*.

Theorem 3.13 (Euler's theorem)**.** *A connected graph has an Euler cycle if and only if every vertex has an even number of incident edges.*

Proof. Let v be any vertex. Consider a closed path starting and ending at v that uses every edge at most once. Such a path exists. For example, we can take the path consisting of the vertex v only. But we don't want this very short path; instead, we consider a longest closed path W starting at v, using every edge at most once.

We want to show that this path W is Eulerian. Suppose not. Then there is at least one edge e that is not used by W. We claim that we can choose this edge so that W passes through at least one of its endpoints. Indeed, if p and q are the endpoints of e and W does not pass through them, then we take a path from p to v (such a path exists since the graph is connected), and look at the first vertex r on this path that is also on the path W. Let $e' = sr$ be the edge of the path just before r. Then W does not pass through e (because it does not pass through s), so we can replace e by e', which has one endpoint on W. So let e be an edge that is not used by W but has an endpoint p that is used by W. Then we start a new path W' at p. We start through e, and continue path as we please, only taking care that (i) we don't use the edges of W, and (ii) we don't use any edge twice.

Sooner of later we get stuck, but where? Let u be the vertex where we get stuck, and suppose that $u \neq p$. Vertex u has even degree; W uses up an even number of edges incident with u; every previous visit of the new path to this vertex used up two edges (in and out); our last entrance used up one edge; so we have an odd number of edges that are edges neither of W nor of W'. But this means that we can continue our path!

So the only vertex we can get stuck in is vertex p. This means that W' is a closed path. Now we take a path as follows. Starting at v, we follow W to p; then follow W' all the way through, so that eventually we get back to p; then follow W to its end at v (Figure 7.11(b)). This new path starts and ends at v, uses every edge at most once, and is longer than W, which is a contradiction. $\square$

Observations If a vertex v has odd degree, then every Eulerian path must either start or end at v. Similarly, we can see that if a vertex v has even degree, then every Eulerian path either starts and ends at v, or starts and ends somewhere else.

Theorem 3.14.

1. *If a connected graph has more than two vertices with odd degree, then it has no Eulerian path.*
2. *If a connected graph has exactly two vertices with odd degree, then it has an Eulerian path. Every Eulerian path must start at one of these and end at the other one.*
3. *If a connected graph has no vertices with odd degree, then it has an Eulerian path. Every Eulerian path is closed.*

3.3.1 Constructing Eulerian paths and cycles

Consider a graph known to have all edges in the same component and at most two vertices of odd degree.

Fleury's algorithm Fleury's algorithm is an elegant but inefficient algorithm that dates to 1883.

- The algorithm starts at a vertex of odd degree, or, if the graph has none, it starts with an arbitrarily chosen vertex.
- At each step it chooses the next edge in the path to be one whose deletion would not disconnect the graph, unless there is no such edge, in which case it picks the remaining edge left at the current vertex.
- It then moves to the other endpoint of that edge and deletes the edge.
- At the end of the algorithm there are no edges left, and the sequence from which the edges were chosen forms an Eulerian cycle if the graph has no vertices of odd degree, or an Eulerian path if there are exactly two vertices of odd degree.

Hierholzer's algorithm Hierholzer's 1873 paper provides a different method for finding Euler cycles that is more efficient than Fleury's algorithm:

- Choose any starting vertex v, and follow a path of edges from that vertex until returning to v. It is not possible to get stuck at any vertex other than v, because the even degree of all vertices ensures that, when the trail enters another vertex w there must be an unused edge leaving w. The tour formed in this way is a closed tour, but may not cover all the vertices and edges of the initial graph.
- As long as there exists a vertex u that belongs to the current tour but that has adjacent edges not part of the tour, start another trail from u, following unused edges until returning to u, and join the tour formed in this way to the previous tour.
- Since we assume the original graph is connected, repeating the previous step will exhaust all edges of the graph.

By using a data structure such as a doubly linked list to maintain the set of unused edges incident to each vertex, to maintain the list of vertices on the current tour that have unused edges, and to maintain the tour itself, the individual operations of the algorithm (finding unused edges exiting each vertex, finding a new starting vertex for a tour, and connecting two tours that share a vertex) may be performed in constant time each, so the overall this algorithm takes linear time, $O(|E|)$.

3.4 Hamiltonian Path Problem and Hamiltonian Cycle Problem

A question similar to the problem of the Bridges of Königsberg was raised by another famous mathematician, the Irish William R. Hamilton, in 1856.

Definition 3.15. A *Hamiltonian cycle* is a cycle that in a graph that visits each vertex exactly once. Similarly, a *Hamiltonian path* is a path in a graph that visits each vertex exactly once.

The Hamilton cycle (resp. path) problem is the problem of deciding whether or not a given graph has a Hamiltonian cycle (resp. path).

Hamiltonian cycles sound quite similar to Eulerian walks: Instead of requiring that every edge be used exactly once, we require that every vertex be used exactly once. But much less is known about them than about Eulerian walks. Euler told us how to decide whether a given graph has an Eulerian walk; but no efficient way is known to check whether a given graph has a Hamiltonian cycle, and no useful necessary and sufficient condition for the existence of a Hamiltonian cycle is known.

Example 3.16. There are some cases we know Hamiltonian path exists.

1. A complete graph with more than two vertices is Hamiltonian.
2. Every cycle graph (a graph that consists of a single cycle) is Hamiltonian.

The computational problems of determining whether such paths and cycles exist in graphs are in general very difficult. They belong to the class of NP-complete problem (roughly speaking, the hardest of the problems to which solutions can be verified quickly).

3.4.1 Traveling Salesman Problem

The travelling salesman problem (TSP) asks the following question: "Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the origin city?"

The TSP is an NP-hard problem (again means it is hard and it takes a very long time to figure out a solution). Although there is no fast algorithm that produces the optimal solution, there are some algorithms that deliver approximated solutions in a reasonable time. However, greedy algorithm on the traveling salesman problem does not produce optimal solution.

For instance, the nearest neighbour (NN) algorithm (a greedy algorithm) lets the salesman choose the nearest unvisited city as his next move. This algorithm quickly yields an effectively short route.

3.5 Trees

Definition 3.17. A *tree* is a connected graph with no cycles.

Theorem 3.18. 1. *A graph is a tree if and only if it is connected, but deleting any of its edges results in a disconnected graph.*

2. *A graph is a tree if and only if it contain no cycles, but adding any new edge creates a cycle.*

Definition 3.19. A subgraph of a graph G with the same vertex set that is a tree is called a *spanning tree*.

Definition 3.20. A degree one vertex in a tree is called *leaf*.

Proposition 3.21. *Every tree has (at least two) leaves.*

Theorem 3.22. *Every tree with n vertices has $n-1$ edges.*

Sketch of Proof. Remove one leaf from a tree with n vertices will result in a tree with one less vertices and use induction on the number of vertices. □

3.5.1 Minimal Spanning Tree

Definition 3.23. A *weighted graph* is a graph where edges are labeled with weights. The *weight* of a tree is the sum of weights of its edges. A *minimal spanning tree* is a spanning tree with minimal weight.

The Algorithm of Christofides and Serdyukov for TSP

1. Find a minimum spanning tree for the problem.
2. Create a (minimal weight) matching for the problem with the set of cities of odd order.
3. Find an Eulerian tour for this graph.
4. Convert to TSP using shortcuts.

3.6 Euler's Formula

In the first week we consider the following question that now we can cast in the language of graph theory: If we draw some special graphs in the plane, into how many parts do these graphs divide the plane? Indeed, we start with a set of lines; we consider the intersections of the given lines as vertices of the graph, and the segments arising on these lines as the edges of our graph.

More generally, we study a planar map: a graph that is drawn in the plane so that its edges are nonintersecting continuous curves. We also assume that the graph is connected. Such a graph divides the plane into certain parts, called *countries*. Exactly one country is infinite, the other countries are finite.

3.6.1 Formula on Maps

Let's first look at the examples of maps. If we denote the number of vertices by v, the number of edges by e and the number of countries by f (we also count the outside region as an exterior country), staring at these numbers for a little while, one discovers that in every case the following relation holds:

$$v + c = e + 2.$$

and this is *Euler's formula*.

There are many different proofs of this formula, and we will discuss an induction proof by removing good choices of counties step by step. In your homework, you will need to use properties of trees to give an alternative proof of Euler's formula.

Definition 3.24. A map on an island is OK if every interior country contains at least one vertex and if every interior country is a distorted disk (i.e., a simply connected region).

Theorem 3.25. *If v, e, and c are the number of vertices, edges, and countries (respectively) of an OK map on an island, then $v - e + c = 2$.*

Sketch of proof. We will argue by induction on the number of countries on an island that the Euler's formula holds and there is a country that shares just one edge with the island's shoreline (possibly after removing degree 2 vertices).

Base Case Consider the case where the island contains exactly one interior country with at least one vertex. This country is enclosed by a simple closed curve (a distorted circle without self-intersection) that separates the interior from the exterior.

- The number of vertices equals the number of edges cut by those vertices along the boundary of this interior country.
- Thus, we have
$$v - e + c = c = 1 + 1 = 2.$$
- We can remove redundant degree 2 vertices to simplify the structure, ensuring that there is only one vertex and one edge. This edge is the only shared boundary between the interior and exterior countries.

Inductive Hypothesis Assume that for an island with $n - 1$ interior countries (i.e., n total countries), assume $v - e + f = 2$. Additionally, assume that there exists at least one country that shares exactly one edge with the island's shoreline (possibly after removing degree 2 vertices).

Inductive Step Now, consider an island with n interior countries (i.e., $n+1$ total countries). We need to show that:

1. Euler's formula still holds.
2. There exists a country that shares exactly one edge with the island's shoreline (after possibly removing degree 2 vertices).

To do this:

- Identify a country that shares exactly one edge with the island's shoreline (possibly simplifying the structure by removing k degree 2 vertices).
- Remove this edge, reducing the number of countries by one.
- The resulting island now has $n-1$ interior countries.
- By the inductive hypothesis, the updated map satisfies:

$$2 = (v-k) - (e-1-k) + (n-1+1) = v - e + (n+1) = v - e + c.$$

- Expanding, we obtain: $v - e + c = v - e + (n+1) = 2$.

Thus, by induction, Euler's formula holds for all OK maps, and there is always at least one country that shares exactly one edge with the island's shoreline (after possibly removing degree 2 vertices).

□

3.6.2 Formula for Polyhedral

For every convex polyhedral, if we remove one of its face and flatten it out on the plane, we will get a map (or a planer graph as we will discuss in the next lecture). Therefore we have a similar formula between the number of vertices, edges and faces of a polyhedron.

To see this as a map on the sphere, imagine that our polyhedron is made out of rubber. Punch a hole into one of the faces and blow it up like balloon. The most familiar solids will be blown up to spheres (for instance the cube and prism).

Now grab the rubber sphere at the side of the hole and stretch it until you get a huge rubber plane. If we paint the edges of the original solid with black ink, then we will see a map on the plane. The vertices of this map are the vertices of the solid, the edges are the edges of the solid, and the countries are the faces of the body. Therefore, if we use Euler's Formula for maps, we get Euler's Formula for polyhedral.

3.6.3 Maps on Sphere

All the three-dimensional shapes (convex polyhedral) we've been looking at correspond to maps on the surface of the sphere. Take the surface of a cube and imagine it's made of rubber and that you can inflate it like a balloon. You'll get a sphere. The imprint of the cube's faces, edges, and vertices forms the countries, edges, and vertices of a map on the sphere.

Formula for Maps on a Sphere If v is the number of vertices, e the number of edges, and c the number of countries of an OK map on a sphere, then

$$v - e + c = 2.$$

3.6.4 Planer Graph

Definition 3.26. A graph is called *planar* if it can be drawn as a map in the plane, that is, if we can represent its vertices by different points in the plane, and its edges by curves connecting the appropriate points in such a way that these curves don't intersect each other.

Theorem 3.27 (Euler's formula for planer graphs)**.** *Let G be a planer graph. If we denote the number of vertices by v, the number of edges by e and the number of faces by f, then the following relation holds:*

$$v + c = f + 2.$$

and this is the Euler's formula for planer graphs.

You will prove this theorem in the homework.

Which graphs can be drawn as planar maps? This question is important not only because we want to know to which graphs we can apply Euler's Formula, but also in many applications of graph theory, for example, placing a network on a printed circuit board.

Example 3.28 (Non-example)**.** The complete graph K_5 and bipartite complete graph $K_{3,3}$ are not planer.

Example 3.29. The graph obtained by removing one of the edges of K_5 is planer.

Definition 3.30. An *edge contraction* is an operation that removes an edge from a graph while simultaneously merging the two vertices it used to connect. An (undirected) graph H is a minor of another (undirected) graph G if a graph isomorphic to H can be obtained from G by contracting some edges, deleting some edges, and deleting some isolated vertices.

Theorem 3.31 (Wagner's theorem)**.** *A finite graph is planar if and only if it does not have K_5 or $K_{3,3}$ as a minor.*

3.6.5 Dual Graphs

The dual graph of a planar graph G is a graph that has a vertex for each face of G. The dual graph has an edge for each pair of faces in G that are separated from each other by an edge.

Because the dual graph depends on a particular embedding, the dual graph of a planar graph is not unique, in the sense that the same planar graph can have non-isomorphic dual graphs.

3.7 Map Coloring Revisited

Theorem 3.32. *If a nice map has every vertex of order three or more, then there must be at least one country whose number of edges is five or less.*

Question 3.33. *Suppose a map on the sphere (all vertices of order three or more) has no countries with exactly three borders and none with exactly four borders. What can you say about the number of countries with exactly five borders?*

Order-Three Reduction For Map Coloring Replace every vertex of order greater than three by a little country. Every vertex of the altered map is now of order three.

Assume we can color the altered map in six colors using some method. Keep this same coloring of the map while shrinking the little countries you added.

How to color a map with six colors?

- Start with a map on the sphere where every vertex is of order three.
- Find one country with at most five edges.
- Remove one edge of this country to get a map with one fewer countries.
- Remove any vertices of order two that may have come up because the new map should have the same properties as the original.
- All vertices should be of order three.
- Repeat the procedure above to decrease the number of countries to six.
- Leave the dotted lines in to show where you've gotten rid of edges along the way, and keep track, somehow, of the order in which you have eliminated edges, vertices, and countries.
- Color the last map you get – the one with six countries – properly with six or fewer colors.

- Restore the countries, vertices, and edges you've eliminated one country at a time, in reverse order, coloring the restored countries as you go.
- Leaving the rest of the map colored as it is, color the last country you eliminated with a color not used by its neighbors.
- A color will be available. The last country has five or fewer edges so there are at most five countries surrounding it; at most five colors are used to color those neighbors so there is a color left over (from the six colors) to color the restored country.

Theorem 3.34. *Every map on a sphere can be properly colored in six or fewer colors.*

How to color a map with five colors? We can do better. Consider the procedure of restoring a country with five or less edges.

- If the number of edges is less than or equal to four, we can color with at most five colors.
- If the number of edges is equal to five, you can remove two edges and get a map with two fewer countries.
  - The two edges you remove can't be contiguous.
  - And, you have to make sure the end result is an OK map.
- When you restore the country, at most four colors have been used for its bordering countries.

Theorem 3.35. *Every map on a sphere can be properly colored in five or fewer colors.*

3.8 The Four Color Theorem

How to color a map with four colors? There were several early failed attempts at proving the theorem. One proposed proof was given by Alfred Kempe in 1879, which was widely acclaimed; another was given by Peter Guthrie Tait in 1880. It was not until 1890 that Kempe's proof was shown incorrect by Percy Heawood, and in 1891, Tait's proof was shown incorrect by Julius Petersen—each false proof stood unchallenged for 11 years.

During the 1960s and 1970s, German mathematician Heinrich Heesch developed methods of using computers to search for a proof. Notably he was the first to use discharging for proving the theorem, which turned out to be important in the unavoidability portion of the subsequent Appel–Haken proof. He also expanded on the concept of reducibility and, along with Ken Durre, developed a computer test for it. Unfortunately, at this critical juncture, he was unable to procure the necessary supercomputer time to continue his work.

Others took up his methods, including his computer-assisted approach. While other teams of mathematicians were racing to complete proofs, Kenneth Appel and Wolfgang Haken at the University of Illinois announced, on June 21, 1976, that they had proved the theorem. They were assisted in some algorithmic work by John A. Koch.

In short, the actual proof of the four color theorem is too complicated so we are not going to discuss it in class. Instead we will go through a false argument in class to see why this problem is difficult.

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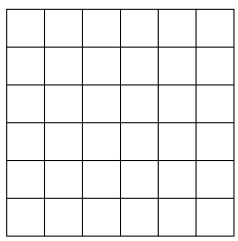
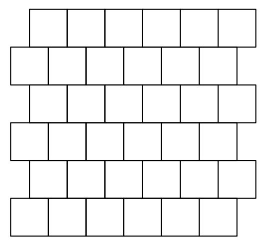
4 Tesselations

4.1 Regular and Semiregular Tilings

A *tesselation* or tiling is a pattern of repeated figures on the plane. The simplest example is the tesselation of squares. This is called a *regular tiling* or *tesselation*, one that consists of repeated copies of a single regular polygon, meeting edge to edge so that every vertex has the same configuration. Four squares must meet at each vertex, and since each square contributes a 90° angle, they join to form a 360° angle, thus lying flat on the plane. The figure below on the right does not count as a regular tiling, since the junction of two of the squares forces a vertex at the midpoint of the edge of the adjoining square.





Exercise 4.1. Which other regular polygons can tile the plane?

A *semiregular* or *Archimedean tesselation* is one in which more than one type of regular polygon is used and each vertex has the same configuration. There are exactly three regular tilings (see Exercise 1), and eight semiregular ones. An example is the tiling where two regular octagons and a square, all with the same edge length, meet at each vertex. Each octagon has a 135° vertex angle and the square a 90° angle. Since $2 \times 135^\circ + 90^\circ = 360^\circ$, they fit together to surround a vertex, and the pattern can be extended to form a tiling.

Exercise 4.2. Fill in the following table carefully. You will need the answers later.

So far, we know four tilings: the three regular tilings found in Exercise 4.1 and the octagon-square shown above. This latter semiregular tiling is denoted by 4.8.8, to indicate the vertex configuration (one square and two octagons). The regular tiling of squares is denoted 4.4.4.4.

The rest of this section is devoted to finding all the other semiregular tilings. We begin by deriving the rules as found in 1785 by The Rev. Mr. Jones.

Rule 1 Every regular and semiregular tiling must have the angles of the polygons meeting at a vertex sum to exactly 360°.

Polygon	Sides	Angle
Triangle	3	
Square	4	
Pentagon	5	
Hexagon	6	
Heptagon	7	
Octagon	8	
Nonagon	9	
Decagon	10	
Dodecagon	12	
Pentakaidecagon	15	
Octakaidecagon	18	
Icosagon	20	
Tetrakaicosagon	24	

Table 1: Vertex Angles of Regular Polygons

Exercise 4.3. What is the largest number of regular polygons that can fit around a vertex?
Hint: If you're going to have lots of polygons at a vertex, then the angles had better be small. What is the smallest angle measure among the regular polygons? Only one tiling has this many. What is it?

From Exercise 4.3, we have a limit on how many polygons to look for. Clearly, we must have at least three polygons meet at each vertex. Thus, we have another rule that all semiregular tilings must obey:

Rule 2 Every regular and semiregular tiling must have at least 3 polygons and no more than six meeting at each vertex.

Next, let us think about how many different types of polygons can occur at each vertex.

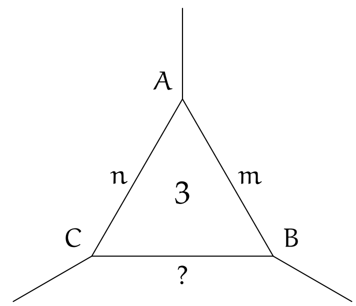
Exercise 4.4. Can there be four different polygons at a vertex? The angles of each would have to be rather small. Explain your answer.

Exercise 4.5. Therefore, if there cannot be four different polygons at a vertex, then when there are four or more polygons at a vertex, what can you say about them?

This gives us another rule:

Rule 3 No semiregular tiling can have four different types of polygons meeting at a vertex.

A little additional thought gives us further limits on types of vertex configurations that can occur. Note that if we try to arrange one equilateral triangle and two other different polygons at a vertex, to make pattern 3.n.m they would be arranged as below:



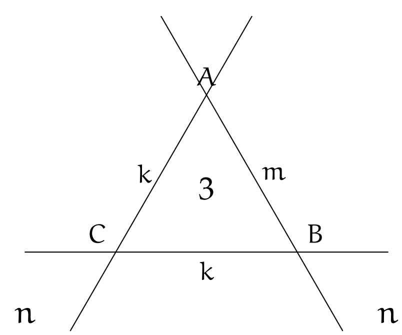
At vertex A, we have the vertex configuration $3.n.m$. A triangle and an m-sided polygon meet at B, so the polygon marked "?" ought to have n sides. But at vertex C there are already a triangle and an n-sided polygon, so the polygon marked "?" should have m sides. Thus, we cannot have any vertex configuration of the form $3.n.m$ if $n \neq m$.

Exercise 4.6. Show that one cannot have a vertex configuration of the form $5.n.m$ where $n \neq m$. Generalize the result to any vertex configuration of the form $k.n.m$ where k is odd and $n \neq m$.

This gives us the following rule:

Rule 4 No semiregular tiling can have vertex configuration $k.n.m$ where k is odd and $n \neq m$.

Another configuration that cannot occur is $3.k.n.m$ unless $k = m$ as shown in the illustration below. At vertices A and B, we have the configuration $3.k.n.m$. This forces the configuration $3.k.n.k$ at vertex C. If this is to be a semiregular tiling, we must have $k = m$.



Rule 5 No semiregular tiling can have vertex configuration $3.k.n.m$ unless $k = m$.

Now we have set some parameters for our search. Next we will experiment to find out which polygons can fit around a vertex. Try fitting cardboard polygons together, or you can trace the polygons on the last page of this section, but check your answers by using the angles you calculated in Exercise 4.2. Write your answers to Exercises 7, 8, and 9 in the table below.

Symbol	# polygons	Comment	Symbol	# polygons	Comment
	6	Exercise 3	3.7 .42	3	Exercise 9
	5	Exercise 7		3	Exercise 9 a
	5	Exercise 7		3	Exercise 9 a
4.4 .4 .4	4	Exercise 1		3	Exercise 9 b
	4	Exercise 8		3	Exercise 9 b
	4	Exercise 8		3	Exercise 9 c
	4	Exercise 8		3	Exercise 9 c
6.6 .6	3	Exercise 1		3	Exercise 9 c
4.8 .8	3	text			

Table 2: Vertex Configurations for Possible Tilings

Exercise 4.7. There are two ways to fit five regular polygons around a vertex so that the angle sum is 360°. What are they?
Hint: Use lots of triangles.

Exercise 4.8. One tiling with four polygons at a vertex is 4.4.4.4, four squares meeting at each vertex. There are three other ways to fit four polygons around vertex. Find them. Remember what you figured out in Exercise 4.4 and Exercise 4.5.

Exercise 4.9. There are, unfortunately, a lot of ways to fit three polygons around a vertex. One such pattern is the regular tiling consisting of three hexagons meeting at a vertex, each contributing 120°, to form $3 \times 120^\circ = 360^\circ$. We designate this pattern by 6.6.6. We also already know about 4.8.8. Another that I will give you, since I do not want to draw a 42-sided figure, is 3.7.42, consisting of an equilateral triangle (60°), a heptagon $\left(128\frac{4}{7}^\circ\right)$, and a 42-gon $\left(171\frac{3}{7}^\circ\right)$, so $60^\circ + 128\frac{4}{7}^\circ + 171\frac{3}{7}^\circ = 360^\circ$. Now there are seven more to figure out.

(a) There are two others that have duplicate polygons, besides 6.6.6 and 4.8.8. Find these.

(b) Two of the remaining patterns contain a square. Find these.

(c) The last three each contain one equilateral triangle. Find the patterns and list on the table.

In the exercises above we found 17 ways to fit regular polygons around a vertex. Unfortunately, not all of these extend to tilings of the plane. We have only found possible tilings: we know only that these vertex configurations add up to 360° at each vertex. Extending the configuration to cover the plane is another question.

For example, if we consider 3.7.42, the pattern starts like the picture below. The 42-gon is drawn with radial lines to help distinguish the sides, since a 42-sided polygon looks very much like a circle. We must have a triangle, a heptagon, and a 42-gon at each vertex. If we arrange this at vertex A and vertex B, then we are forced to have the triangle and two heptagons at vertex C. There is no room to squeeze in a 42-gon at vertex C; therefore, this pattern cannot be extended to a tiling of the plane.

This is a geometric explanation of Rule 4, which already told us that $n \neq m$. We have 16 other vertex configurations to check. All of the regular tilings work out, so we'll do those first.

Exercise 4.10. Draw a section (at least 2 inches square) of each of the three regular tilings found in Exercise 4.1.

That takes care of three. And we already know that 4.8.8 tiles the plane, and that 3.7.42 does not, so we have 12 left to check. Answer the next set of exercises by fitting together cardboard polygons, by tracing the patterns on the last page of the section, or by applying Rules 4 and 5.

Exercise 4.11. There are 10 patterns with three polygons at a vertex. We know that 6.6.6 and 4.8.8 tile the plane and 3.7.42 does not. Of the seven remaining, only two extend to a tiling.

1. Use Rule 4 to eliminate the ones that do not work.
2. Draw sections (at least 2 inches square) of the two remaining tilings of this type.

When we turn to the examples with four and five polygons at a vertex, there is an additional difficulty. One of the four solutions to Exercise 8 should be 3.4.4.6. There are two different ways that a triangle, two squares, and a hexagon can be arranged around a vertex: we designate these by 3.4.4.6 and 3.4.6.4. TODO

Note that the pattern 3.4.4.6 violates Rule 5, but 3.4.6.4 does not.

Exercise 4.12. Draw a section of the tiling 3.4.6.4.

Exercise 4.13. Only one of the other two patterns found in Exercise 4.8 and their rearrangements with four polygons at a vertex can be extended to cover the whole plane.

(a) Find the one which does not work, and show why. Be sure to check all possible arrangements.

(b) The other configuration from Exercise 4.8 has one rearrangement that does not tile, and one that does. Show why the first fails and draw a section (at least two inches square) of the tiling.

Exercise 4.14. Both of the patterns of five polygons around a vertex found in Exercise 4.7 can be extended to tilings. One gives two different tilings, depending on the order. Draw sections of these three tilings.

You should have found and drawn the three regular tilings and the eight semiregular tilings!

When we turn to the examples with four and five polygons at a vertex, there is an additional difficulty. One of the four solutions to Exercise 8 should be 3.4.4.6. There are two different ways that a triangle, two squares, and a hexagon can be arranged around a vertex: we designate these by 3.4.4.6 and 3.4.6.4. TODO

Note that the pattern 3.4.4.6 violates Rule 5, but 3.4.6.4 does not.

Exercise 4.15. Draw a section of the tiling 3.4.6.4.

Exercise 4.16. Only one of the other two patterns found in Exercise 4.8 and their rearrangements with four polygons at a vertex can be extended to cover the whole plane.

(a) Find the one which does not work, and show why. Be sure to check all possible arrangements.

(b) The other configuration from Exercise 4.8 has one rearrangement that does not tile, and one that does. Show why the first fails and draw a section (at least two inches square) of the tiling.

Exercise 4.17. Both of the patterns of five polygons around a vertex found in Exercise 4.7 can be extended to tilings. One gives two different tilings, depending on the order. Draw sections of these three tilings.

You should have found and drawn the three regular tilings and the eight semiregular tilings!

4.1.1 Dual Tilings

For each tiling we can associate another dual tiling, which might not be regular or semiregular. For example, we will see in class the tiling 4.8.8 and its dual tiling, which is a tiling by nonequilateral triangles.

The process of forming the dual is as follows: place a vertex at the center of gravity of each polygon in the original tiling. Whenever two polygons share an edge in the original tiling, draw a dual edge connecting the new vertices at the centers of those polygons. In order to carry out this procedure, you will need to locate the centers of gravity for various regular polygons.

Exercise 4.18. Find the center of gravity for a square and a regular hexagon. Figure out a general method for finding the center of gravity for any regular polygon with an even number of sides.

Exercise 4.19. Find the center of gravity for an equilateral triangle and a regular pentagon. Figure out a general method for finding the center of gravity for any regular polygon with an odd number of sides.

Exercise 4.20. Suppose somewhere in a tiling there are two regular polygons, one with n sides and one with k sides, that share an edge. Where will the line connecting the centers of the two polygons intersect the shared edge? What angle will this line make with the shared edge? Explain your answers.

Exercise 4.21. In the dual tiling of 4.8.8, what are the angles of the triangles?

Exercise 4.22. Draw and describe the dual tilings for each of the three regular tilings found in Exercise 4.1.

We will also see the tiling 3.3.4.3.4 and its dual. Note that the dual contains only one type of tile: an irregular pentagon/

To find the angles of this irregular pentagon, note that it has two types of vertices: Two of the vertices are at the centers of squares of the original tiling and the other three are at the centers of equilateral triangles. Since the lines of the dual tiling divide the square into four equal sectors, these two angles must be $\frac{360°}{4} = 90°$. Since the dual tiling divides the triangles into three equal sectors, those angles must be $\frac{360°}{3} = 120°$. Thus, the irregular pentagon has angles $120° - 120° - 90° - 120° - 90°$, which sum to $540°$, as they should for a pentagon.

Exercise 4.23. Draw the dual tiling for each of the tilings found in Exercise 4.11. Describe the types of polygons formed and find their vertex angles.

Exercise 4.24. Draw the dual tiling for the tiling 3.4.6.4 of Exercise 4.15. Describe the types of polygons formed and find their vertex angles.

Exercise 4.25. Draw the dual tiling for each of the tilings found in Exercise 4.16. Describe the types of polygons formed and find their vertex angles.

Exercise 4.26. Draw the dual tiling for each of the tilings found in Exercise 4.17. Describe the types of polygons formed and find their vertex angles.

Exercise 4.27. Explain why the dual tiling of a regular or semiregular tiling will always have only one type of polygon, though it may be irregular.

Exercise 4.28. Of the examples and exercises above, which tilings have duals formed by triangles? squares? pentagons? Find a rule for the type of polygon in the dual.

4.2 k-uniform Tilings

Euclidean tilings by convex regular polygons may be classified by the number of orbits of vertices, edges and tiles. If there are k orbits of vertices, a tiling is known as k-uniform.

There are twenty (20) 2-uniform tilings of the Euclidean plane. We will see four of them in the class and ask you to draw another two in your homework.

4.3 Irregular Tilings

In tiling, we need not restrict ourselves to regular polygons. It is easy to tile the plane with copies of any rectangle or parallelogram.

Exercise 4.29. Show that any triangle can tile the plane.

Exercise 4.30. Show that any trapezoid can tile the plane.

Exercise 4.31. Show that any quadrilateral can tile the plane.

Reptile A *reptile* (short for repeating tile) is a tile that can be arranged to form a larger copy of itself. This larger copy must be an exact scaled replica of the original.

For example, a square is easily seen to be a reptile, since four squares (or, for that matter, 9 squares, or 16 , etc.) can be arranged to form a larger square.

Exercise 4.32. There is a trapezoidal reptile. Find the angles of the trapezoid.

There are lots of irregular tilings, or tesselations. For example, one can tile the plane with any parallelogram, as above. Any reptile will tile the plane, since if, for example, four copies of the figure fit together to make a larger copy of itself, then four of the bigger copies will fit together to make an even larger one, etc.

Although we showed that the regular pentagon does not tile the plane, it is easy to tile with the irregular pentagon.

The Dutch artist M.C. Escher created many tiling pictures, initially inspired by the tiles at the Alhambra, in Spain. Many of these involved repetitions of pictures, rather than simply polygons, though the polygons are there in the background. In his notebooks, one sees the construction of the grid underlying many of these pictures. In this section, we will outline some processes for generating irregular tilings.

Recall that the regular tilings of the plane are by squares, equilateral triangles, and hexagons. The simplest irregular tilings are by rectangles or parallelograms. These grids can be modified to form other irregular tilings.

Parallel Translation The first method we will use is a variation on a square, rectangular, parallelogram, or hexagonal grid. We want to make use of the parallel pairs of sides each of these has. Cut a basic unit (square, rectangle, parallelogram, or hexagon) out of cardboard, and use it to trace faint lines on a piece of paper, laying out your basic grid.

Figure 3: Work of M.C. Escher

Now take the tile, and cut a section out of one side. Tape this piece onto the parallel side, as shown below. This move is called parallel translation. The cut-off piece moves in a straight line, without any rotation or reflection, to a position parallel to its original position.

Lay the new tile on your grid and trace it, to get a new irregular tiling. This process can be repeated for the other pair of parallel sides.

For the tiling given by the construction above, we can find a minimal parallelogram grid to generate the tiling by translations.

Glide Reflection Another way to modify a tiling when the original tile has a pair of parallel sides is called glide reflection: Take the original tile and cut something off of one side. Flip the cut piece over and tape it onto the parallel side. Then tile the plane with the new tile. You will have to turn over every other tile.

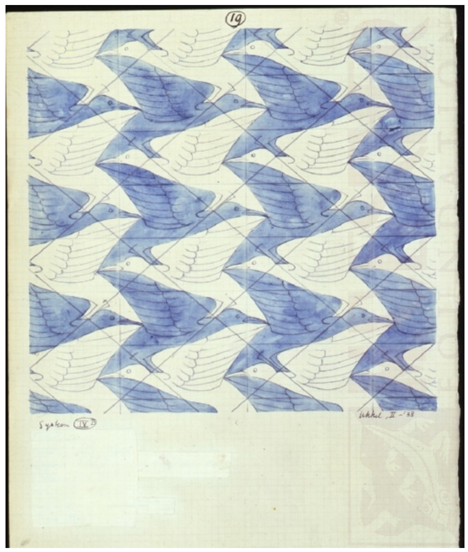


Figure 4: Work of M.C. Escher

Midpoint Reflection Another way of creating an irregular tiling from one of the basic grids is midpoint rotation. For this one can use a basic grid formed by any quadrilateral or triangle, since we will not need parallel sides. Take one side of the tile and find its midpoint. Cut a pattern out of the tile that begins at one endpoint and ends at the midpoint of the same side.

Side Rotation The fourth technique we will investigate, side rotation, works nicely only on the regular grids: squares, equilateral triangles, and regular hexagons. Cut a pattern out of the tile that begins at one end of one of the sides and ends at the other end of that side.

Then take the piece you cut out, rotate it about the endpoint, and tape it back along an adjacent edge of the tile.

Then tile the plane with the new tile.

Many of Excher's tiling pictures make use of more than one tile. For example, in the drawing on the next page there are both fish and birds. Thus, the basic repeating unit must contain one fish and one bird. The parallelogram grid chosen gives one of each after a translation is applied in each pair of parallel sides.

Figure 5: Work of M.C. Escher

What are the shapes that cannot tile the plane? There are many shapes, in fact most shapes, that cannot tile the plane, for example, regular pentagons, the modified rhombus of Exercise 15 , etc. Here is one rule for deciding whether a figure will tile the plane. However, note that there are lots of figures (such as the one you developed in Exercise 9) that do not fulfill the criterion but succeed in tiling.

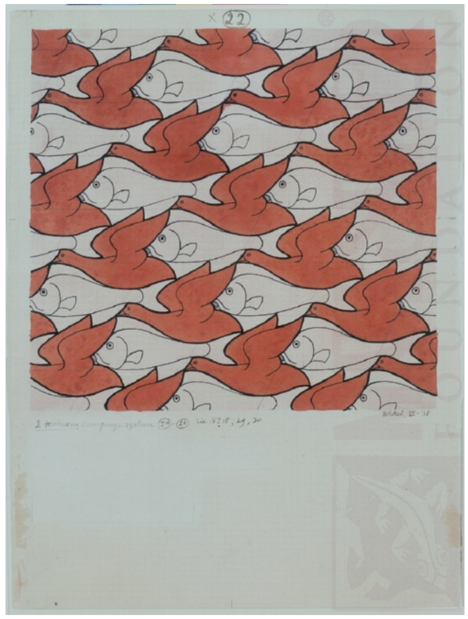


Figure 6: Caption

Exercise 4.33. Analyze the tiling shown below by M.C. Escher. Describe the underlying grid and the polygonal tile, and how this tile was modified to form a single copy of the reptile.

Conway Criterion A simple region (the boundary must form a loop without crossings and there can be no punctures) will tile the plane if the boundary can be divided into six arcs by six points labeled A, B, C, D, E, and F in order as one travels around the boundary, and
(1) The arc AB from A to B is the parallel translate of the arc ED.
(2) The arcs BC, CD, EF, and FA have rotational symmetry about their midpoints.

An extension of the Conway criterion allows some of the points A, B, C, D, E, and F to be the same, as long as there are at least three distinct points. Of course, if $A = B$, then the arc AB is a single point, and then we must have $D = E$. In this case there are no translated sides, and we have a quadrilateral $ACDF$ with a midpoint rotation on each side.

4.4 Penrose Tiling

The tilings with which we are most familiar tend to be periodic: that is, there is a fixed pattern that is repeated over and over again in a predictable way to

Figure 7: Caption

cover the plane. A tiling is periodic if there is a finite section of the tiling that can be translated in two nonparallel directions to recreate the entire tiling. In other words, if you can make a rubber stamp out of some tiles and use it to cover the rest of the plane without rotating or reflecting the stamp and with no gaps or overlap, then you have a periodic tiling. Most of the tilings from the previous sections are periodic. The regular tilings of the plane by squares and by hexagons are periodic, and the simplest rubber stamp would be a single tile. The regular tiling by triangles is also periodic, but the rubber stamp would be made of two triangular tiles (a parallelogram) since you are not allowed to rotate the stamp. In general, if a tiling is periodic, your rubber stamp will need at least one of each type of tile in your tiling and more than one if there are rotations or glide reflections in your pattern.

Note that these regions are not unique. For the octagon-square tiling, we could choose several different regions for the rubber stamp.

In 1937 the Dutch artist M.C. Escher began to experiment with the metamorphosis of his tiling patterns. Note that the Escher metamorphosis etchings or parquet deformations are not periodic, even though small sections seem to be. Your stamp is not allowed to change as you move along the tiling. However, these prints are only tilings of finite sections of the plane and so periodicity is not an issue.

The question arose whether all tilings are periodic. The answer is no, as the figure below demonstrates.

The next logical question is whether there is a set of tiles that can tile the plane only in a nonperiodic way. Upon hearing the question, most people suspect that there is not an aperiodic set of tiles. Most mathematicians agreed until 1964, when Robert Berger produced such a set. His original set contained

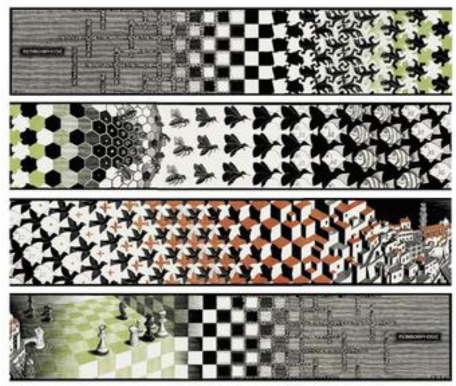


Figure 8: Escher, Metamorphosis II

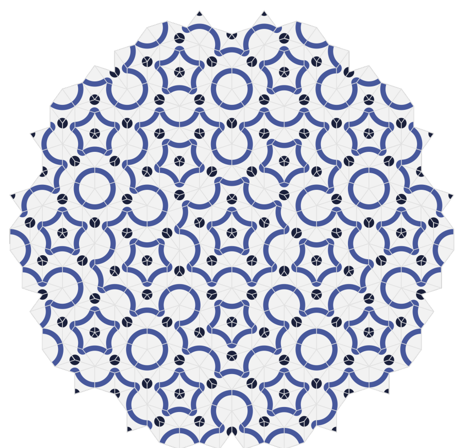


Figure 9: Rhombus Penrose tiling with arcs

over 20,000 different tiles. He continued to work on the problem and produced a set of 104 tiles that could cover the plane only in a nonperiodic way. Another mathematician, Raphael Robinson, found an aperiodic set containing only six tiles.

The most well-known set of aperiodic tiles are the Penrose tiles. The set was introduced by Roger Penrose in 1974 and contains only two tiles along with a set of rules for how the tiles must be put together. Penrose began his search for an aperiodic set of tiles by looking at pentagons. While it is true that the plane cannot be tiled by regular pentagons, Penrose studied the gaps left when one tried. He then took smaller pentagons and tried to fill in the holes. After several subdivisions, he found that the holes could have only a few shapes. These are pictured below, and he called the shapes diamonds, paper boats, and stars. Significant insight and refinement led to Penrose's first aperiodic set of six tiles and finally to the set of two tiles most commonly known as the kite and dart, names suggested by John Conway, another mathematician who has contributed a great deal to what is known about Penrose tilings.

Anyone who has studied stars and pentagons should expect to find the golden ratio somewhere in a situation involving these two shapes, and indeed it makes several appearances here. In any infinite tiling of the plane by kites and darts, the number of kites used is the golden ratio times the number of darts. In any tiling of a finite section of the plane, the ratio of kites to darts will approximate the golden ratio. The approximation improves as the area tiled increases. The area of a kite is the golden ratio times the area of a dart. With these facts in mind, it is probably not surprising that the golden ratio is part of the actual measurements of the tiles. The kite and dart are cut from a rhombus

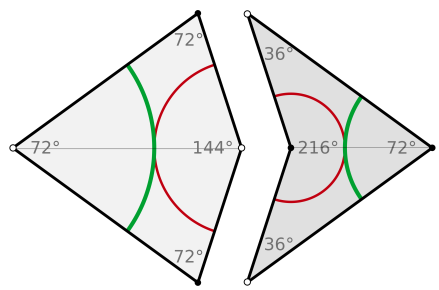


Figure 10: Kite and Dart

with side lengths equal to the golden ratio ϕ and main diagonal length equal to the golden ratio plus one, $\phi + 1$. Connect the vertices at the obtuse angles to the main diagonal at a distance of the golden ratio from an acute angle (and hence a distance of one from the other acute angle). The large piece is the kite; the small piece is the dart. Notice that the kite is made of two isosceles triangles with the golden ratio for the length of the equal sides. The length of the third side of these triangles is equal to one. These triangles are sometimes called golden triangles. Consequently, the kite is also made of two isosceles triangles with equal sides of length one and third side of length equal to the golden ratio.

Exercise 4.34. The Pythagorean pentagram (pictured below) is a familiar figure involving both the five-pointed star, or pentagram, and the pentagon. The length of any line segment in this figure is the golden ratio times the length of the next smaller line segment. Using this information and what you know about the lengths and angles of the Penrose tiles, find both a kite and a dart in the Pythagorean pentagram.

The rules for constructing a nonperiodic tiling with the Penrose tiles are

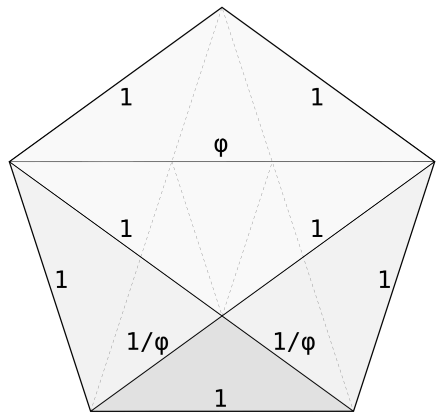


Figure 11: Rhomb Pentagon

simple. However, following them consistently is not necessarily easy. First, as with many tilings, only sides of the same length can be put together. This rule ensures that no vertex of one tile can occur in the middle of the side of another tile. The second rule requires a certain direction along the sides of the tiles. To enforce the direction rule, some people have put notches and bumps on the tiles, some use dots or holes, and some reshape the tiles to fit only in the correct way. John Conway puts arcs on the tiles and requires that the arcs of the same color must meet to form continuous curves. Thus, in constructing a nonperiodic tiling, dark arcs must join dark arcs, and light arcs must join to light. Conway went on to prove a number of results involving the way the arcs connect.

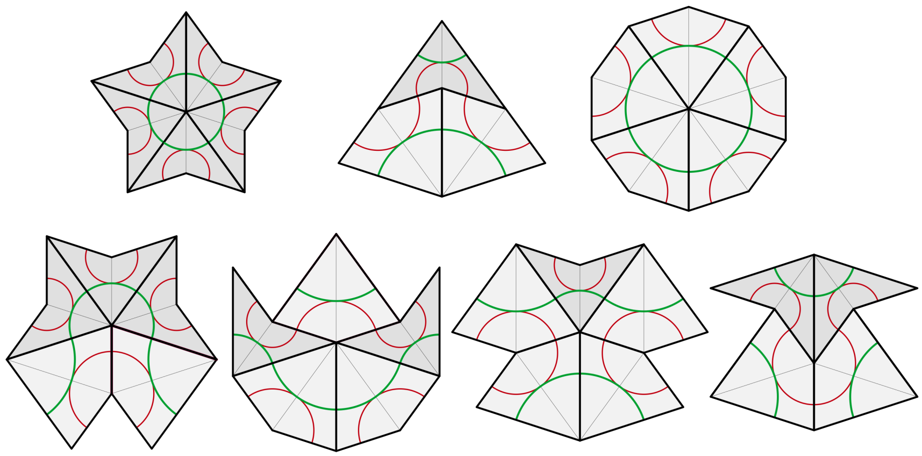


Figure 12: Penrose Vertex Figures

4.5 References

1. Tessellation Wikipedia.
2. Symmetry, Shape and Space: An Introduction to Mathematics Through Geometry, by L. Christine Kinsey and Teresa E. Moore. Chapter 4 Tesselations.
   - 4.1: Regular and Semiregular Tilings.
   - 4.2: Irregular Tilings.
   - 4.3: Penrose Tilings.

5 Topology

5.1 Universes

In the first one and half months we talked about maps and graphs and coloring problems and tessellations. Most of the time when we discuss about these topics, we are working with geometric shapes that lie in some ambient space, which, for the most of the time is the Euclidean space $\mathbb{R}^2$, and occasionally the sphere $\mathbb{S}^2$ or the hyperbolic disc $\mathbb{H}$ (even though we didn't really explain very rigorous about the geometry on the hyperbolic disc $\mathbb{H}$).

One thing we've seen in class is that the possible tessellations of different types depends on this ambient space where we construct our tiling. And similarly we can ask the question whether the Euler's formula and the graph/map coloring problem depends on the ambient space we are working with.

To answer these questions, we will study the space where the story happens, which we call the universes, and see how things would change when we change our universe.

5.1.1 Dimensions

The dimension of a universe is an intrinsic property that help us to determine the local information of the space. For instance, the Euclidean spaces $\mathbb{R}^n$ has dimension n as we need n coordinates to determine the location of a point.

The notion of dimension in fact depends on our definition of universes, so our first question would be what are the permissible universes? In this course, we will restrict ourselves to shapes which are locally look like Euclidean (so that we have a local definition of dimensions based on our knowledge about Euclidean space), and we glue all the locally small pieces together in some nice way so that this local behavior will not mess up after gluing. This is in fact the notion of manifolds.

We are not going to start with general manifolds of dimension n, as they can be quite difficult to understand and visualize. We will first discuss $1, 2$ and 3-dimensional universes, and understand the local properties based on our knowledge of the Euclidean spaces, and see how to obtain global information to distinguish different universes.

Some properties that we can discuss to distinguish different universes are whether the universe has finite length/area/volume, and whether it has boundary or not.

1-dimensional universes are finite lines (finite length, with boundary), rays (infinite length, with boundary), infinite lines (infinite length, no boundary) and circumferences of circles (finite length, no boundary). And all 1-dimensional universes are the disjoint union of components of these four types.

2-dimensional universes are slightly more complicated, but similar basic types are the interior of circles (finite area, with boundary), upper half plane (infinite area, with boundary given by the x-axis), infinite plane (infinite area, no boundary) and spheres (finite area, no boundary).

Example 5.1. Find some examples of 3-dimensional universes with finite/infinite volume and with/without boundaries.

5.2 Cylinder Tic-Tac-Toe

We obtain a cylinder by gluing two parallel sides of a square (in the same direction). Play the cylinder tic-tac-toe in class.

5.2.1 Equivalent Games

Two games are considered *equivalent* if they yield the same game on a cylinder.

Question 5.2. *What are the operations that will yield equivalent games?*

Hint Translations, horizontal and vertical reflections and rotations by 180 degrees.

5.2.2 Winning Strategy

Question 5.3. *In the traditional tic-tac-toe, two good player will always play to a draw. Is that also true in cylindrical tic-tac-toe?*

Hint. Identify equivalent steps and analyze all possible cases. □

5.3 Torus

Construction We obtain a torus by gluing the two pairs of parallel sides along the same directions.

5.3.1 Torus Games

Play games on the torus. See how they differ from classical ones.

Example 5.4. Torus tic-tac-toe.

Example 5.5. Torus mazes.

Example 5.6. Torus chess game.

Example 5.7. Torus word search.

Example 5.8. Torus Gomoku.

Example 5.9. Torus jigsaw puzzle.

Example 5.10. Torus pool.

Question 5.11. *What are the differences of the game rules if we switch from the standard version to torus version? How would you adjust your strategy when you play the games on the torus?*

5.3.2 Fundamental Domain and Tiling View

The square we use to glue to get the torus is called the fundamental domain of the torus. If we repeat the square and tile it on the plane, we get a tiling view of the torus. Play the same games in the tiling view and see if you find it easier or more difficult.

5.3.3 Graphs on Torus

Similar to the case of spheres, we can draw graphs and maps on the torus. Is there any difference between those two cases?

Five Terminal Problems The five terminal problem is stated as follows: Take five points. Connect each pair of points with an edge so that the edges don't cross.

The solution to this problem depends on the universe where we are going to draw the points and edges. For instance, we know that the complete graph K_5 and the bicomplete graph $K_{3,3}$ are not planer, i.e. they cannot be drawn on the plane or sphere with no crossing edges, so the five terminal problem does not have a solution on the plane or sphere.

We want to reexamine this question on torus and see if there is any difference.

Exercise 5.12. Draw the complete graph K_5 on the doughnut torus with no crossing.

Hint. In your homework, I've asked you to draw the graph obtained from K_5 by removing one edge. This is a planer graph. Can you put this graph on torus and add the missing edge on the torus without crossing? □

Six Terminal Problems When our space is torus, we can do better than just the complete graph K_5.

Exercise 5.13. Draw the complete graph K_6 on the doughnut torus with no crossing.

Seven Terminal Problems

Exercise 5.14. Draw the complete graph K_7 on the doughnut torus with no crossing.

Complete Bipartite Graphs We can also ask whether certain complete bipartite graphs, which cannot be embedded on the sphere without crossings, might instead be drawn on the torus without any crossings.

Example 5.15. Draw the complete bipartite graph $K_{3,3}$ on the doughnut torus with no crossing.

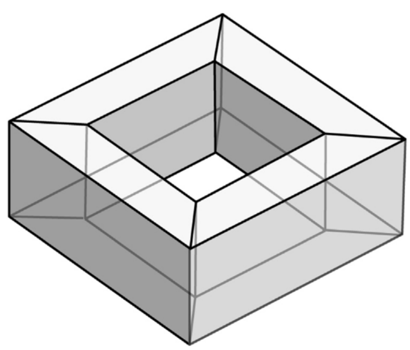
Exercise 5.16. Can you draw the complete bipartite graph $K_{3,4}$ on the doughnut torus with no crossing?

Exercise 5.17. Can you draw the complete bipartite graph $K_{4,4}$ on the doughnut torus with no crossing?

5.3.4 Euler's Formula on Torus

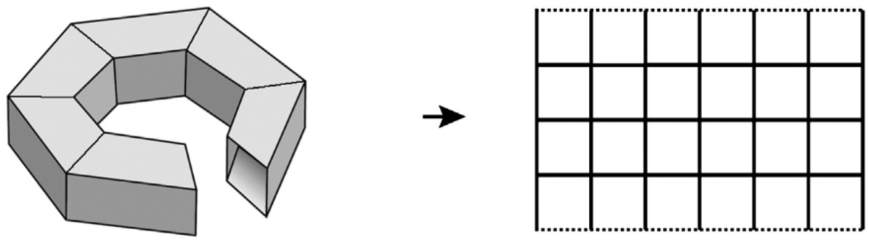
We showed the complete graph K_5 is not planer using Euler's formula. Since we can draw K_5 on the torus without crossing, this makes us to rethink about Euler's formula on the torus.

Question 5.18. *To start with, let's first consider the following torus polyhedron and count the number of vertices, edges and faces, what is the alternating sum in this case?*



Answer. There are 16 vertices, 32 edges and 16 faces. So $V - E + F = 0 \neq 2$. □

This is different from the case of planes or spheres. Draw a few more torus polyhedral or maps on torus and count, do you see any pattern?

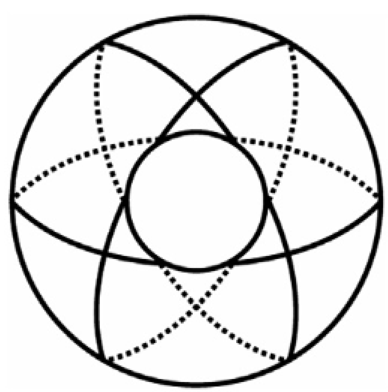


Imagine those shapes are made of cardboard so they aren't solid. We can cut them open like this. Then maybe we could use what we know about sphere or island maps. We need to do some stretch-shrinking to get it to flatten out.

The resulting picture is indeed a map on the plane. The only difference is that we do not have an "outside" country, so the Euler's formula should read as $V - E + F = 1$. This is not precisely the result when we counted on the dougnnut torus. What is the issue?

Anything along the cut of the box will be duplicated on the island's border. So we have to subtract those duplications in the island count to get the correct doughnut count. This includes both vertices and edges. Notice we have the same amount of vertices and edges, so after removing the duplicate vertices and edges, we will have the same relation $V - E + F = 1$.

Furthermore, look at the vertices on the corners of the rectangle. They come from the same vertex on the original map. When we counted V on the original, that vertex only counted for one, but when we count V on the square, we count four. Even after subtracting the duplicated ones as above, there is still one extra duplicate copy we need to remove. After removing this duplicate point gives up the right formula $V - E + F = 0$. The reason above for polyhedral

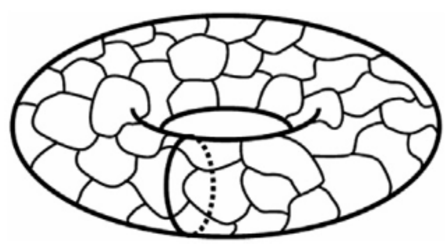


would probably give us an argument if we were able to cut the surface open along edges of the map. However, it is not true that we can always do that and get something nice, like a square with opposite sides identified. Above's an example.

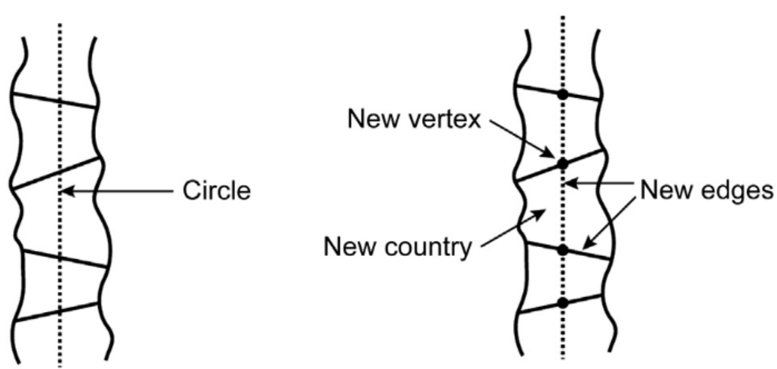
Exercise 5.19. To understand this map better, try to draw an equivalent graph on the hexagon shaped torus using straight line segments and cut it open to draw it on a square.

However, we are not too far from a complete proof.

Start with a map on the doughnut. Suppose that V, E, and F are its data. Look at a circle that goes around the doughnut's vertical "waist."



The circle might go through a vertex or two, and it might coincide with large segments of the map's edges at certain instances. However, if you "nudge" the circle a little at those spots, you can get it to avoid the vertices and intersect edges transversely—at single points—so that a narrow "belt" surrounding the "circle" would look like this.



The idea is to add this closed curve (circle) to the map, along with the new vertices, edges, and faces it creates. All the new stuff can be determined by what happens in the belt. Suppose there are k edges from the original map that the circle crosses. Then as you travel along the circle and it crosses an edge, it creates a new vertex, two new edges, and a new country. So, as you travel along the entire circle, the circle creates k new vertices, $2k$ new edges, and k new faces. With all of this added to the original map, the new data has the same alternating sum of number of vertices, edges and faces.

Similarly we can add a new horizontal "waist". This also will not change the Euler's formula. Cut it open along the added vertical circle and the horizontal circle. We get a square. This is the case we just proved so we know $V - E + F = 0$. This is the Euler's formula on the torus.

An alternative (but not too different) way to see this, is to cut alone only the vertical circle and we get a map on cylinder. Note we can flatten out a cylinder and view it as an annulus on the plane. You can view this as the map of an island with a lake, and the number of countries will be 2 less than what you have for the standard Euler's formula if you remove both the lake and the outside country, which is our desired formula.

5.3.5 Coloring Problem on Torus

The complete graph K_7 tells us that there is a graph on the torus which we need at least 7 colors to color the vertices (or a map with 7 colors to color the countries).

Example 5.20 (Tessellated Seven-Color Tori)**.** See Figure 13.

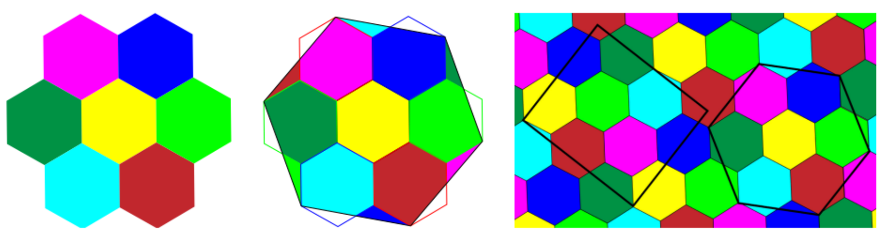


Figure 13: The seven hexagons in seven colors (left) are scissors congruent to a single large hexagon (middle) and can be cut from a tiling of hexagons in layouts that sew into seven-color torus maps (right)

Exercise 5.21. Draw a 7-colored doughnut torus.

Exercise 5.22. Draw a 7-colored flat torus.

Example 5.23. Here are some demonstrations of 7-colored tori "in real life".

1. Torus coloring from Wolfram.
2. Seven-Color Tori.
3. 7-color colored torus using Tomoko Fuse's honeycomb unit.

Example 5.24. The 7-color map of birds in Tessellated Seven-Color Tori.

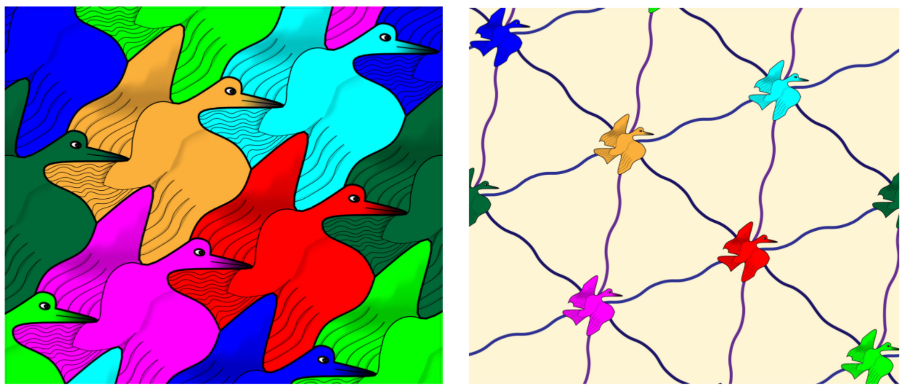


Figure 14: (a) 7-color map of birds using a regular hexagon as the base tile for the tessellation. To make it into the toroidal scarf shown in Figure 4, sew together the left and right edges and give the resulting cylinder a half twist before sewing together the cylinder ends (i.e., the top and bottom edges).
(b) K7 graph with no link crossings using scaled down tiles as the nodes, sewn together the same way.

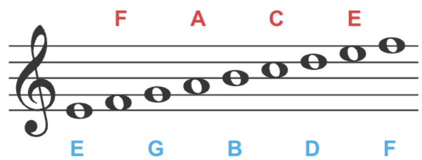
5.4 The Space of 2-Chords

Let's take a short break from math and consider music (well in fact math in music, so we are not completely away from math).

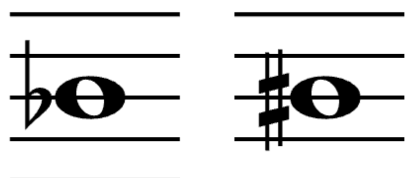
5.4.1 Notes

There are some basic musical notations we are interested in, in particular notes. There are several features that determines the note:

- Pitches: different pitches are named by letters. The musical alphabet is, in ascending order by pitch, A, B, C, D, E, F and G. After G, the cycle repeats going back to A. Each line and space on the staff represents a different pitch. The lower on the staff, the lower the pitch of the note. Notes are represented by little ovals on the staff. Depending on the clef, the position of each note on the staff corresponds to a letter name.



- Accidentals: Accidentals modify the pitch of a note by increasing or decreasing it by one half step. Accidentals stay in effect for all notes of the same pitch for the rest of the measure. When these same symbols appear at the very beginning of the music they are specifying a key signature. Flats (left side of the picture) lower the pitch of the note by one half step. Sharps (right side of picture) raise the pitch of the note by one half step.



- There are some other components, for instance durations. All notes have length. We are not going to consider those factors for now and will only consider pitches and accidentals.

There are 12 notes in traditional western music:

$$\begin{array}{cccccc} C & C\sharp = D\flat & D & D\sharp = E\flat & E & F \\ F\sharp = G\flat & G & G\sharp = A\flat & A & A\sharp = B\flat & B \end{array}$$

In this class we prefer to deal with numbers rather than the musical letters, and the convention is to identify

$$\begin{array}{cccccc} C = 0, & C\sharp = 1, & D = 2, & D\sharp = 3, & E = 4, & F = 5, \\ F\sharp = 6, & G = 7, & G\sharp = 8, & A = 9, & B\flat = 10, & B = 11. \end{array}$$

After 12 half steps (an octave) the notation repeats, because we hear those notes as very similar. For the purpose of thinking about chords we won't distinguish between a higher sounding C and a lower sounding C. If we identify all notes that represent the same sound up to octaves, we can think of the notes living around a circle, repeating over and over again.

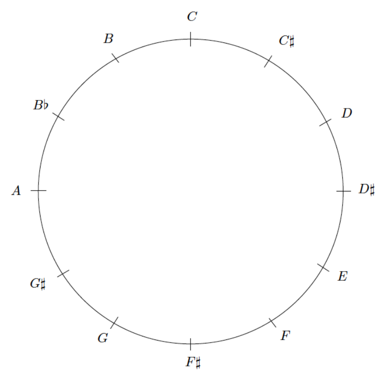


Figure 15: Writing the Notes around the Circle

5.4.2 The Space of 2-Chords

A *chord* is a number of notes sounding at the same time, there could be a 2-chord with 2 notes sounding but also a 5-chord with 5 notes sounding simultaneously.

If we look at the spaces of two chords with notes that represents the same sound up to octaves, what is the space that 2-chords live? If we use the tuple

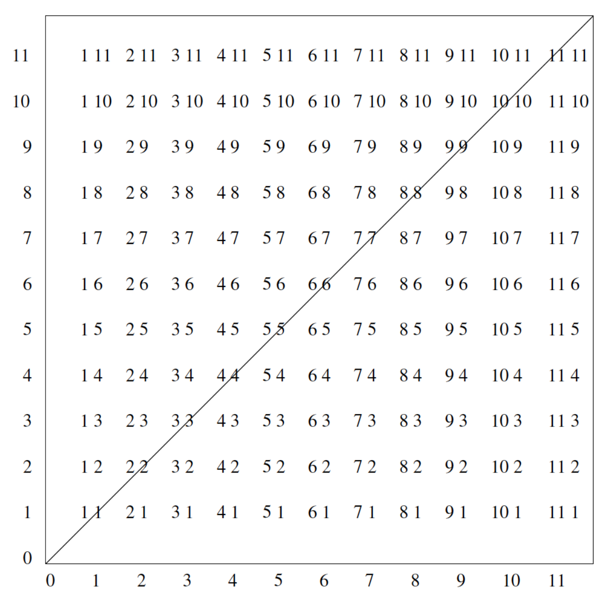


Figure 16: Fundamental Domain of Musical Torus with Triangles

(i, j) to denote a pair of notes, we can draw them as in Figure 16, and the mod 12 congruence relation gives us precisely the fundamental domain of a torus. However, the delicate difference for the space of two chords is that we do not really care about the order of the pair (i, j), so we'd better identify the two triangles by the reflection along the diagonal as in Figure 16.

All the 2-chords in that triangle are the same as the ones above the diagonal, if we ignore the order in which the notes are played. For instance $(1, 9)$ is the same as $(9, 1)$. Unfortunately, when we look at just the lower triangle, it is difficult to figure out what is the result space.

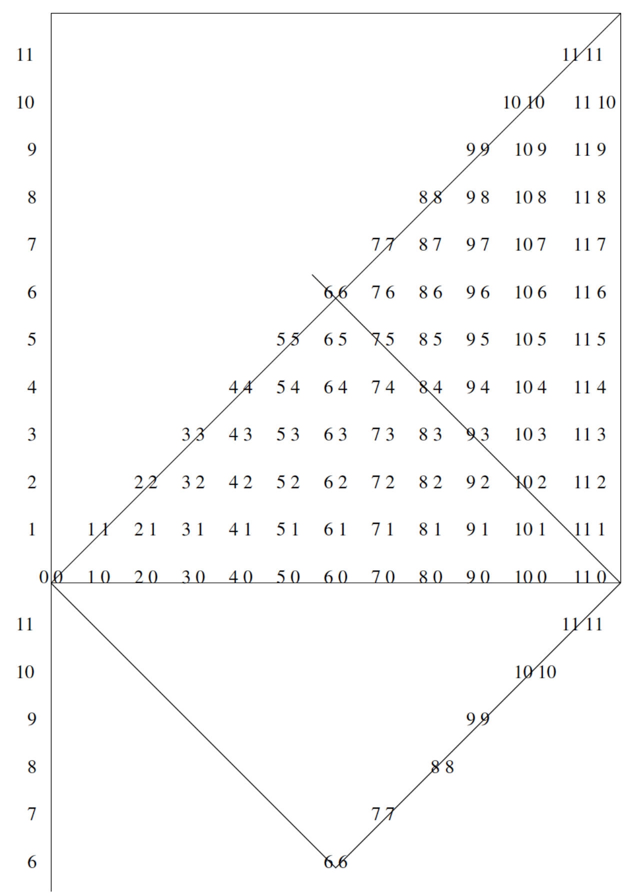


Figure 17: A Different Fundamental Domain of 2-Chords

We cut our triangle under the diagonal in Figure 17 into two pieces and compare the right one of the two pieces with the triangle that you can see under the x-axis. Convince yourself that the 2-chords in these two areas are the same!

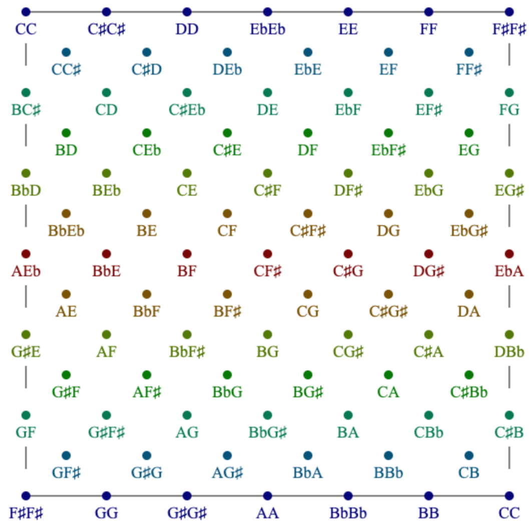


Figure 18: The Dyadic Möbius Strip

Use the numerical identification, we have the following picture of the space of 2-chords.

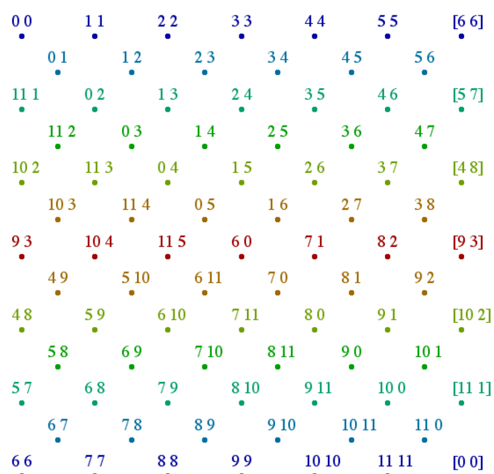


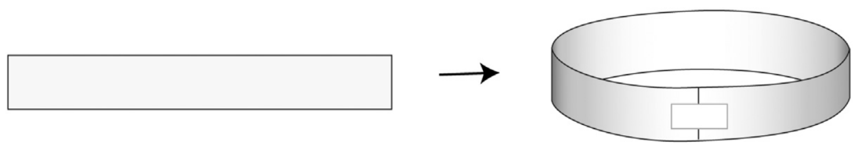
Figure 19: The Dyadic Space

We will see in the next class that the space of 2-chords is a Möbius strip!
Enjoy Chord Geometries Demo: Chopin on a Möbius strip!

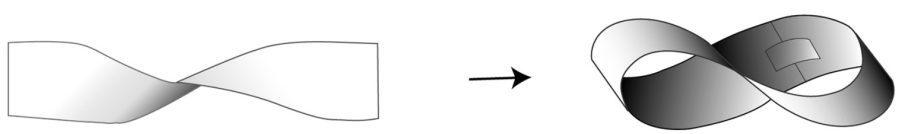
5.5 The Möbius Strip

5.5.1 Basic Construction

Let's take a step back and reexamine the cylinder construction. When we glue two parallel edges of a square in the same direction, we get a cylinder. What if

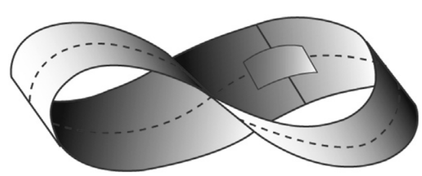


we glue them together in the opposite direction? This is the space of 2-chords, which is called a *Möbius strip*.



Exercise 5.25. Take a pen and draw a line starting anywhere on the strip, continuing along the strip. Will you connect again with your original line? Why or why not?

Exercise 5.26. Make a Möbius strip and cut it along the dotted line below. What happens?



5.5.2 Twisted Strips

Twisted Strips		
No. of Half Twists	**No. of Divisions**	**Results**
1	2 (halves)	1 strip twice as long
2	2	2 strips, same size, linked
1	3 (thirds)	?
2	3	
1	4 (fourths)	
3	2	
3	3	
4	2	

Question 5.27. *What would happen if we made strips with different numbers of twists – or rather half twists – in them and then cut them down the middle?*

Question 5.28. *For the strips you have, paint along the edge of strips until you get to where you started. How many edges do you have?*

Fact 5.29. Facts about Twisted Strips.

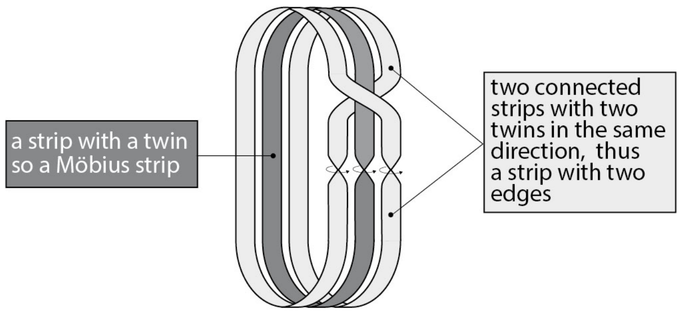
- A cylinder (zero-half-twists strip) has two sides and two edges.
- A Möbius strip (one-half-twists strip) has one side and one edge.
- A two-half-twists strip has two sides and two edges.

Question 5.30. *Can you find a surface that has one side and two edges?*

Twisted Strips		
No. of Half-Twists	**No. Divisions**	**Results**
0	2	2 strips, same size, unlinked; each one like the original
1	2 (halves)	1 strip twice as long
2	2	2 strips, same size, linked
1	3 (thirds)	?

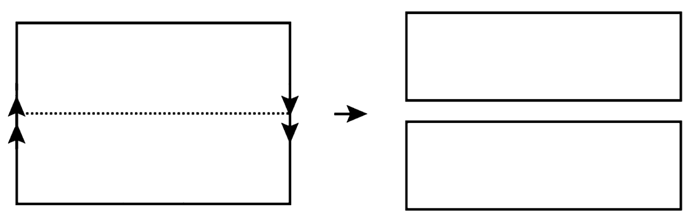
Exercise 5.31. For each line, there is an original strip and strips resulting from cutting the original. Figure out how many edges the original strip has and how many edges each resulting strip possesses. Add this new information to the right column of your table. Look for patterns and make predictions.

Example 5.32. Explanation for cutting a Möbius strip at one-third its width.

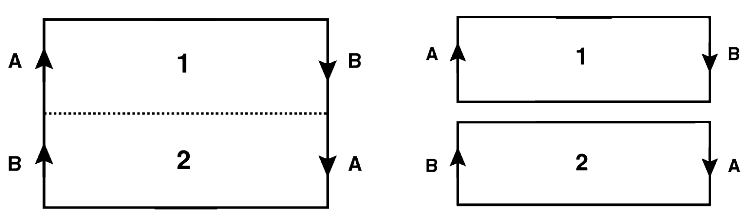


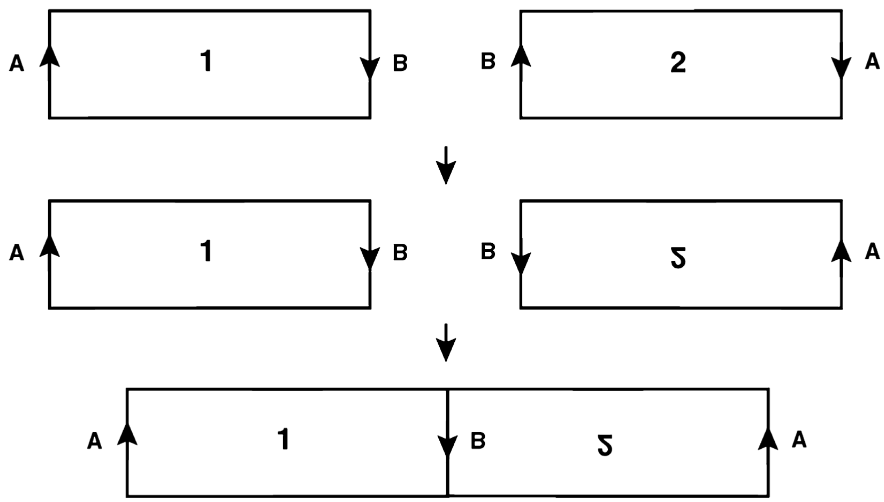
5.5.3 Cutting a Twisted Strip

After playing with twisted strips on paper, let's do some analysis to summarize our observations.



If we cut the Möbius strip in half before gluing, and identify the edges glued together, we will get the same result as cutting a Möbius strip in half.





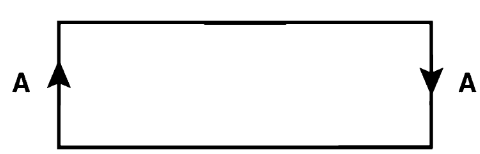
Cutting a Möbius Strip in Half If you cut a Möbius strip in half down the middle, then the result is a strip half as wide and twice as long with an even number of half twists.

Cutting a Strip with an Odd Number of Half-Twists in Two Cut a strip with an odd number of half-twists down the middle. The result is a strip half as wide and twice as long with an even number of half-twists (double the number of half twists).

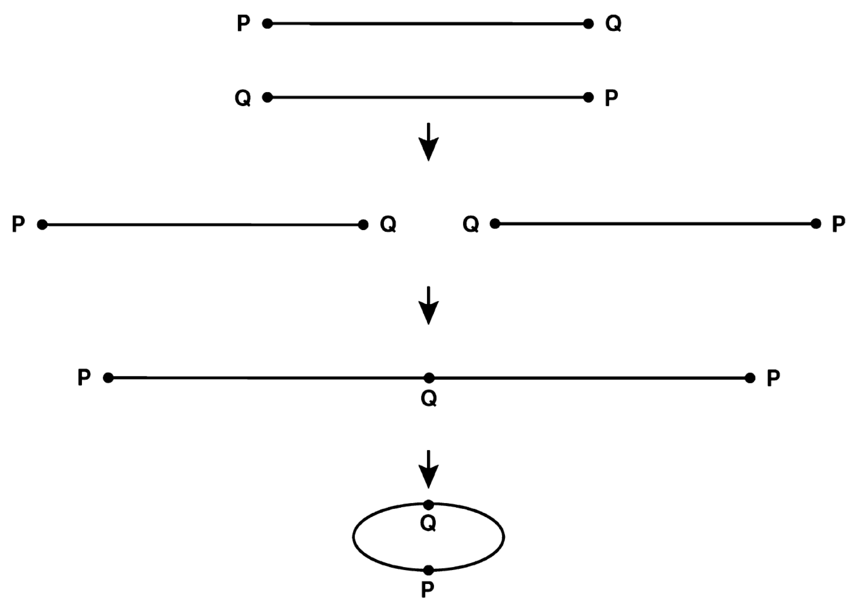
Cutting a Strip with an Even Number of Half-Twists in Two Cut a strip with an even number of half-twists down the middle. The result are two linked strips half as wide with the same number of half twists.

5.5.4 Edges of Twisted Strips

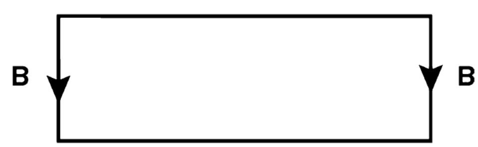
Patterns for Odd-Number Twisted Strips A pattern for a strip with an odd number of half twists is



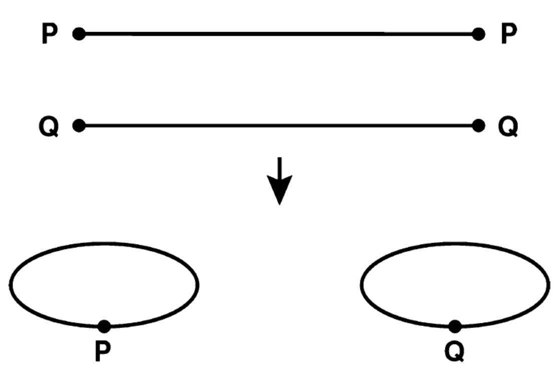
After gluing we will have only one edge.



Patterns for Even-Number Twisted Strips A pattern for a strip with an even number of half twists is

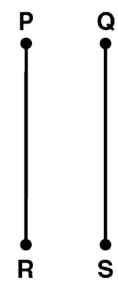


After gluing we will have two edges.

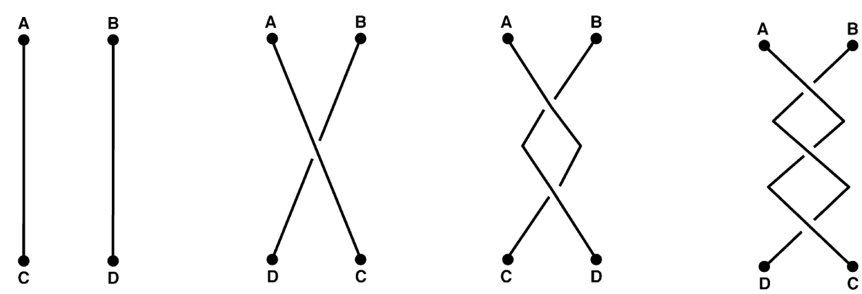


Shapes of Edges of Twisted Strips We would like to know what the edge (or edges) would look like for a specific strip.

Use the same idea as above, take the rectangle that will become the twisted strip and look at the two edges that will become the edge (or edges) of the twisted strip.



Then, without gluing, let's put twists in the strip and looks just at these two edges. Here is what we get in the first few cases.

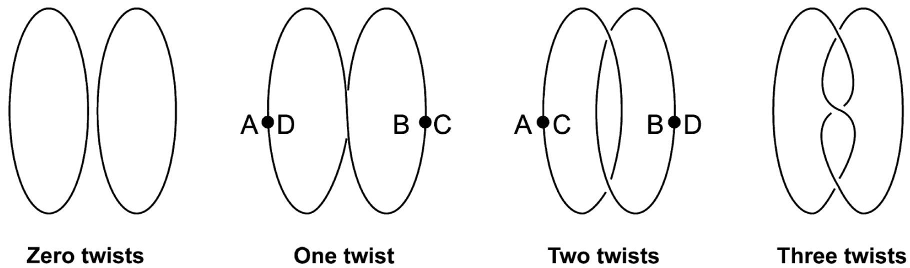


Zero twists One twists Two twists Three twists

Then we glue the ends together. In the cases above, here is what happened to the original edges of the rectangle.

Edges of Twisted Strips

- The edges of a strip with zero half twists are two unlinked, unknotted closed curves.
- The edge of a strip with one half twists is an unknotted closed curve.
- The edges of strip with two half twists are two unknotted closed curves, linked.



- The edge of a strip with three half twists is a knotted closed curve.

5.5.5 Tic-Tac-Toe

We can play the Tic-Tac-Toe game on Möbius strip as well. How is it different from the cylinder case?

Equivalent Games

Exercise 5.33. Identify equivalent games on Möbius strip.

Tiling View

Exercise 5.34. Draw the tiling view of the Tic-Tac-Toe game on Möbius strip.

5.6 Klein Bottle

5.6.1 Games on Klein Bottle

Play games on the Klein Bottle by changing the game mode to Klein Bottle. See how they differ from the torus games.

Example 5.35. Klein bottle tic-tac-toe.

Example 5.36. Klein bottle mazes.

Example 5.37. Klein bottle chess game.

Example 5.38. Klein bottle word search.

Example 5.39. Klein bottle Gomoku.

Example 5.40. Klein bottle jigsaw puzzle.

Example 5.41. Klein bottle pool.

5.6.2 Construction

If we change the direction of one pair of parallel edges that are glued together to get a torus, we will get a Klein bottle. We can visualize a torus as a doughnut in $\mathbb{R}^3$, what about the Klein bottle?

How to visualize Klein Bottle in $\mathbb{R}^3$? The standard way to see Klein bottle in $\mathbb{R}^3$ is in Figure 20 and a Figure Eight picture is given in Figure 21.

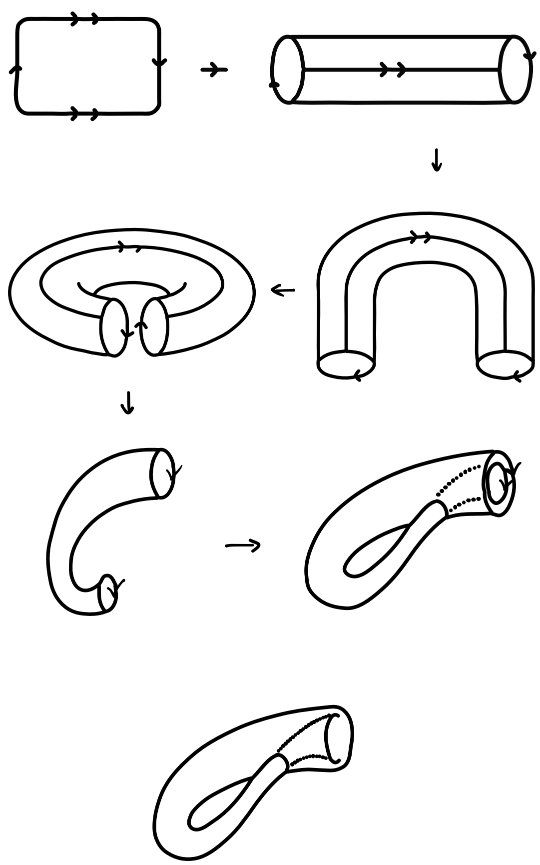


Figure 20: Standard Klein Bottle

Example 5.42. Check out the Klein Bottle Lego series made by Andrew Lipson.

Why Klein Bottle Cannot Embed in $\mathbb{R}^3$? This is an elementary proof available on mathoverflow: Why can't the Klein bottle embed in $\mathbb{R}^3$?. Here in the notes

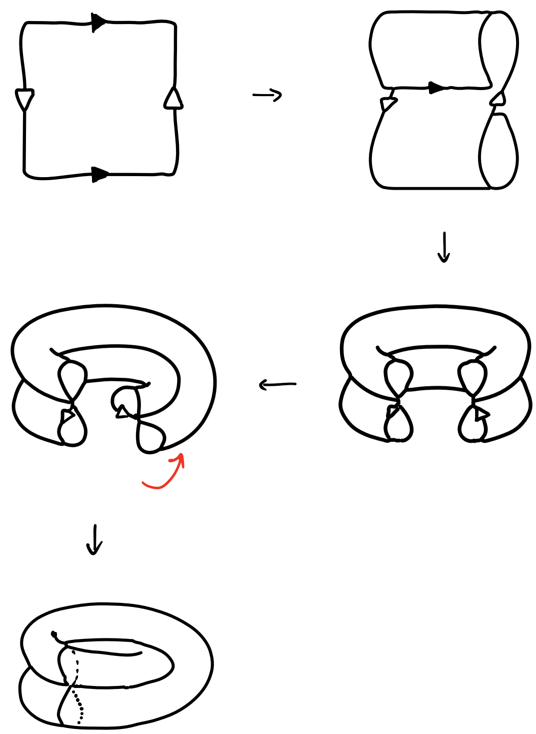


Figure 21: Figure 8 Klein Bottle

we only give the sketch of the proof and a more detail explanation can be found in the worksheet.

For simplicity let's assume that the embedded surface S is polyhedral, you can prove that it is orientable by the following argument.

Fix a direction (nonzero vector) which is not parallel to any of the faces. For every point v in the complement of S, consider the ray starting at v and going in the chosen direction.

- If this ray does not intersect edges of S, count the number of intersection points of the ray and the surface. If this number is even, you say that v is black, otherwise v is white.
- If the ray intersects an edge of S, you paint v the same color as some nearby point whose ray does not intersect edges.

The color does not change along any path in the complement of S (it suffices to consider only polygonal paths avoiding points whose rays contain vertices of S).

Now take points p and q near the surface such that the segment pq is parallel to the chosen direction. Then they are of different colors. But if the surface is non-orientable, you can go from p to q along a Möbius strip contained in the surface. This contradicts the above fact about paths in the complement of S.

5.6.3 Application

The Inscribed Square Problem If you have a closed curve in 2D space, will you always be able to find four points on this curve that make up a square?

This is known as the Toeplitz' conjecture. By "closed curve", we mean you squiggle some line through space in a potentially crazy way and end at the point where you started.

If your closed loop is a circle, for example, it's quite easy to find an inscribed square. Infinitely many squares, in fact. If your loop was an ellipse, it's still pretty easy to find an inscribed square.

This problem has been solved for some special cases, for example curves which are smooth1 are known to always contain inscribed squares. However, the general case where the only assumption you make is that the curve is continuous, which includes much wilder curves like fractals, remains unsolved.

The Inscribed Rectangle Problem The current tools of math can neither confirm nor deny the existence of a loop that has no inscribed square. But what about a slightly weaker question? Can we prove that any closed continuous curve always has an inscribed rectangle?

It's still pretty hard, but there is a beautiful solution described in the video This open problem taught me what topology is by 3Blue1Brown using the fact that Klein bottle cannot be embedded in $\mathbb{R}^3$.

5.6.4 How to Cut A Klein Bottle

Question 5.43. *How to cut a Klein bottle to get one piece of Möbius strip?*

Question 5.44. *How to cut a Klein bottle to get two pieces of Möbius strips?*

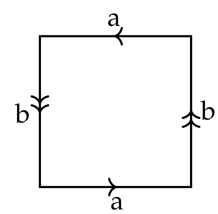
5.6.5 Paper Klein Bottle

So far we've seen several constructions of surfaces via gluing polygons and different ways to visualize them in $\mathbb{R}^3$. We will take a break and in this class we are going to spend some time asking you to make paper Klein bottle on your own.

5.7 Gluing A Square

We discussed two ways of gluing squares along parallel pairs of edges, what are the other possibilities?

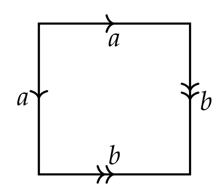
Question 5.45. *What would you get when you glue the parallel pairs of edges both in the opposite direction?*



Answer. The projective plane $\mathbb{RP}^2$ which is obtained by identify antipodes $((x, y, z) \sim (-x, -y, -z))$ on the sphere S^2.

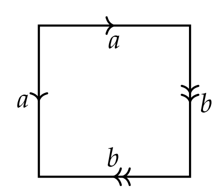
Question 5.46. *What would you get when you identify adjacent pairs of edges?*

(a). All pairs are glued along one clock wise and one anti-clockwise direction.



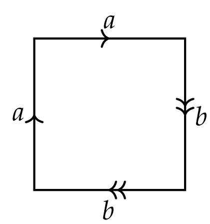
This is a sphere S^2.

(b). All pairs are glued along one clock wise and one anti-clockwise direction.



This is $\mathbb{RP}^2$.

(c). All pairs are glued along clockwise direction.



Cut along the two diagonals will give you two different ways to see this space.
One choice allows you to identify this space with Klein Bottle.
The other choice allows you to see this is the space obtained by gluing two copies of Möbius strip along their edge (in either directions).

5.8 References

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2. Jeffrey R. Weeks, Exploring the Shape of Space, Chapter 3: Cylindrical Tic-Tac-Toe.
3. Jeffrey R. Weeks, Exploring the Shape of Space, Chapter 4: Torus Games.
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12. David Gay, Explorations in Topology: Map Coloring, Surfaces and Knots, Chapter 6: Doughnuts.
13. David Gay, Explorations in Topology: Map Coloring, Surfaces and Knots, Chapter 7: The Möbius Strip.

14. David Gay, Explorations in Topology: Map Coloring, Surfaces and Knots, Chapter 8: New Worlds: Klein Bottles and Other Surfaces.

15. 3Blue1Brown, This open problem taught me what topology is.

6 More Surfaces

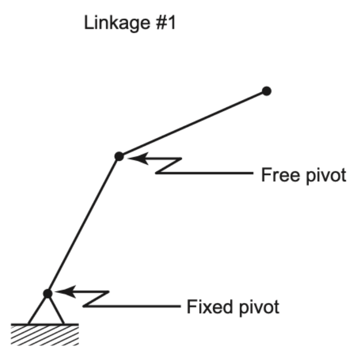
So far we examined all possibilities by gluing a square. What about other polygons? Are we able to describe the resulting space by gluing along the boundary edges in certain ways?

6.1 Examples of Surfaces

To start with, let's take a look at some examples which produces us some examples of surfaces given by gluing edges of polygons.

6.1.1 Linkage System: First Example

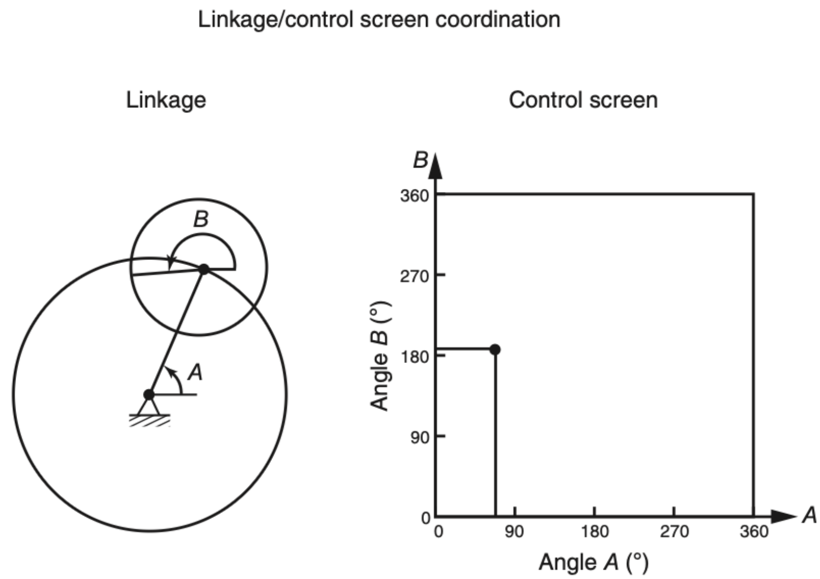
Consider the following linkage system. This linkage has just two rods. At the



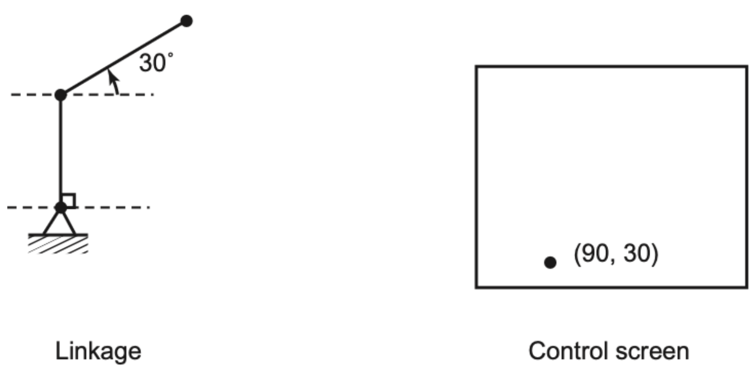
end of the first rod, there is a fixed pivot. Between the first and second rods, there is a connecting pivot. As you can see, the entire linkage can rotate freely about the fixed pivot, and the second rod can rotate freely about the connecting pivot. These are idealized rods so we allow the second rod pass over the first.

We can control the movement of the linkage by moving a cursor on a computer screen. We call it a control screen. Here's how the control screen works. Each rod makes an angle to the horizontal. We impose a coordinate system on the screen so that the lower left-hand corner is the origin, the bottom of the screen is the positive x-axis, and the left edge of the screen is the positive y-axis. When the cursor is at the point (s, t), the angle of the first rod to the horizontal is s and the angle of the second is t. Thus, as the cursor moves from left to right along the bottom of the screen, the angle the first rod makes to the horizontal increases smoothly to 360.

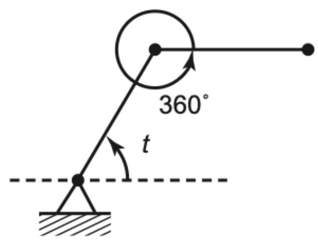
The first coordinate parameterizes the angle of the first rod to the horizontal. The second rod remains parallel to the horizontal. Similarly, if the cursor moves

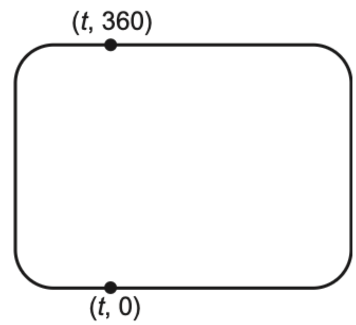


from the lower left corner up the left side of the screen, the angle the second rod makes to the horizontal increases smoothly to 360. At the same time, the first rod lies parallel to the horizontal line. Consequently, the second coordinate parameterizes the angle of the second rod. So, for example, if the cursor is at (90, 30), then the first rod is perpendicular to the horizontal and the second makes an angle of 30 to the horizontal.



If the cursor is at the top of the screen, the second rod has rotated 360. It's back where it started. The point on the top of the screen produces the same position of the linkage as the corresponding point at the bottom. So if you want

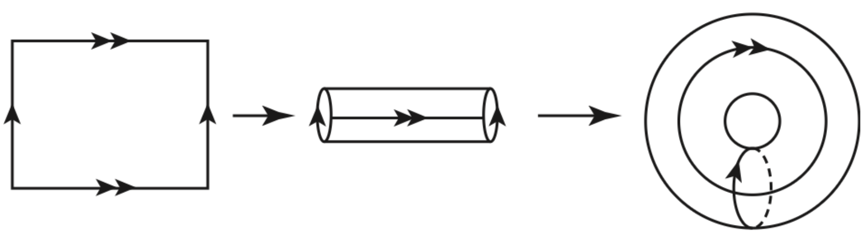




to do negative angles, you start at the bottom of the screen and come out the top going down.

Similarly, if the cursor hits the right of the screen, then the first rod has rotated 360 from the horizontal. It's back where it started. The point at the right edge and the point directly to its left on the left edge both correspond to the same position of the linkage.

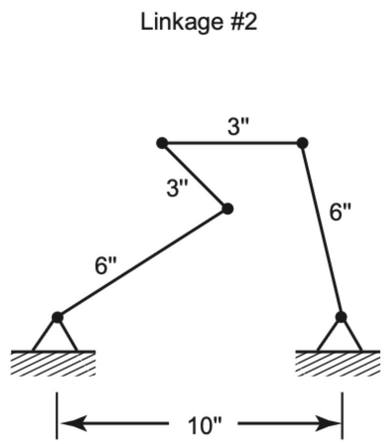
If you manipulate the linkage itself and spin the first rod, keeping the second rod at the same angle to the horizontal, you'd see the cursor going from left to right across the screen. Each time the cursor hit the right edge, it would reappear at the left again. The right edge is identified with the left edge. The top is identified with the bottom. This is a torus!



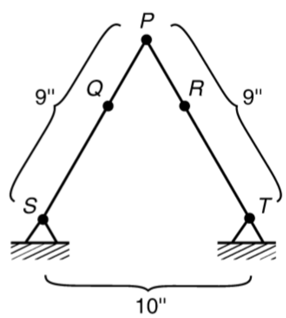
To summarize, the torus smoothly parameterizes the positions of the linkage: to every point on the torus, there corresponds a position of the linkage, and conversely, to every position of the linkage, there corresponds a point on the torus. The word "smoothly" means moving the cursor smoothly around the screen produces a smooth motion of the linkage. There are no lurches or bumps.

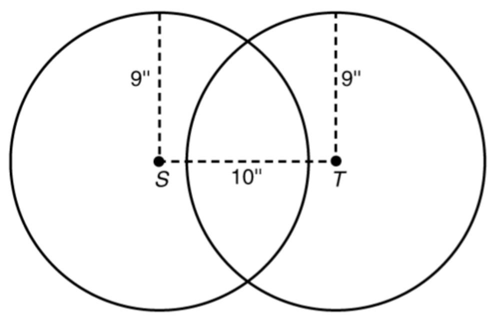
6.1.2 Linkage System: Second Example

Now let's consider a linkage that's a little bit more complicated. This time there are four arms, three free pivots, and two fixed pivots. The goal is to describe a



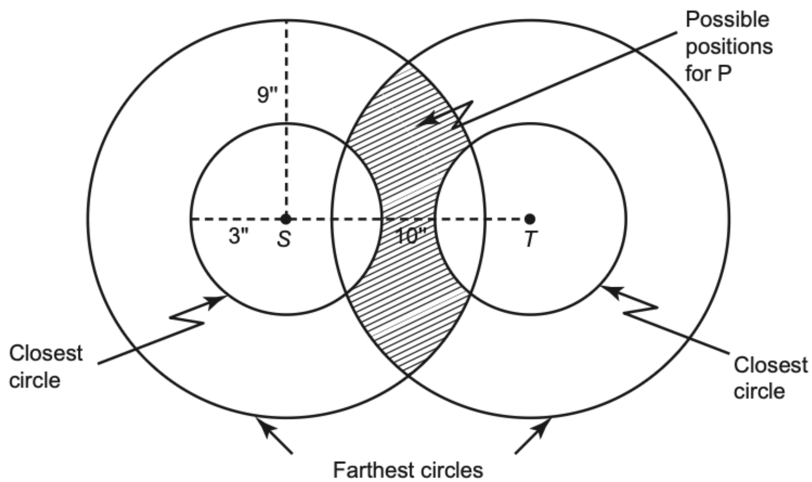
set of points corresponding to the positions (or states) of the linkage. We chose to look at the central free pivot and describe its possible positions.



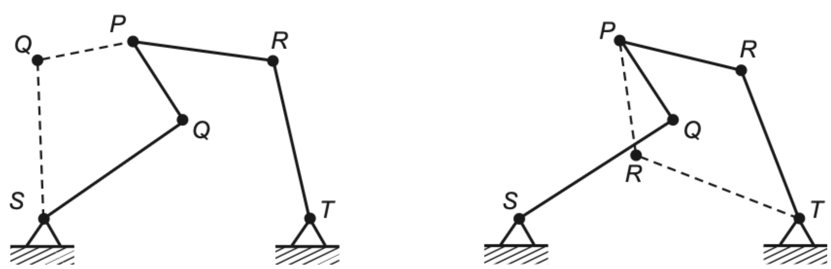


First let us label the linkage: P is the central free pivot, Q and R are the other free pivots, S and T are the fixed pivots. We started with the rods set up like an isosceles triangle with base of 10 inches and two sides of 9 inches each. So 9 inches is the farthest P can be from T and from S. The position of P has to lie inside two circles, each of radius 9 inches, one with center S and the other with center T. We thought of these circles as “farthest” circles.

How close could P be to T (or to S)? Instead of extending PRT as before, we can bend it as much as we could, so P would actually lie on top of the rod RT. This is the closest P could be to T. So P must lie outside a circle with a radius of 3 inches with center T. Similarly, P would have to lie outside the same size circle with center S. These are the “closest” circles. Now we know P is inside both of the farthest circles and outside both of the closest circles. Here’s the picture: The curvy hexagon is the set of positions of the point P.

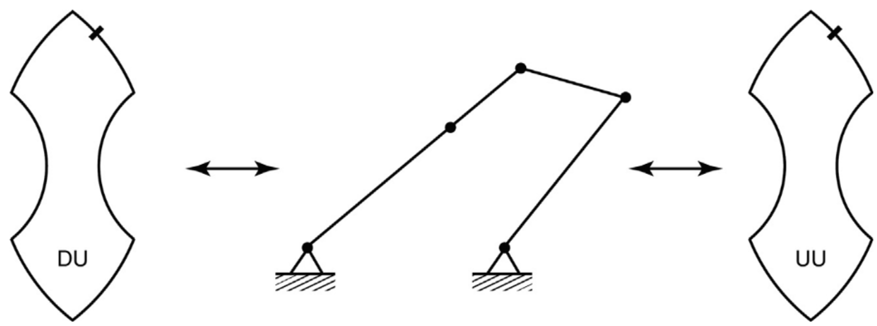


But there's more to the linkage than the position of point P. In fact, every position of P corresponds to four different states of the linkage, depending on how the secondary elbows are bent. The left elbow could be bent up or down; the right elbow could be bent up or down. Here's an example.

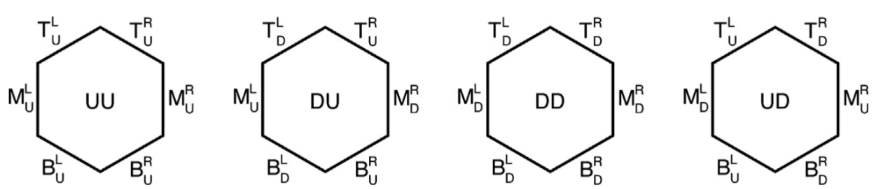


To cover these possibilities, make four copies of the hexagon and label them UU, UD, DU, and DD. The first letter describes the state of the left elbow (up or down); the second describes the state of the right elbow. So the state of the linkage in the example just shown corresponds to a point in hexagon UD with the left elbow up and the right elbow down. The four hexagons are connected. Each edge of a hexagon gets identified with the edge of another hexagon.

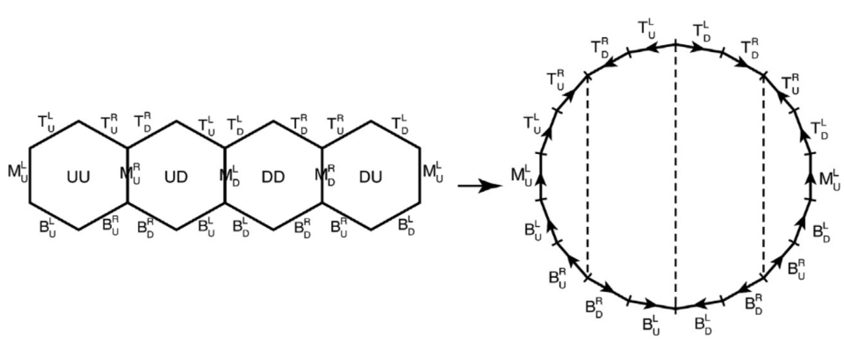
To start, a point along the top right edge (T^R) of hexagon UU is the same as the corresponding point on the top right edge of hexagon DU. These are points where the left elbow is fully extended and not bent so that "crossing" this edge from UU to DU (or DU to UU) smoothly switches the left elbow from up to down (or down to up, respectively). Similarly, a point along the top left edge



(TL) of UU is the same as the corresponding point on the top left edge of UD. Crossing edges T^R, B^R, and M^L amounts to switching the left elbow smoothly from up to down (and vice versa). Thus, when two hexagons are labeled with the same second letter, the corresponding edges labeled $T^R, B^R, andM^L$ get identified. Call them T^R_U, B^R_U, and M^L_U when the second letter is U and T^R_D, B^R_D, and M^L_D when the second letter is D. Crossing the remaining three sides of the hexagon - T^L, B^L, and M^R - amounts to switching the right elbow from up to down and back. Combining all this information, we obtain four hexagons having their edges labeled like this:



We can glue the hexagons together side by side. That gave us a single



shape—with 18 edges left unglued. Then we can stretch the whole thing into an 18-gon. The space we obtain after gluing along the labeled directed edges is a very complicated surface we have not seen before.

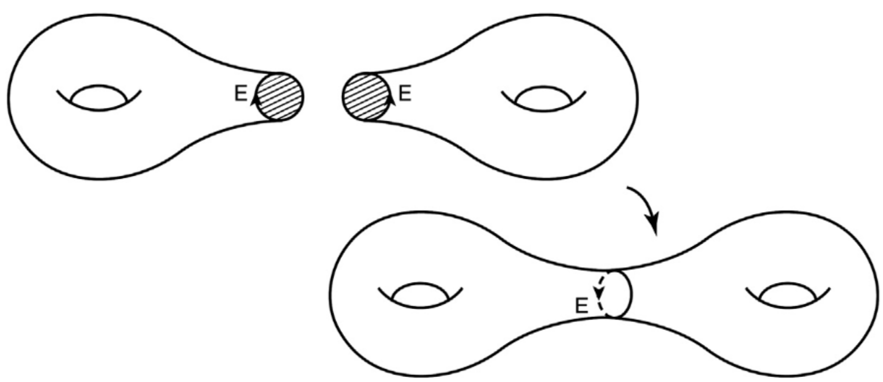
Definition 6.1. A surface S is an object on which you can draw a map; that is, you can express the set S as a union of vertices, edges, and countries of an OK map.

For the surfaces we've seen so far, we are always able to cut them open and flatten them up to get a polygon with gluing data given by labeled directed edges, we call such a polygon with gluing date a pattern.

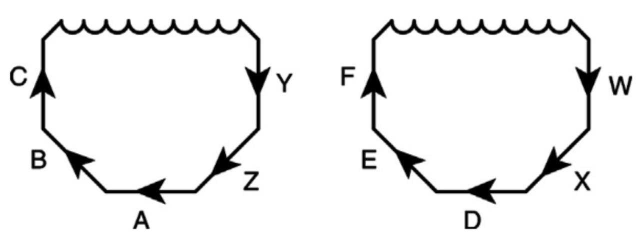
Theorem 6.2. *Given a surface, you can cut it open and flatten it out to get a pattern. Conversely, given a pattern, it can be assembled into a surface.*

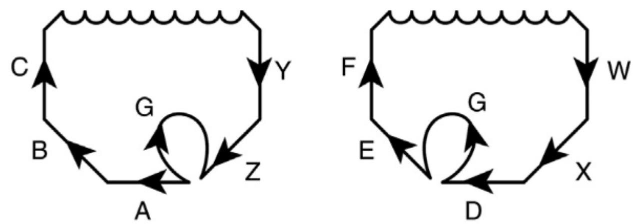
6.2 Connected Sum

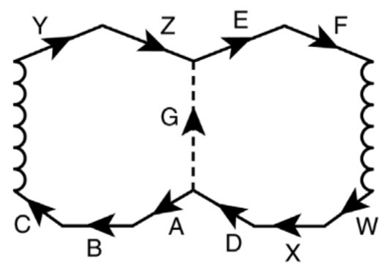
Definition 6.3. A connected sum of two surfaces S and T is a surface formed by deleting a disk inside each surface and gluing together the boundary circles.



Suppose patterns for surfaces S and T are as follows:

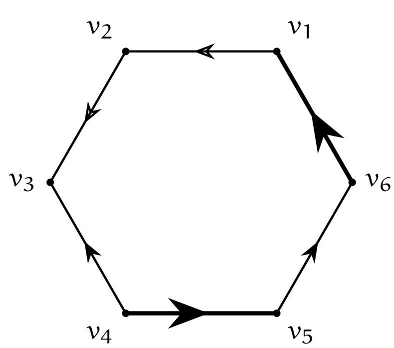






So patterns for S-with-disk-removed and T-with-disk-removed are given by the above two patterns. Then a pattern for S#T is

Exercise 6.4. Identify the following pattern as the connected sum of two surfaces.



Cut along the line segment connecting v_1 and v_2, the triangle component is $\mathbb{RP}^2$-with-disk-removed, or equivalently a Möbius strip, and other component is a torus-with-disk-removed, so this is the connected sum $\mathbb{RP}^2\#T$.

Exercise 6.5. Draw the pattern for $\mathbb{RP}^2\#K$.

Exercise 6.6. Use cut and repaste to show that $\mathbb{RP}^2\#T = \mathbb{RP}^2\#K$.

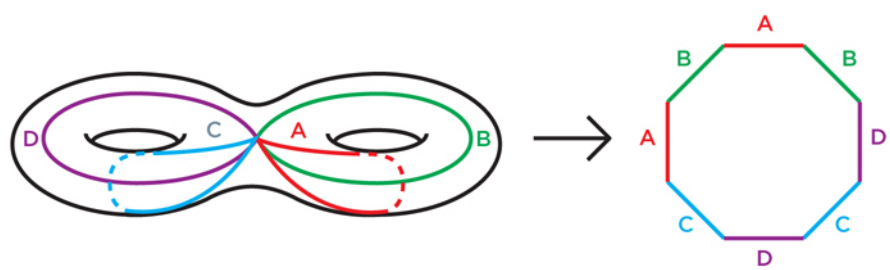
Example 6.7. Let M be a Möbius strip. Draw a graph to show that $M\#T = M\#K$.

Example 6.8. There are four octagon gluing patterns (up to rotation and relabelling) which give a double torus.

- a b c d A B C D
- a b A B c d C D
- a b A c B d C D
- a b c A B d C D

To convert these into pictures, start from one corner of the octagon and walk around it in the counterclockwise direction. When you see lower case, write that letter and a counterclockwise arrow. When you see upper case, write the lower case letter and a clockwise arrow.

The second one is the standard one that you can get from the pattern of connected sums. We will discuss in class how to see the first pattern can be seen



as equivalent to the second via cut and paste.

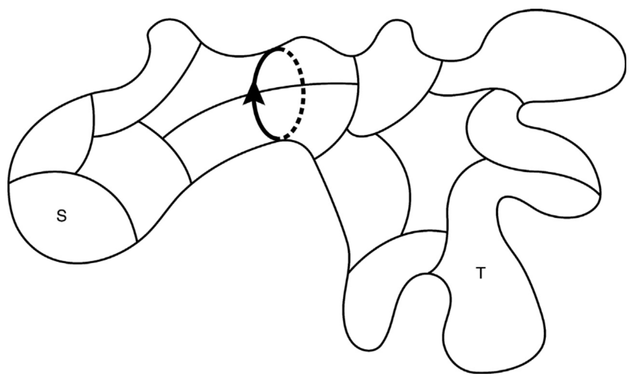
Exercise 6.9. Draw the pattern for an n-torus $nT = \underbrace{T\#\cdots\#T}_{n}$.

6.3 Euler Number

Definition 6.10. Suppose that S is a surface and that, for every OK map on the surface, $V - E + C$ is always the same. Then call that common number the *Euler number* of the surface. Denote the Euler number by $N(S)$.

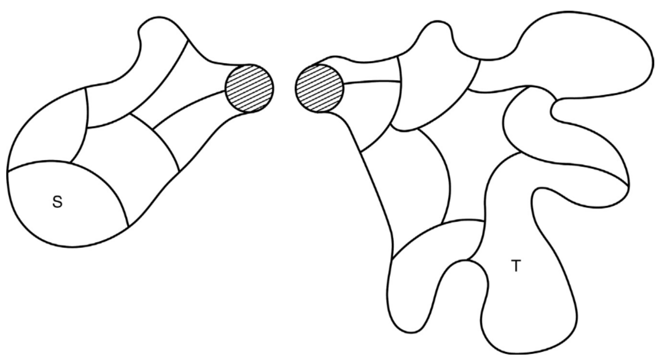
Example 6.11. $N(S^2) = 2$ and $N(T) = 0$.

Now, suppose S and T are two surfaces with Euler numbers $N(S)$ and $N(T)$. (We will show their existence later) So, take a map on $S\#T$. If we were to count V, E, and C, would there be some way we could predict what $V - E + C$ is if we knew $N(S)$ and $N(T)$?



We did something like this for the torus. We cut it open along its "waist," flattened it out into an island-with-lake/island-with-disk-removed, and then used the Euler number for that.

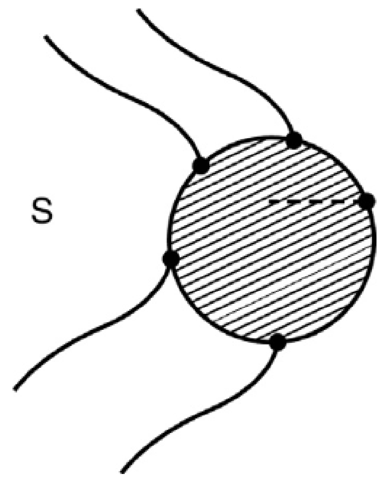
We drew a circle around the waist and added to the map the new vertices, edges, and countries it created. We showed that doing that didn't change $V - E + C$. We could try the same thing with S#T. So we add the circle. The



same argument as before says $V - E + C$ stays the same. Now cut along the circle. We get S-with-lake and T-with-lake with maps having data V_S, E_S, C_S and V_T, E_T, C_T, respectively.

Also, $V_S - E_S + C_S = N(S) - 1$, and $V_T - E_T + C_T = N(T) - 1$. Now, $V - E + C = V_S - E_S + C_S + V_T - E_T + C_T$, except that the vertices and edges along the added circle have been counted twice. We need to subtract those off from the V and E counts.

The circle has the same number of vertices as edges. We'd be subtracting n (vertices) and adding n (edges) to $V - E + C$. They cancel each other out! So our



equation holds:

$$V - E + C = V_S - E_S + C_S + V_T - E_T + C_T = N(S) + N(T) - 2.$$

Theorem 6.12. *If* S *and* T *are two surfaces having Euler numbers* $N(S)$ *and* $N(T)$, *then the surface sum has an Euler number given by the formula*

$$N(S\#T) = N(S) + N(T) - 2.$$

Exercise 6.13. Suppose that S is a surface and that S' is gotten by removing a disc from S. (Thus S' is S-with-lake.) Consider the following questions:

1. If S has an Euler number, does S'? If the answer is yes, are the two numbers related? How?
2. If S' has an Euler number, does S? If the answer is yes, are the two numbers related? How?
3. What about S-with-n-lakes?

Exercise 6.14. What is the Euler number of an n-torus?

Exercise 6.15. What is the Euler number of the Möbius strip?

Exercise 6.16. What is the Euler number of the projective plane $\mathbb{RP}^2$?

Exercise 6.17. What is the Euler number of the Klein bottle?

6.4 Classification of Surfaces

6.4.1 When are two surfaces the same?

Definition 6.18. Start with maps M_1 and M_2. Denote the sets of vertices, edges, and countries of M_j by V_j, E_j, and C_j, respectively, for $j = 1, 2$. Suppose F is a 1-1 onto function $F : V_1 \cup E_1 \cup C_1 \to V_2 \cup E_2 \cup C_2$ such that G F takes vertices to vertices, edges to edges, and countries to countries.

- If v and v' are two vertices in M_1 joined by edge e, then $F(v)$ and $F(v')$ are two vertices in M_2 joined by edge $F(e)$.
- If e and e' are two edges in M_1 that meet in vertex v, then $F(e)$ and $F(e')$ are two edges of M_2 that meet in vertex $F(v)$.
- If c and c' are two countries in M_1 that border along edge e, then $F(c)$ and $F(c')$ are two countries of M_2 that border along edge $F(e)$.

Then maps M_1 and M_2 are the same.

Definition 6.19. Suppose that S_1 and S_2 are two surfaces such that S_1 has map M_1 on it and S_2 has map M_2 on it. If maps M_1 and M_2 are the same, then surfaces S_1 and S_2 are the same.

Theorem 6.20. *If surfaces S_1 and S_2 are the same, then they have the same pattern.*

Theorem 6.21. *If surfaces S_1 and S_2 have the same pattern, then they are the same surface.*

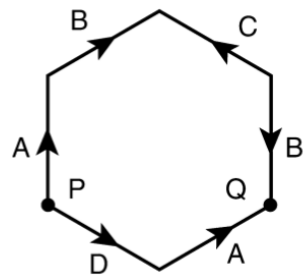
Equivalent Patterns and Surfaces Patterns P_1 and P_2 are equivalent if you can cut up P_1, label the cuts, reassemble the pieces, and glue them to get pattern P_2.

Theorem 6.22. *Two surfaces are the same if and only if they have equivalent patterns.*

6.4.2 Surface Symbol

Take a pattern for the surface. Pick a vertex of the polygon and travel clockwise around its perimeter, writing down the edge labels as you go, one after the other. If an edge is labeled A and the arrow on the edge is also going clockwise, write A; if the arrow is going counterclockwise, write down A^{-1}. Do this until you arrive back at the vertex where you started. The result is the surface symbol.

Example 6.23. Start with vertex P on the following pattern. The symbol is $ABC^{-1}BA^{-1}D^{-1}$.



Equivalent Symbols It shouldn't matter at which vertex you start. If you start at vertex Q on the pattern, then you'd get the symbol $A^{-1}D^{-1}ABC^{-1}B$. This is a different symbol from the one we got before. But they're for the same surface. So we call them equivalent and write

$$ABC^{-1}BA^{-1}D^{-1} \sim A^{-1}D^{-1}ABC^{-1}B.$$

There are other symbols we'd call equivalent. Like the symbols we'd get by using different letters. If instead of using A, B, C, and D, we used W, X, Y, Z (respectively), we'd get $WXY^{-1}XW^{-1}Z^{-1}$. So we'd want to write

$$ABC^{-1}BA^{-1}D^{-1} \sim WXY^{-1}XW^{-1}Z^{-1}.$$

Also going counterclockwise instead of clockwise should not change the pattern.

And we will get another symbol if you changed the direction of all the arrows. But it's really the same pattern. So

$$ABC^{-1}BA^{-1}D^{-1} \sim A^{-1}B^{-1}CB^{-1}AD.$$

We also want the symbols we get from equivalent patterns to be equivalent themselves.

Rules for Equivalent Symbols

- $ABC...Z \sim BC...ZA$ (Cyclically permute letters.)
- $AB...YZ \sim ZY...BA$ (Write letters in the opposite direction.)
- $\alpha A\beta A\gamma \sim \alpha A^{-1}\beta A^{-1}\gamma$ and $\alpha B\beta B^{-1}\gamma \sim \alpha B^{-1}\beta B\gamma$ (Reverse arrows on all occurrences of a single letter. Assume a Greek letter denotes a string of letters.)
- $\alpha A\beta A\gamma \sim \alpha X\beta X\gamma$ and $\alpha A\beta A^{-1}\gamma \sim \alpha X\beta X^{-1}\gamma$ (Replace all occurrences of a letter by another letter not already being used.)

- $ABAB^{-1} \sim CCDD$ (Different symbols for a Klein bottle.)
- $\alpha EE^{-1}\beta \sim \alpha\beta$ (Connected sum of sphere with surface S is equivalent to surface S.)

Example 6.24. Here is a list of surfaces you and they have worked with before. Find a surface symbol for each one. You may know more than one pattern for a surface; in that case, there would be more than one symbol. Write down these extra symbols, too.

1. Sphere.
2. Cylinder.
3. Möbius strip.
4. Torus.
5. Klein bottle.
6. Projective plane P.
7. Double torus.
8. $P\#P\#P$.

Occurrence of Letters in Surface Symbol In a surface symbol, a letter can occur at most twice. A letter that occurs only once must be the edge of a lake or part of the edge of a lake.

We would like to find some universal equivalent presentation of the symbols to help us understand the classification of surfaces.

The first two rules that we shall get out of cut and reassembling deals with the cases where we have "...X...X..." showing up in the symbols.

Rule 1 $\alpha X\beta\gamma X\delta \sim \alpha X\gamma X\beta^{-1}\delta$. This rule tell us how to move symbols on the right hand side of the first copy of X to the right hand side of the second copy of X.

Rule 2 $\alpha\beta X\gamma X\delta \sim \alpha X\gamma\beta^{-1}X\delta$. This rule tell us how to move symbols on the left hand side of the first copy of X to the left hand side of the second copy of X.

Combining those two rules tells us how to to move pairs XX to the beginning of the symbols, and in particular this works all possible pairs that shows up as we can do this inductively.

Assembling the Projective Planes The pair XX corresponds to a connected sum component $\mathbb{RP}^2$.

Theorem 6.25. *If a surface symbol S has a pair of letters occurring as ...A...A... or ...A^{-1}...A^{-1}..., then $S \sim XXYY...ZZ\sigma$, where σ is a symbol string having the property that, if a letter B occurs twice in it, it occurs as ...B...B^{-1}... or ...B^{-1}..B.... So S is the connected sum of several projective planes and something else.*

Exercise 6.26. Consider the following two symbols for $\mathbb{RP}^2\#K$.

1. Find an equivalent symbol of $XXYZYZ^{-1}$.
2. Find an equivalent symbol of $XXYZY^{-1}Z$.

Your results for either presentation should be equivalent.

The following two rules deals with the cases where we have "$\ldots X \ldots X^{-1} \ldots$" or "$\ldots X \ldots X^{-1} \ldots$" showing up in the symbols.

Rule 3 $\alpha X\beta\gamma X\delta \sim \alpha X\gamma X\beta^{-1}\delta$. This rule tell us how to move symbols on the right hand side of the first copy of X to the right hand side of the second copy of X.

Rule 4 $\alpha\beta X\gamma X\delta \sim \alpha X\gamma\beta^{-1}X\delta$. This rule tell us how to move symbols on the left hand side of the first copy of X to the left hand side of the second copy of X.

Combining those two rules tells us how to to swap the orders of words between $X \ldots X^{-1}$ and move words on the left hand side of $X \ldots .X$ to the right hand side.

In particular, we are able to rewrite symbols of the form

$$\ldots X \ldots Y \ldots X^{-1} \ldots Y^{-1} \ldots$$

to an equivalent one of the form $\ldots XYX^{-1}Y^{-1} \ldots$.

Theorem 6.27. *If two pairs of like letters in a surface symbol are meshed as*

$$\ldots X \ldots Y \ldots X^{-1} \ldots Y^{-1} \ldots,$$

then the surface is the sum of a torus and something else. In fact,

$$\alpha X\beta Y\gamma X^{-1}\delta Y^{-1}\pi \sim XYX^{-1}Y^{-1}\beta\pi\alpha\delta\gamma.$$

Assembling the Tori The word $XYX^{-1}Y^{-1}$ corresponds to a connected sum component of torus T.

Theorem 6.28. *(A) If S is a surface symbol such that every letter 4 that occurs twice occurs as $\ldots A \ldots A^{-1} \ldots$ or $\ldots A^{-1} \ldots A \ldots$, then*

$$S \sim X_1 Y_1 X_1^{-1} Y_1^{-1} \ldots X_n Y_n X_n^{-1} Y_n^{-1} \sigma$$

where σ is a string of letters in which every letter occurring twice occurs as $\ldots A \ldots A^{-1} \ldots$ or $\ldots A^{-1} \ldots A \ldots$.So the corresponding surface is the sum of n tori and something else. In addition, no pairs of letters X, Y occurring twice in σ also mesh; that is, $\ldots X \ldots Y \ldots X^{-1} \ldots Y^{-1} \ldots$ will not occur in σ.

(B) Moreover, if S is any surface symbol, then

$$S \sim U_1 U_1 \ldots U_m U_m X_1 Y_1 X_1^{-1} Y_1^{-1} \ldots X_n Y_n X_n^{-1} Y_n^{-1} \sigma$$

where σ satisfies the same properties as σ in part A. Thus any surface is the sum of m projective planes, n tori, and something else, where the something else has symbol σ.

Exercise 6.29. The surface symbol for the second linkage system "control screen" is

$$ABCD^{-1}B^{-1}EDC^{-1}E^{-1}A^{-1}F^{-1}G^{-1}HFJ^{-1}H^{-1}GJ.$$

Use Rules 3 and 4 to figure out what it is.

Exercise 6.30. Show that $XXABA^{-1}B^{-1} \sim CCDDEE$. How might this modify the part B in the previous theorem?)

Question 6.31. *Suppose a surface symbol S has these properties:*

- *Every letter in the symbol occurs twice.*
- *Every letter occurs as $\ldots A \ldots A^{-1} \ldots$ or $\ldots A^{-1} \ldots A \ldots$.*
- *No pair of letters X, Y mesh; that is, $\ldots X \ldots Y \ldots X^{-1} \ldots Y^{-1} \ldots$ does not occur in S.*

An example is $S = AB^{-1}CC^{-1}BD^{-1}DA^{-1}$. Investigate all such surface symbols. Make up more examples of your own. What can you say about the surface that corresponds to such a symbol?

Question 6.32. *Suppose a surface symbol S has these properties:*

- *There may be letters occurring only once.*
- *Every letter occurring twice occurs as $\ldots A \ldots A^{-1} \ldots$ or $\ldots A^{-1} \ldots A \ldots$.*

- *No pair of letters* X, Y *mesh; that is,* $\ldots X \ldots Y \ldots X^{-1} \ldots Y^{-1} \ldots$ *does not occur in* S.

(The only difference between the symbols here and those in the previous questions is that, here, there may be letters occurring only once.) Investigate all such surface symbols. Make up more examples of your own. What can you say about the surface that corresponds to such a symbol?

Question 6.33. *Combine what we discussed so far to come up with a theorem of the following form: "every surface is equivalent to a surface on the following list:..."*

Question 6.34. *Earlier, there was a question: does every surface have an Euler number? Given what you know so far, what is the status of this question?*

6.4.3 Classification of Surfaces

We get a complete list of surfaces

1. Sphere.
2. Sphere with r lakes (or equivalently, with r disks removed).
3. Connected sum of s tori.
4. Connected sum of s tori with r lakes.
5. Connected sum of s projective planes.
6. Connected sum of s projective planes with r lakes.

How do we know all of those are different?

Note for the surfaces above, we know how to compute their Euler number.

In particular, the Euler numbers of the torus and the Kline bottle are both zero. The Euler number of a sphere with two lakes is also zero.

These spaces are different, and how shall we tell the difference?

Orientability

Definition 6.35. A surface is nonorientable if and only if it has a pattern whose symbol is of the form $\ldots X \ldots X \ldots$.

Theorem 6.36. *If* S *is a nonorientable surface, then every symbol has the property that if a letter* X *occurs twice, it occurs as* $\ldots X \ldots X \ldots$. *In particular, if surface* S *has a symbol with the property that every letter occurring twice occurs as* $\ldots X \ldots X \ldots$, *then that surface can't be nonorientable; that is, it must be orientable.*

Corollary 6.37. *The connected sum of projective planes (with lakes) must be different from spheres (with lakes) and connected sum of tori (with lakes).*

How to compute Euler number from a pattern of a surface? A pattern on a surface gives us a map with one country (face), and the number of edges is given by the number of distinctly label edges, so if we can compute the number of distinct vertices on the polygon, we know how to compute its Euler number.

In class, we will discuss an algorithm to count the number of vertices. This algorithm shall also tell us how to count the number of lakes from a pattern.

Theorem 6.38. *If a surface has two symbols S and S', then the number of lakes counted for S equals the number of lakes counted for S'.*

Theorem 6.39. *Two surfaces are the same if and only if the following three conditions are satisfied:*

- *The two surfaces are either both orientable or both nonorientable.*
- *The two surfaces have the same Euler number.*
- *The two surfaces have the same number of lakes.*

Genus The *genus* g of a finite surface without boundary is the maximal number of disjoint circular cuts that can be made in a surface without disconnecting it.

Theorem 6.40. *For a finite surface S without boundary, $\chi(S) = 2 - 2g$ where $\chi(S)$ is the Euler number of S and g is the genus of S.*

6.5 References

1. David Gay, Explorations in Topology: Map Coloring, Surfaces and Knots, Chapter 9: Surface Sums and Euler Numbers.

2. David Gay, Explorations in Topology: Map Coloring, Surfaces and Knots, Chapter 10: Classification of Surfaces.

3. David Gay, Explorations in Topology: Map Coloring, Surfaces and Knots, Chapter 11: Classification (Part II), Existence and Four-Space.

7 Knots

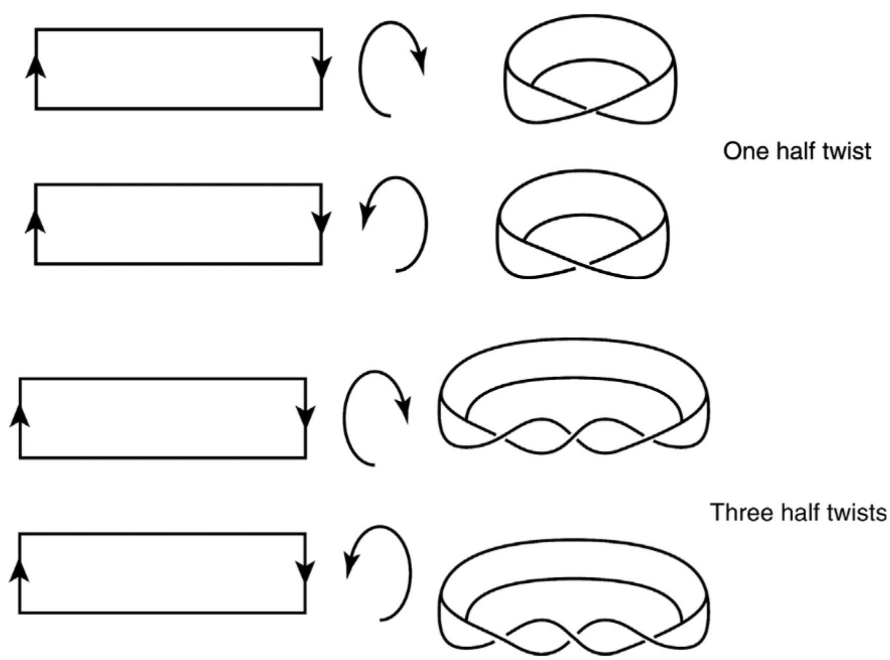
We now have a complete classification of finite (connected) surfaces with or without boundaries, and you might wonder why we never mention the story of 1-dimensional spaces. It turns out there are only two of them: an infinite line and a circle.

While the intrinsic topology of 1-dimensional spaces might be boring, the extrinsic topology of 1-dimensional spaces in $\mathbb{R}^3$ is anything but boring. There are infinitely many different ways to put a knot in $\mathbb{R}^3$ such that you cannot deform one into another. In general, identifying equivalent knots in $\mathbb{R}^3$ is a very challenging problem.

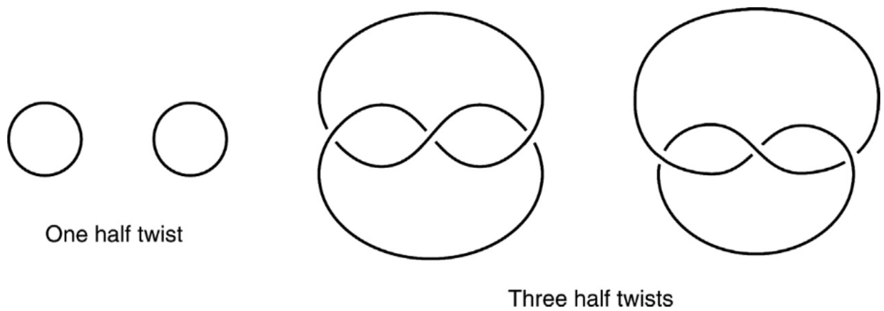
Even though it might sound like it's not a theory,*knot theory* is an important area of lower dimensional topology that has some surprising applications, including the study of DNA and proteins. The general goal of knot theory is to develop methods to determine whether a given knot can be deformed to another knot, in particular to an ordinary circle.

Definition 7.1. A *knot* is a circle loop in $\mathbb{R}^3$, and a *link* as one or more disjoint loops in $\mathbb{R}^3$. We say two knots or links are equivalent or the same if one can be deformed to the other in $\mathbb{R}^3$.

We've seen some knots and links when we study twisted strips.



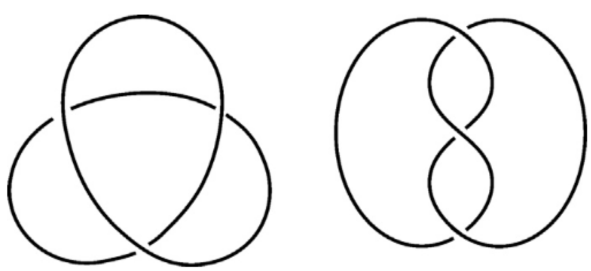
We can draw the projection of these edges of the strips in the following way.



Exercise 7.2. Use rope to make the two overhand knots above. (Be sure to secure the ends.) See if you can manipulate one into the other. (No cutting!) What happens?

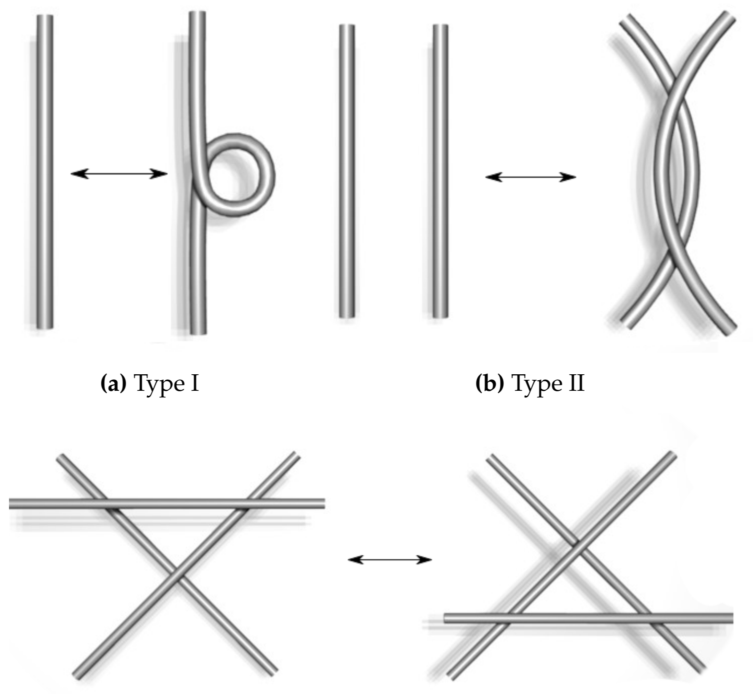
Since the pages of books and blackboards are only 2 dimensional, we need to represent-knots and links in a 2-dimensional way. A *projection* is a 2-dimensional drawing of a knot or link where we indicate undercrossings (i.e., places where part of the knot or link passes under another part in $\mathbb{R}^3$. The diagram we see above are examples of knot diagrams.

Example 7.3. The following two knots are the same. This is called the *trefoil knot*.



Exercise 7.4. Make one knot from Example 7.3 and see if you can transform it into another. What are the movements that you need to use?

If you think about the movements you were doing to transform one knot diagram into another, you will notice that essentially there are three moves that are essential. These are called Reidemeister moves, see Figure 22.



(a) Type I **(b)** Type II

(c) Type III

Figure 22: Reidemeister moves

7.1 Knot Invariants

In general it's very difficult to tell whether two projection diagram of knots are the same, so we want to create something that would help us to distinguish knots by looking at their diagrams.

A knot invariant is a "quantity" that is the same for equivalent knots. For example, if the invariant is computed from a knot diagram, it should give the same value for two knot diagrams representing equivalent knots. An invariant may take the same value on two different knots, so by itself may be incapable of distinguishing all knots.

7.2 References

1. David Gay, Explorations in Topology: Map Coloring, Surfaces and Knots, Chapter 13: Knots.

2. Erica Flapan, Knots, Molecules, and the Universes: An Introduction to Topology.

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